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# SOBRIETY VIA $\theta$-OPEN SETS

BY

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**Abstract.** SÜNDERHAUF studied the important notion of sobriety in terms of nets. In this paper, by the same token, we present and study the notion of $\theta$-sobriety by utilizing the notion of $\theta$-open sets.

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**Key words:** $\theta$-open, $\theta$-closed, $\theta$-compact space, $\theta$-sobriety.

**1. Introduction.** In 1943, FOMIN [4] (see, also [5]) introduced the notion of $\theta$-continuity. The notions of $\theta$-open subsets, $\theta$-closed subsets and $\theta$-closure were introduced by VELIČKO [11] for the purpose of studying the important class of $H$-closed spaces in terms of arbitrary filterbases. DICKMAN and PORTER [2], [3], JOSEPH [7] continued the work of Veličko. Recently NOIRI and JAFARI [9] have also obtained several new and interesting results related to these sets.

In what follows $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) denote topological spaces. Let $A$ be a subset of $X$. We denote the interior and the closure of a set $A$ by $Int(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a $\theta$-cluster point of $A$ if $A \cap Cl(U) \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure of $A$. A subset $A$ is called $\theta$-closed if A and its $\theta$-closure coincide. The complement of a $\theta$-closed set is called $\theta$-open. We denote the collection of all $\theta$-open sets by $\theta(X, \tau)$. It is shown in [8] that the collection of $\theta$-open sets in a space $X$ form a topology denoted by $\tau_\theta$. A topological space $(X, \tau)$ is called $\theta$-compact [6] if every cover of the space by $\theta$-open sets has a finite subcover. We denote the filter of $\theta$-open neighbourhoods [1] of some point $x$ in $X$ by $\Omega_\theta(x)$.

**Definition 1.** *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\theta$-continuous if for each $x \in X$ and each $V$ in $Y$ containing $f(x)$, there exists an open set $U$ in $X$ containing $x$ such that $f(Cl(U)) \subset Cl(V)$.*

**Definition 2.** *Two topological spaces $(X, \tau)$ and $(Y, \sigma)$ are $\theta$-homeomorphic* [11] *if there exists a one-to-one and onto function $f : (X, \tau) \to (Y, \sigma)$ such that $f$ and $f^{-1}$ are both $\theta$-continuous.*

## 2. $\theta$-sobriety

**Definition 3.** *A space $(X, \tau)$ is said to be $\theta$-sober if it is $\theta$-homeomorphic with the space of points of its frame of $\theta$-open sets.*

Recall that the $\theta$-saturated set and the $\theta$-kernel of a set $A$ [1] are the same, i.e. $\cap\{O \in \theta(X, \tau) \mid A \subset O\}$.

$\theta$-sobriety implies the existence of LUB(= least upper bound) of subsets which are directed with respect to the order of $\theta$-specialization, i.e. if $(X, \tau)$ is a topological space, then the order of $\theta$-specialization of $X$ is defined by $x \leq_r y \Leftrightarrow x \in Cl_\theta(\{y\})$.

**Theorem 1.** *Let $X$ be a $\theta$-sober space for which finite intersection of $\theta$-compact $\theta$-saturated subsets are $\theta$-compact.*

(1) *Every cover of a $\theta$-compact set by $\theta$-open sets contains a finite subcover.*

(2) *If the intersection of $\theta$-compact $\theta$-saturated sets is contained in a $\theta$-open set, then the same is true for an intersection of finitely many of them.*

Now we offer a new notion called $\theta$-observative net by which we characterize $\theta$-sobriety.

**Definition 4.** *A net $(x_i)_i \in I$ on a space $X$ is $\theta$-observative if for all $i \in I$ and for all $U \in \theta(X, \tau)$ we have that $x_i \in U$ implies that the net is eventually in the set $U$.*

Recall that a filter base $\mathcal{F}$ is called $\theta$-convergent [12] to a point $x$ in $X$ if for any open set $U$ containing $x$ there exists $B \in \mathcal{F}$ such that $B \in Cl(U)$.

**Definition 5.** *A $\theta$-observative $(x_i)_i \in I$ strongly $\theta$-converges to a point $x$ in $X$ if it $\theta$-converges to $x$ with respect to $\theta(X, \tau)$, and also if it satisfies that $x$ is an element of every $\theta$-open set which eventually contains the net. We denote it by $x_i \xrightarrow{\theta} x$.*

**Lemma 1.** *If $(x_i)_i \in I$ is a $\theta$-observative net on a space $(X, \tau)$, then $x_i \xrightarrow{\theta} x$ if and only if $x_i \longrightarrow x$ with respect to the $\tau_\theta$.*

**Proof.** Let $x_i \xrightarrow{\theta} x$ and $x \in A$ for some $\theta$-closed set $A$. If the net is not eventually contained in $A$, then it is frequently in the $\theta$-open set $X - A$. By hypothesis, the net is $\theta$-observative and therefore $[x]_{\geq i} \subseteq X - A$ for some tail. Hence $x \in X - A$ as a consequence of strong $\theta$-convergence. But this is a contradiction and hence the claim.

Now suppose that $x_i \longrightarrow x$ with respect to the $\tau_\theta$. Then a $\theta$-open set which eventually contains the net but does not contain $x$ establishes a $\theta$-neighbourhood $X - U$ of $x$ which has been forgotten by the net. Therefore strong $\theta$-convergence follows readily. $\square$

Here we establish the $\theta$-derived filter $\mathcal{F}(x)\mathcal{I}$ for a net $(x_i)_i \in I$ as follows:

$$\mathcal{F}(x)\mathcal{I} = \{U \in \theta(X, \tau) \mid \exists i \in I \cdot [x]_{\geq i} \subseteq U\}.$$

**Definition 6.** *A filter $\mathcal{F} \subseteq \tau$ is called $\theta$-completely prime if for every $O \in \mathcal{F}$ and for any family of $\theta$-open sets $(O_i)_{i \in I}$ such that $O \subseteq \bigcup_I O_i$, then $O_k \in \mathcal{F}$ for some $k \in I$.*

**Theorem 2.** *A filter derived from a $\theta$-observative is $\theta$-completely prime.*

**Proof.** Let the net $(x_i)_i \in I$ be a $\theta$-observative and $[x]_{\geq i} \subseteq \bigcup_{j \in J} U_j$ for some collectection of $\theta$-open sets and some index $i \in I$. Hence $x_i \in \bigcup_{j \in J} U_j$. Therefore there is some $j_0 \in J$ with $x_j \in U_{j_0}$. Since the net is $\theta$-observative, then it follows that some tail is contained in $U_{j_0}$. Thus the set is a filter.$\square$

**Proposition 1.** *If $(x_i)_i \in I$ is a $\theta$-observative net, then $x_i \xrightarrow{\theta} x$ if and only if $\mathcal{F}(x)\mathcal{I} = \Omega_\theta(x)$.*

**Proof.** Since $(x_i)_i \in I$ strongly $\theta$-converges to $x$ if and only if it is the case that $x \in U$ is equivalent to the existence of some $i \in I$ with $[x]_{\geq i} \subseteq U$.

But how can we deal with the situation where a space is $\theta$-sober if all its $\theta$-observative nets strongly $\theta$-converge?

In such situation, we need the following construction:

Assign to each $\theta$-completely prime filter $\mathcal{F}$ a $\theta$-observative net such that $\mathcal{F}(x)\mathcal{I} = \mathcal{F}$. $\square$

**Theorem 3.** *Let $\mathcal{F}$ be a filter of $\theta$-open subsets of the space $(X, \tau)$. Then $\mathcal{F}$ is $\theta$-completely prime if and only if for all $U \in \mathcal{F}$, there exists $x \in U$ with the property that $x \in G$ implies $G \in \mathcal{F}$ for every $G \in \theta(X, \tau)$.*

**Proof.** Suppose that $\mathcal{F}$ has this property and $\bigcup_{j\in J} U_j \in \mathcal{F}$. Take $x \in \bigcup_{j\in J} U_j$ with $x \in G \Rightarrow G \in \mathcal{F}$. Clearly, $x \in U_{j_0}$ for some $j_0 \in J$. Therefore $U_{j_0} \in \mathcal{F}$. This means that the filter is $\theta$-completely prime.
Conversely. assume that $U \in \mathcal{F}$ has not this property. It follows that for each $x \in U$, this is $G_x \in \theta(X,\tau)$ with $G_x \notin \mathcal{F}$. Put $U_x := G_x \cap U$. Now we have $U_x \notin \mathcal{F}$ for all $x \in U$ and $U = \bigcup_{x\in U} U_x \in \mathcal{F}$. But this is against our hypothesis that $\mathcal{F}$ is $\theta$-completely prime and hence the claim. □

Now we give a new appropriate construction. Let $\mathcal{F}$ be a $\theta$-completely prime filter of $\theta$-open sets on $(X,\tau)$. Take $\mathcal{F}$ with reserved set inclusion as order to be the index set of our net. If $U \in \mathcal{F}$, pick $x_U \in U$ with the property that $x_U \in G$ implies $G \in \mathcal{F}$. This is possible by the above Theorem. A net established in this way is called a $\theta$-derived net from the filter.

**Lemma 2.** *A $\theta$-derived net from a $\theta$-completely prime filter is $\theta$-observative.*

**Proof.** Let $U \in \mathcal{F}$ and $x_u \in G$. Then $G \in \mathcal{F}$ by choice of $x_U$. If $V \subseteq G$, then $x_v \in V \subseteq G$. Hence $[x]_{\geq i} \subseteq G$. Therefore the net is $\theta$-observative. □

**Theorem 4.** *Every $\theta$-completely prime filter equals the $\theta$-derived filter of any of its $\theta$-derived nets.*

**Proof.** Clearly, $[x]_{\geq i} \subseteq U$ for $U \in \mathcal{F}$. Thus $U \in \mathcal{F} \Rightarrow \mathcal{F}(x)\mathcal{U}$. Conversely, $U \in \mathcal{F}(x)\mathcal{U} \Rightarrow [x]_{\geq i} \subseteq U$ for some $G \in \mathcal{F}$. Therefore, $x_U \in U$ which implies that $U \in \mathcal{F}$ by choice of $x_G$. □

**Theorem 5.** *A topological space is $\theta$-sober if and only if every $\theta$-observative net strongly $\theta$-converges to a unique point.*

**Proof.** Obvious. □

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