ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVI, 2010, f.2
DOI: 10.2478/v10157-010-0017-5

ARCHIMEDEAN RESIDUATED LATTICES

BY

DUMITRU BUŞNEAG, DANA PICIU and ANTONETA JEFLEA

Abstract. For a residuated lattice A we denote by $Ds(A)$ the lattice of all deductive systems (congruence filters) of A. The aim of this paper is to put in evidence new characterizations for maximal and prime elements of $Ds(A)$ and to characterize archimedean and hyperarchimedean residuated lattices; so we prove some theorems of Nachbin type for residuated lattices. These results generalize to the case of residuated lattices some results earlier obtained by Buşneag and Piciu for the case of BL-algebras.

Mathematics Subject Classification 2000: 03G10, 06B20.

Key words: residuated lattice, boolean algebra, archimedean, hyperarchimedean residuated lattice, deductive system, irreducible element, prime deductive system, maximal deductive system.

1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([18]), Dilworth ([10]), Ward and Dilworth ([24]), Ward ([23]), Balbes and Dwinger ([1]) and Pavelka ([21]).

In [15], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK-latices* in [14], *full BCK-algebras* in [18], *FL_{ew}-algebras* in [19], and *integral, residuated, commutative l-monoids* in [4].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [3], [7], [9], [10], [17], [20], [23], [24]); for rules of calculus in residuated lattices see [7] and [9].

The paper is organized as follows.

In Section 2 we review some basic definitions and results of residuated lattices, with more details and more examples; also, this section contains some results relative to the lattice of deductive systems of a residuated lattice.

Section 3 contains new characterizations for prime deductive systems of a residuated lattice (Proposition 25, Corollary 26, Corollary 27, Theorem 28) and completely meet-irreducible deductive systems (Theorem 33, Theorem 34, Corollary 35).

In Section 4 we introduce the notions of *archimedean* and *hyperarchimedean* residuated lattices (Lemma 46), and we have proved a theorem of *Nachbin type* for residuated lattices (see Theorem 50), which give characterizations for archimedean and hyperarchimedean residuated lattices.

2. Definitions and preliminaries

In this section we review the basic definitions of residuated lattices, with more details and examples.

Definition 1. A *residuated lattice* ([3], [22]) is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

(LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

(LR_2) $(A, \odot, 1)$ is a commutative ordered monoid;

(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\rightarrow$ expressed by (LR_3), is a particular case of the *law of residuation* ([3]). Namely, let A and B two posets, and $f : A \rightarrow B$ a map. Then f is called *residuated* if there is a map $g : B \rightarrow A$, such that for any $a \in A$ and $b \in B$, we have $f(a) \leq b$ iff $b \leq g(a)$ (this is, also expressed by saying that the pair (f, g) is a *residuated pair*).

Now setting A a residuated lattice, $B = A$, and defining, for any $a \in A$, two maps $f_a, g_a : A \rightarrow A$, $f_a(x) = x \odot a$ and $g_a(x) = a \rightarrow x$, for any $x \in A$, we see that $x \odot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \rightarrow y$ for every $x, y \in A$, that is, for every $a \in A$, (f_a, g_a) is a pair of residuation.

The symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence, respectively.

Proposition 1 ([15])**.** *The class $\mathcal{RL}$ of residuated lattices is equational.*

Example 1. Let p be a fixed natural number and $I = [0, 1]$ the real unit interval. If for $x, y \in I$, we define $x \odot y = 1 - \min\{1, [(1-x)^p + (1-y)^p]^{1/p}\}$ and $x \to y = \sup\{z \in [0, 1] : x \odot z \leq y\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice.

Example 2. If we preserve the notation from Example 1, and we define for $x, y \in I$, $x \odot y = (\max\{0, x^p + y^p - 1\})^{1/p}$ and $x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ becomes a residuated lattice called *generalized Łukasiewicz structure*. For $p = 1$ we obtain the notion of *Łukasiewicz structure* ($x \odot y = \max\{0, x+y-1\}, x \to y = \min\{1, 1-x+y\}$).

Example 3. If on $I = [0, 1]$, for $x, y \in I$ we define $x \odot y = \min\{x, y\}$ and $x \to y = 1$ if $x \leq y$ and y otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called *Gödel structure*).

Example 4. If we consider on $I = [0, 1]$, $\odot$ the usual multiplication of real numbers and for $x, y \in I, x \to y = 1$ if $x \leq y$ and y/x otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called *Products structure* or *Gaines structure*).

Example 5. If $(A, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then if we define for every $x, y \in A, x \odot y = x \wedge y$ and $x \to y = x' \vee y$, then $(A, \vee, \wedge, \odot, \to, 0, 1)$ becomes a residuated lattice.

Definition 2 ([22])**.** A residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ is called *BL-algebra*, if the following two identities hold in A :

(BL_1) $x \odot (x \to y) = x \wedge y$;

(BL_2) $(x \to y) \vee (y \to x) = 1$.

If $x^{**} = x$ for every $x \in A$ (where $x^* = x \to 0$), then we obtain the notion of *MV-algebra* (see [22], p.46).

Remark 1. 1. Łukasiewicz structure, Gödel structure and Product structure are BL-algebras;

2. Not every residuated lattice, however, is a BL-algebra. Consider, for example (see [22], p.16) a residuated lattice defined on the unit interval I, for all $x, y \in I$, such that $x \odot y = 0$ if $x + y \leq \frac{1}{2}$ and $x \wedge y$ elsewhere,

$x \to y = 1$ if $x \leq y$ and $\max\left\{\frac{1}{2} - x, y\right\}$ elsewhere. Let $0 < y < x$, $x+y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$, but $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. Therefore (BL_1) does not hold.

3. ([22]) A residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ is an MV-algebra iff it satisfies the additional condition: $(x \to y) \to y = (y \to x) \to x$, for any $x, y \in A$.

Example 6 ([16])**.** We give an another example of a finite residuated lattice, which is not a BL-algebra. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A becomes a residuated lattice relative to the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

.

The condition $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$, for all $x, y \in A$ is not verified, since $c = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (b \to b) \wedge (a \to a) = 1$, hence A is not a BL-algebra.

Example 7 ([17])**.** We consider the residuate lattice A with the universe $\{0, a, b, c, d, e, f, 1\}$. Lattice ordering is such that $0 < d < c < b < a < 1$, $0 < d < e < f < a < 1$ and elements $\{b, f\}$ and $\{c, e\}$ are pairwise incomparable. The operations of implication and multiplication are given by the tables below :

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	1	a	b	c	d	e	f	1

,

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Clearly, A contains $\{a, b, c, d, e, f\}$ as a sublattice, and that is a copy of the so-called *benzene ring*, which shows that A is not distributive, and even not modular. But it is easy to see that $a^* = d, b^* = e, c^* = f, d^* = a$, $e^* = b$ and $f^* = c$.

Example 8 ([17])**.** Let A be the residuate lattice with the universe $\{0, a, b, c, d, 1\}$ such that $0 < b < a < 1$, $0 < d < c < a < 1$ and c and d are incomparable with b. The operations of implication and multiplication are given by the tables below :

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

Then A is obtained from the nonmodular lattice N_5, called the *pentagon*, by adding the new greatest element 1. Then A is another example of nondistributive residuated lattice.

Example 9 ([16])**.** We give an example of a finite residuate lattice which is an non-linearly MV-algebra. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1, 0 < b < d < 1$, but a, b and, respective c, d are incomparable. We define

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

and so A become a BL-algebra. We have in A the following operations:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

,

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

It is easy to see that $0^* = 1$, $a^* = d$, $b^* = c$, $c^* = b$, $d^* = a$, $1^* = 0$ and $x^{**} = x$, for all $x \in A$, hence A is an $MV-$ algebra which is not chain.

Example 10 ([16]). We give another example of a finite residuate lattice $A = \{0, a, b, c, d, e, f, g, 1\}$, which is non-linearly MV-algebra, with $0 < a < b < e < 1$, $0 < c < f < g < 1$, $a < d < g$, $c < d < e$, but $\{a, c\}$, $\{b, d\}$, $\{d, f\}$, $\{b, f\}$ and, respective $\{e, g\}$ are incomparable. We define

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

,

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

and so A becomes a residuated lattice. We have $0^* = 1$, $a^* = g$, $b^* = f$, $c^* = e$, $d^* = d$, $e^* = c$, $f^* = b$, $g^* = a$.

In what follows we denote by A a residuated lattice; for $x \in A$ and a natural number n, we define $x^* = x \rightarrow 0$, $(x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Theorem 2 ([9], [17], [22]). *Let $x, y, z \in A$. Then we have the following rules of calculus:*

$c_1:$ $1 \rightarrow x = x, x \rightarrow x = 1$, $y \leq x \rightarrow y$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$;

$c_2:$ $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$ *and* $x \odot 0 = 0$;

$c_3:$ $x \leq y$ *iff* $x \rightarrow y = 1$;

$c_4:$ $x \odot (x \rightarrow y) \leq y$, $x \leq (x \rightarrow y) \rightarrow y$, $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;

$c_5:$ $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$ *and* $y^* \leq x^*$;

$c_6:$ $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$.

$c_7:$ $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

$c_8 :\ x \leq x^{**} \leq x^* \to x;$

$c_9 :\ 1^* = 0\ ,\ 0^* = 1;$

$c_{10} :\ x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*.$

If A is a complete residuated lattice, $x \in A$ and $(y_i)_{i\in I}$ a family of elements of A, then:

$c_{11} :\ x \odot (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \odot y_i);$

$c_{12} :\ (\bigvee_{i\in I} y_i)^* = \bigwedge_{i\in I} y_i^*.$

Corollary 3 ([7]). *If $x, x', y, y', z \in A$ then:*

$c_{13} :\ x \vee y = 1$ *implies* $x \odot y = x \wedge y;$

$c_{14} :\ x \to (y \to z) \geq (x \to y) \to (x \to z);$

$c_{15} :\ x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$, *hence* $x \vee y^n \geq (x \vee y)^n$ *and* $x^m \vee y^n \geq (x \vee y)^{mn}$, *for any m, n natural numbers;*

$c_{16} :\ (x \to y) \odot (x' \to y') \leq (x \vee x') \to (y \vee y');$

$c_{17} :\ (x \to y) \odot (x' \to y') \leq (x \wedge x') \to (y \wedge y').$

If $B = \{a_1, a_2, ..., a_n\}$ is a finite subset of A we denote $\Pi B = a_1 \odot ... \odot a_n$.

Proposition 4 ([2], [5]). *Let $A_1, ..., A_n$ finite subsets of A.*

$c_{18} :$ *If* $a_1 \vee ... \vee a_n = 1$, *for all* $a_i \in A_i, i \in \{1, ..., n\}$, *then* $(\Pi A_1) \vee ... \vee (\Pi A_n) = 1.$

Corollary 5. *Let $a_1, ..., a_n \in A$.*

$c_{19} :$ *If* $a_1 \vee ... \vee a_n = 1$, *then* $a_1^k \vee ... \vee a_n^k = 1$, *for every natural number k.*

Lemma 6. *For every $a, b \in A$, we have:*

$c_{20} :\ a^{**} \odot b^{**} \leq (a \odot b)^{**}.$

Proof. By c_{10}, $(a\odot b)^* = a \to b^*$, so $(a\odot b)^* \odot a \leq b^*$. By c_5 we deduce that $b^{**} \leq [(a\odot b)^* \odot a]^* = (a\odot b)^* \to a^*$, so $b^{**} \odot (a\odot b)^* \leq a^*$. Then $a^{**} \leq [b^{**} \odot (a \odot b)^*]^* = b^{**} \to (a \odot b)^{**}$, that is, $a^{**} \odot b^{**} \leq (a \odot b)^{**}$. □

Corollary 7. *For every $a \in A$ and $n \geq 1$ we have:*

$c_{21}: \ (a^{**})^n \leq (a^n)^{**}$.

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall (see [13]) that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We will denote $b = a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 8 ([17])**.** *Suppose that $a \in A$ has a complement $b \in A$. Then, the following hold:*

(i) *If c is another complement of a in A, then $c = b$;*

(ii) $a' = b$ *and* $b' = a$;

(iii) $a^2 = a$.

Let $B(A)$ be the set of all complemented elements of the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.

Lemma 9 ([7])**.** *If $e \in B(A)$, then $e' = e^*$ and $e^{**} = e$.*

Remark 2 ([17])**.** If $e, f \in B(A)$, then $e \wedge f, e \vee f \in B(A)$. Moreover, $(e \vee f)' = e' \wedge f'$ and $(e \wedge f)' = e' \vee f'$. So, $e \to f = e' \vee f \in B(A)$.

Lemma 10 ([17])**.** *If $e \in B(A)$, then*

$c_{22}: \ e \odot x = e \wedge x$, *for every $x \in A$.*

Corollary 11 ([17])**.** *The set $B(A)$ is the universe of a Boolean subalgebra of A (called the Boolean center of A).*

Proposition 12 ([7])**.** *For $e \in A$ the following conditions are equivalent:*

(i) $e \in B(A)$;

(ii) $e \vee e^* = 1$.

Proposition 13. *For $a \in A$ and $n \geq 1$, the following conditions are equivalent:*

(i) $a^n \in B(A)$;

(ii) $a \vee (a^n)^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Since $a^n \in B(A)$, by Proposition 12 we deduce that $a^n \vee (a^n)^* = 1$. But $a^n \leq a$, so $1 = a^n \vee (a^n)^* \leq a \vee (a^n)^*$, hence $a \vee (a^n)^* = 1$.

$(ii) \Rightarrow (i)$. Since $a \vee (a^n)^* = 1 \overset{c_{19}}{\Rightarrow} a^n \vee [(a^n)^*]^n = 1$. Since $[(a^n)^*]^n \leq (a^n)^*$, we obtain $1 = a^n \vee [(a^n)^*]^n \leq a^n \vee (a^n)^*$, so $a^n \vee (a^n)^* = 1$. By Proposition 12 we deduce that $a^n \in B(A)$. $\square$

Lemma 14. *If $a \in A$ and $n \geq 1$ then the following hold: $a^n \in B(A)$ and $a^n \geq a^*$, implies $a = 1$.*

Proof. By Proposition 13, $a^n \in B(A) \Leftrightarrow a \vee (a^n)^* = 1$. By hypothesis, $a^n \geq a^*$. By c_5 we obtain $(a^n)^* \leq a^{**}$, so $1 = a \vee (a^n)^* \leq a \vee a^{**} = a^{**}$, hence $a^{**} = 1$, that is, $a^* = 0$. Then $(a \odot a) \to 0 = a \to (a \to 0) = a \to 0 = a^* = 0$, so we deduce that $(a^2)^* = 0$. Recursively we obtain that $(a^n)^* = 0$. Then $a \vee (a^n)^* = a \vee 0 = 1$, hence $a = 1$. $\square$

Definition 3. A totally ordered (linearly ordered) residuated lattice will be called *chain*.

Remark 3. If A is a chain, then $B(A) = \{0, 1\}$.

Definition 4 ([17], [22])**.** A nonempty subset $D \subseteq A$ is called an *implicative filter* (or *congruence filter*) of A if for all $x, y \in A$:

(D_1) If $x, y \in D$, then $x \odot y \in D$;

(D_2) If $x \in D, y \in A, x \leq y$, then $y \in D$.

Remark 4 ([17], [22])**.** A non empty subset $D \subseteq A$ is an implicative filter of A iff the following conditions are satisfied:

(D_1') $1 \in D$;

(D_2') If $x, x \to y \in D$, then $y \in D$,

that is, the notions of implicative filters and deductive systems are the same.

Clearly $\{1\}$ and A are deductive systems ; a deductive system D of A is called *proper* if $D \neq A$.

Remark 5. To avoid confusion we reserve, however in this paper, the name filter to lattice filters and deductive system for implicative (congruence) filters. From c_2 and Remark 4 we deduce that every deductive system of A is a filter for $L(A)$, but filters of $L(A)$ are not, in general, deductive systems for A (see [22]).

We denote by $Ds(A)$ the set of all deductive systems of A.

With any deductive systems D of A we can (see [17], [22]) associate a congruence θ_D on A by defining : $(a, b) \in \theta_D$ iff $a \to b, b \to a \in D$ iff $(a \to b) \odot (b \to a) \in D$. Conversely, for $\theta \in Con(A)$, the subset D_θ of A defined by $a \in D_\theta$ iff $(a, 1) \in \theta$ is a deductive system of A. Moreover the natural maps associated with the above are mutually inverse and establish a bijection between the lattices $Ds(A)$ and $Con(A)$. For $a \in A$, let a/D be the equivalence class of a modulo θ_D. If we denote by A/D the quotient set A/θ_D, then A/D becomes a residuated lattice with the natural operations induced from those of A. Clearly, in A/D, $\mathbf{0} = 0/D$ and $\mathbf{1} = 1/D$.

Proposition 15 ([9]). *Let $D \in Ds(A)$, and $a, b \in A$, then*

(*i*) *$a/D = 1/D$ iff $a \in D$, hence $a/D \neq \mathbf{1}$ iff $a \notin D$;*

(*ii*) *$a/D = 0/D$ iff $a^* \in D$;*

(*iii*) *If D is proper and $a/D = 0/D$, then $a \notin D$;*

(*iv*) *$a/D \leq b/D$ iff $a \to b \in D$.*

For a nonempty subset $S \subseteq A$, the smallest deductive system of A which contains S, i.e. $\cap\{D \in Ds(A) : S \subseteq D\}$, is said to be *the deductive system of A generated by S* and will be denoted by $< S >$. If $S = \{a\}$, with $a \in A$, we denote by $< a >$ the deductive system generated by $\{a\}$ ($< a >$ is called *principal*); we recall that the lattice principal filter generated by a is $[a) = \{x \in A : a \leq x\}$.

For $D \in Ds(A)$ and $a \in A$, we denote by $D(a) =< D \cup \{a\} >$ (clearly, if $a \in D$, then $D(a) = D$).

Proposition 16 ([17], [22]). *Let $S \subseteq A$ a nonempty subset of A, $a \in A$, $D, D_1, D_2 \in Ds(A)$. Then*

(*i*) *$< S >= \{x \in A : s_1 \odot ... \odot s_n \leq x$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$. In particular, $< a >= \{x \in A : x \geq a^n$, for some $n \geq 1\}$;*

(*ii*) $D(a) = \{x \in A : x \geq d \odot a^n,$ *with* $d \in D$ *and* $n \geq 1\}$;

(*iii*) $< D_1 \cup D_2 >= \{x \in A : x \geq d_1 \odot d_2$ *for some* $d_1 \in D_1$ *and* $d_2 \in D_2\}$.

Definition 5. We recall ([13], p.93) that a lattice $(L, \vee, \wedge)$ is called *Brouwerian* if it satisfies the identity $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$ (whenever the arbitrary unions exists).

Proposition 17 ([9])**.** *The lattice* $(Ds(A), \subseteq)$ *is a complete Brouwerian lattice* (*hence distributive*)*, the compact elements being exactly the principal deductive systems of* A*. For a family* $(D_i)_{i\in I}$ *of deductive systems,* $\wedge\{D_i : i \in I\} = \bigcap_{i\in I} D_i$ *and* $\vee\{D_i : i \in I\} =< \bigcup_{i\in I} D_i >$ *.*

For D_1, $D_2 \in Ds(A)$ we put $D_1 \to D_2 = \{a \in A : D_1 \cap < a > \subseteq D_2\}$.

Lemma 18 ([9])**.** *If* D_1, $D_2 \in Ds(A)$ *then*

(*i*) $D_1 \to D_2 \in Ds(A)$;

(*ii*) *If* $D \in Ds(A)$*, then* $D_1 \cap D \subseteq D_2$ *iff* $D \subseteq D_1 \to D_2$*, that is,* $D_1 \to D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\}$.

Corollary 19 ([9])**.** $(Ds(A), \vee, \wedge, \to, \{1\}, A)$ *is a Heyting algebra, where for* D_1, $D_2, D \in Ds(A)$, $D_1 \wedge D_2 = D_1 \cap D_2$, $D_1 \vee D_2 =< D_1 \cup D_2 >$, $D^* = D \to \mathbf{0} = D \to \{1\} = \{x \in A : x \vee y = 1,$ *for every* $y \in D\}$, *hence for every* $x \in D$ *and* $y \in D^*, x \vee y = 1$*. In particular, for every* $a \in A, < a >^* = \{x \in A : x \vee a = 1\}$.

3. The spectrum of a residuated lattice

This section contains new characterizations for meet-irreducible and completely meet-irreducible deductive systems of a residuated lattice A.

For a lattice L we denote by $Ir(L)$ $(Irc(L))$ the set of all meet-irreducible (completely meet-irreducible) elements of L.

Proposition 20. *Let* $D \in Ds(A)$ *and* $a, b \in A$ *such that* $a \vee b \in D$*. Then* $D(a) \cap D(b) = D$*.*

Proof. Clearly, $D \subseteq D(a) \cap D(b)$. To prove the converse inclusion, let $x \in D(a) \cap D(b)$. Then there are $d_1, d_2 \in D$ and $m, n \geq 1$ such that $x \geq d_1 \odot a^m$ and $x \geq d_2 \odot b^n$. Then, by c_{15}, $x \geq (d_1 \odot a^m) \vee (d_2 \odot b^n) \geq (d_1 \vee d_2) \odot (d_1 \vee b^n) \odot (d_2 \vee a^m) \odot (a \vee b)^{mn}$, hence $x \in D$, that is, $D(a) \cap D(b) \subseteq D$, so we obtain the desired equality. □

Corollary 21. *For $D \in Ds(A)$ the following conditions are equivalent:*

(i) *If $D = D_1 \cap D_2$ with $D_1, D_2 \in Ds(A)$, then $D = D_1$ or $D = D_2$;*

(ii) *For $a, b \in A$, if $a \vee b \in D$, then $a \in D$ or $b \in D$.*

Proof. $(i) \Rightarrow (ii)$. If $a, b \in A$ such that $a \vee b \in D$, then, by Proposition 20, $D(a) \cap D(b) = D$, hence $D = D(a)$ or $D = D(b)$, so $a \in D$ or $b \in D$.

$(ii) \Rightarrow (i)$. Let $D_1, D_2 \in Ds(A)$ such that $D = D_1 \cap D_2$. If by contrary $D \neq D_1$ and $D \neq D_2$ then there are $a \in D_1 \backslash D$ and $b \in D_2 \backslash D$.

If denote $c = a \vee b$, then $c \in D_1 \cap D_2 = D$, so $a \in D$ or $b \in D$, a contradiction. □

Definition 6. We say that $P \in Ds(A)$ is *prime* if $P \neq A$ and P is a prime element in the lattice $(Ds(A), \subseteq)$.

Remark 6. Since the lattice $(Ds(A), \subseteq)$ is distributive, $P \in Ds(A)$, $P \neq A$ is prime iff P is a meet-irreducible deductive system in the lattice $(Ds(A), \subseteq)$, so we have for prime deductive systems in a residuated lattice the characterization given by Corollary 21.

We denote by $Spec(A)$ the set of all prime deductive systems of A and by $Irc(A)$ the set of all completely meet-irreducible elements of $Ds(A)$ (clearly, $Irc(A) \subseteq Spec(A)$).

Example 11. Consider the residuated lattice $I = [0, 1]$ from Remark 1, (2), which is not a BL-algebra. If $x \in [0, 1], x > \frac{1}{4}$, then $x + x > \frac{1}{2}$, hence $x \odot x = x \wedge x = x$, so $< x >= [x, 1]$. If $a, b \in I$ and $a \vee b \in < x >= [x, 1]$, then $a \vee b = \max\{a, b\} \geq x$, hence $a \geq x$ or $b \geq x$. So, $a \in < x >$ or $b \in < x >$, that is, $< x > \in Spec(I)$.

Example 12. Consider the residuated lattice $A = \{0, a, b, c, 1\}$ from Example 6. It is immediate that $Ds(A) = \{\{1\}, \{1, c\}, \{1, a, c\}, \{1, b, c\}, A\}$ and $Spec(A) = \{\{1\}, \{1, a, c\}, \{1, b, c\}\}$. Since $\{1, c\} = \{1, a, c\} \cap \{1, b, c\}$, then $\{1, c\} \notin Spec(A)$. Since $\odot$ coincides with $\wedge$, the deductive systems of A coincide with the filters of the associated lattice $L(A)$.

Proposition 22. *For a proper deductive system P of A we consider the following assertions:*

(1) $P \in Spec(A)$;

(2) *If* $a, b \in A$, *and* $a \vee b = 1$, *then* $a \in P$ *or* $b \in P$;

(3) *For all* $a, b \in A$, $a \to b \in P$ *or* $b \to a \in P$;

(4) A/P *is a chain.*

Then

(i) $(1) \Rightarrow (2)$ *but* $(2) \not\Rightarrow (1)$;

(ii) $(3) \Rightarrow (1)$ *but* $(1) \not\Rightarrow (3)$;

(iii) $(4) \Rightarrow (1)$ *but* $(1) \not\Rightarrow (4)$.

Proof. (i).$(1) \Rightarrow (2)$ is clear by Corollary 21, (since $1 \in P$).

$(2) \not\Rightarrow (1)$ Consider A from Example 6. Then $D = \{1, c\} \notin Spec(A)$. Clearly, if $x, y \in A$ and $x \vee y = 1$, then $x = 1$ or $y = 1$, hence $x \in D$ or $y \in D$, but $D \notin Spec(A)$.

(ii). To prove $(3) \Rightarrow (1)$, let $a, b \in A$ such that $a \vee b \in P$. By c_6 we obtain $a \vee b \leq [(a \to b) \to b] \wedge [(b \to a) \to a]$, hence $(a \to b) \to b, (b \to a) \to a \in P$. If $a \to b \in P$ then $b \in P$; if $b \to a \in P$, then $a \in P$, that is, $P \in Spec(A)$.

$(1) \not\Rightarrow (3)$ Consider A from Example 6. Then $P = \{1\} \in Spec(A)$. We have $a \to b = b \neq 1$ and $b \to a = a \neq 1$, hence $a \to b$ and $b \to a \notin P$.

(iii). To prove $(4) \Rightarrow (1)$ let $a, b \in A$. Since A/P is supposed chain, $a/P \leq b/P$ or $b/P \leq a/P \Leftrightarrow$ (by Proposition 15) $a \to b \in P$ or $b \to a \in P$ and we apply (ii).

$(1) \not\Rightarrow (4)$ Consider A from Example 6 and $P = \{1\} \in Spec(A)$. Then $A/P = A$ is not a chain. □

Remark 7 ([6])**.** If A is a BL-algebra, then all assertions from the above proposition are equivalent.

Remark 8. 1. In general, in a residuated lattice A, if P is a prime deductive system and Q is a proper deductive system such that $P \subseteq Q$, then Q is not a prime deductive system. For example, if we consider $A = \{0, a, b, c, 1\}$ from Example 6, we have $P = \{1\}, Q = \{1, c\} \in Ds(A)$, $P \subseteq Q, P = \{1\} \in Spec(A)$ but Q is not a prime deductive system (see Example 12);

2. If the residuated lattice A is a BL-algebra and P is a prime deductive system, Q is a proper deductive system such that $P \subseteq Q$, then Q is a prime deductive system (see [6]).

Remark 9. If P is a prime deductive system of A, then $A\backslash P$ is an ideal in the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.

Proof. Since P is proper, $0 \notin P$, hence we have $0 \in A\backslash P$. If $a \leq b$ and $b \in A\backslash P$, then $a \in A\backslash P$, since P is a deductive system of A. If $a, b \in A\backslash P$ (that is, $a \notin P$ and $b \notin P$), then $a \vee b \in A\backslash P$, since P is a prime deductive system. □

Theorem 23. (*Prime deductive system theorem*) *If $D \in Ds(A)$ and I is an ideal of the lattice $L(A)$ such that $D \cap I = \emptyset$, then there is a prime deductive system P of A such that $D \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let $F_D = \{D' \in Ds(A) : D \subseteq D' \text{ and } D' \cap I = \emptyset\}$. A routine application of Zorn's lemma shows that F_D has a maximal element P. Suppose by contrary that P is not a prime deductive system, that is, there are $a, b \in A$ such that $a \vee b \in P$, but $a \notin P, b \notin P$ (see Corollary 21).

By the maximality of P we deduce that $P(a), P(b) \notin F_D$, hence $P(a) \cap I \neq \emptyset$ and $P(b) \cap I \neq \emptyset$, that is, there are $p_1 \in P(a) \cap I$ and $p_2 \in P(b) \cap I$. By Proposition 16, $p_1 \geq f \odot a^m$ and $p_2 \geq g \odot b^n$, with $f, g \in P$ and m, n natural numbers.

Then $p_1 \vee p_2 \geq (f \odot a^m) \vee (g \odot b^n) \overset{c_{15}}{\geq} (f \vee g) \odot (g \vee a^m) \odot (f \vee b^n) \odot (b^n \vee a^m) \overset{c_{15}}{\geq} (f \vee g) \odot (g \vee a^m) \odot (f \vee b^n) \odot (a \vee b)^{m+n}$. Since $f \vee g, g \vee a^m, f \vee b^n \in P$ we deduce that $p_1 \vee p_2 \in P$; but $p_1 \vee p_2 \in I$, hence $P \cap I \neq \emptyset$, a contradiction. Hence P is a prime deductive system. □

Remark 10. If A is a nontrivial residuated lattice, then any proper deductive system of A can be extended to a prime deductive system.

Remark 11. In general, if A is a residuated lattice, the set of proper deductive systems including a prime deductive system P of A is not a chain, but if the residuated lattice is a BL-algebra, then the set of proper deductive systems including a prime deductive system P of A is a chain, (see [6]). Indeed, we consider the residuated lattice from Example 6 and the prime deductive system $P = \{1\}$. The set of proper deductive systems including a prime deductive system $P = \{1\}$ of A is $\{\{1, c\}, \{1, a, c\}, \{1, b, c\}\}$, but $\{1, a, c\} \nsubseteq \{1, b, c\}$ and $\{1, b, c\} \nsubseteq \{1, a, c\}$, so $\{\{1, c\}, \{1, a, c\}, \{1, b, c\}\}$ is not a chain.

Corollary 24. *Let $D \in Ds\ (A)$ and $a \in A\backslash D$. Then:*

(*i*) *There is $P \in Spec(A)$ such that $D \subseteq P$ and $a \notin P$;*

(*ii*) *D is the intersection of those prime deductive systems which contain D;*

(*iii*) $\cap Spec(A) = \{1\}$.

Proposition 25. *For a proper deductive system $P \in Ds(A)$ the following conditions are equivalent:*

(*i*) $P \in Spec(A)$;

(*ii*) *For every $x, y \in A \backslash P$ there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$.*

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$ and $x, y \in A \backslash P$. If by contrary, for every $a \in A$ with $x \leq a$ and $y \leq a$ then $a \in P$, since $x, y \leq x \vee y$ we deduce that $x \vee y \in P$, hence, $x \in P$ or $y \in P$, a contradiction.

$(ii) \Rightarrow (i)$. Suppose by contrary that there exist $D_1, D_2 \in Ds(A)$ such that $D_1 \cap D_2 = P$ and $P \neq D_1$, $P \neq D_2$. So, we have $x \in D_1 \backslash P$ and $y \in D_2 \backslash P$. By hypothesis there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$. We deduce $z \in D_1 \cap D_2 = P$, a contradiction. $\square$

Corollary 26. *For a proper deductive system $P \in Ds(A)$ the following conditions are equivalent:*

(*i*) $P \in Spec(A)$;

(*ii*) *If $x, y \in A$ and $< x > \cap < y > \subseteq P$, then $x \in P$ or $y \in P$.*

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$ such that $< x > \cap < y > \subseteq P$ and suppose by contrary that x, $y \notin P$. Then by Proposition 25 there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$. Hence $z \in < x > \cap < y > \subseteq P$, so $z \in P$, a contradiction.

$(ii) \Rightarrow (i)$. Let $x, y \in A$ such that $x \vee y \in P$. Then $< x \vee y > \subseteq P$. Since, using [9], $< x \vee y > = < x > \cap < y >$ we deduce that $< x > \cap < y > \subseteq P$, hence, by hypothesis, $x \in P$ or $y \in P$, that is, $P \in Spec(A)$. $\square$

Corollary 27. *For a proper deductive system $P \in Ds(A)$ the following conditions are equivalent:*

(*i*) $P \in Spec(A)$;

(*ii*) *For every $\alpha, \beta \in A/P, \alpha \neq \mathbf{1}, \beta \neq \mathbf{1}$ there is $\gamma \in A/P, \gamma \neq \mathbf{1}$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$.*

Proof. $(i) \Rightarrow (ii)$. Clearly, by Proposition 25 and Proposition 15, since if $\alpha = a/P$, with $a \in A$, then the condition $\alpha \neq \mathbf{1}$ is equivalent with $a \notin P$.

$(ii) \Rightarrow (i)$. Let $\alpha, \beta \in A/P$. Then in A/P, $\alpha = a/P \neq \mathbf{1}$ and $\beta = b/P \neq \mathbf{1}$. By hypothesis there is $\gamma = c/P \neq \mathbf{1}$ (that is, $c \notin P$) such that $\alpha, \beta \leq \gamma$ equivalent with $a \to c, b \to c \in P$. If we consider $d = (b \to c) \to ((a \to c) \to c)$, then by c_4, we deduce that $a \leq (a \to c) \to c \leq (b \to c) \to ((a \to c) \to c) = d$ and $b \leq d$ (since $b \leq (b \to c) \to ((a \to c) \to c) \Leftrightarrow b \odot (b \to c) \leq (a \to c) \to c$, which is true because $b \odot (b \to c) \leq c \leq (a \to c) \to c$). Hence $a, b \leq d$. Since $c \notin P$ we deduce that $d \notin P$, hence by Proposition 25, we deduce that $P \in Spec(A)$. □

Theorem 28. *For a proper deductive system $P \in Ds(A)$ the following conditions are equivalent:*

(i) $P \in Spec(A)$;

(ii) *For every $D \in Ds(A)$, $D \to P = P$ or $D \subseteq P$.*

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$. Since $Ds(A)$ is a Heyting algebra (by Corollary 19) for $D \in Ds(A)$ we have $P = (D \to P) \cap ((D \to P) \to P)$, so $P = D \to P$ or $P = (D \to P) \to P$. If $P = (D \to P) \to P$ then $D \subseteq P$.

$(ii) \Rightarrow (i)$. Let $D_1, D_2 \in Ds(A)$ such that $D_1 \cap D_2 = P$. Then $D_1 \subseteq D_2 \to P$ (see Lemma 18, (ii)) and so, if $D_2 \subseteq P$, then $P = D_2$ and if $D_2 \to P = P$, then $P = D_1$, hence $P \in Spec(A)$. □

We recall that if $(L, \vee, \wedge, ^*, 0, 1)$ is a *pseudocomplemented distributive lattice*, then two subsets associated with L ([1], p.153, [13]) are $Rg(L) = \{x \in L : x^{**} = x\}$ and $D(L) = \{x \in L : x^* = 0\}$. The elements of $Rg(L)$ are called *regular* and those of $D(L)$ *dense*. Note that $\{0, 1\} \subseteq Rg(L), 1 \in D(L)$, $D(L)$ is a filter in L and $Rg(L)$ is a Boolean algebra under the operations induced by the ordering on L ([1], p.157).

Corollary 29. *For a residuated lattice A, $Spec(A) \subseteq D(Ds(A)) \cup Rg(Ds(A))$.*

Proof. Let $P \in Spec(A)$ and $D = P^* \in Ds(A)$; then by Theorem 28, $D \subseteq P$ or $D \to P = P$ equivalent with $P^* \subseteq P$ or $P^* \to P = P$. Since $Ds(A)$ is a Heyting algebra then $P^* \to P = P^{**}$, so $P^{**} = A$ or $P^{**} = P$ equivalent with $P^* = \{1\}$ or $P^{**} = P$, that is $P \in D(Ds(A)) \cup Rg(Ds(A))$.□

Relative to the uniqueness of deductive systems as intersection of primes we have:

Theorem 30. *If every $D \in Ds(A)$ has a unique representation as an intersection of elements of $Spec(A)$, then $(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra.*

Proof. Let $D \in Ds(A)$ and $D' = \cap\{M \in Spec(A) : D \nsubseteq M\} \in Ds(A)$. By Corollary 24, (ii), $D \cap D' = \cap\{M \in Spec(A)\} = \{1\}$; if $D \vee D' \neq A$, then by Corollary 24, (i), there exists $D'' \in Spec(A)$ such that $D \vee D' \subseteq D''$ and $D'' \neq A$. Consequently, D' has two representations $D' = \cap\{M \in Spec(A) : D \nsubseteq M\} = D'' \cap (\cap\{M \in Spec(A) : D \nsubseteq M\})$, which is contradictory. Therefore $D \vee D' = A$ and so $Ds(A)$ is a Boolean algebra. □

Lemma 31. *If $D \in Ds(A)$, $D \neq A$ and $a \notin D$, then there exists $D_a \in Ds(A)$ maximal with the property that $D \subseteq D_a$ and $a \notin D_a$.*

Proof. Let $\mathcal{F}_{D,a} = \{D' \in Ds(A) : D \subseteq D'$ and $a \notin D'\}$; clearly $\mathcal{F}_{D,a} \neq \emptyset$, because $D \in \mathcal{F}_{D,a}$. If $\mathbf{C}$ is a chain in $\mathcal{F}_{D,a}$ then $\cup\mathbf{C} \in \mathcal{F}_{D,a}$. By Zorn's lemma there exists a deductive system D_a which is maximal subject to containing D and $a \notin D_a$. □

Definition 7. $D \in Ds(A)$, $D \neq A$ is called *maximal relative to a* if $a \notin D$ and if $D' \in Ds(A)$ is proper such that $a \notin D'$, and $D \subseteq D'$, then $D = D'$.

If in Lemma 31 we consider $D = \{1\}$ we obtain

Corollary 32. *For any $a \in A, a \neq 1$, there is a deductive system D_a maximal relative to a.*

Theorem 33. *For $D \in Ds(A), D \neq A$ the following are equivalent:*

(i) $D \in Irc(A)$;

(ii) *There is $a \in A$ such that D is maximal relative to a.*

Proof. $(i) \Rightarrow (ii)$. See ([12], p.248) (because by Proposition 17, $Ds(A)$ is an algebraic lattice).

$(ii) \Rightarrow (i)$. Let $D \in Ds(A)$ maximal relative to a and suppose $D = \bigcap_{i \in I} D_i$ with $D_i \in Ds(A)$ for every $i \in I$. Since $a \notin D$ there is $j \in I$ such that $a \notin D_j$. So, $a \notin D_j$ and $D \subseteq D_j$. By the maximality of D we deduce that $D = D_j$, that is, $D \in Irc(A)$. □

Theorem 34. *Let $D \in Ds(A)$ be a deductive system, $D \neq A$ and $a \in A \backslash D$. Then the following conditions are equivalent:*

(i) *D is maximal relative to a;*

(ii) *For every $x \in A \backslash D$ there is $n \geq 1$ such that $x^n \to a \in D$.*

Proof. $(i) \Rightarrow (ii)$. Let $x \in A \backslash D$. If $a \notin D(x) = D \vee < x >$, since $D \subset D(x)$ then $D(x) = A$ (by the maximality of D) hence $a \in D(x)$, a contradiction. We deduce that $a \in D(x)$, hence $a \geq d \odot x^n$, with $d \in D$ and $n \geq 1$. Then $d \leq x^n \to a$, hence $x^n \to a \in D$.

$(ii) \Rightarrow (i)$. We suppose by contrary that there is $D' \in Ds(A), D' \neq A$ such that $a \notin D'$ and $D \subset D'$. Then there is $x_0 \in D'$ such that $x_0 \notin D$, hence by hypothesis there is $n \geq 1$ such that $x_0^n \to a \in D \subset D'$. Thus from $x_0^n \to a \in D'$ and $x_0^n \in D'$, we deduce that $a \in D'$, a contradiction. □

Corollary 35. *For $D \in Ds(A), D \neq A$ the following conditions are equivalent:*

(i) $D \in Irc(A)$;

(ii) *In the set $A/D \backslash \{\mathbf{1}\}$ we have an element $p \neq \mathbf{1}$ with the property that for every $\alpha \in A/D \backslash \{\mathbf{1}\}$ there is $n \geq 1$ such that $\alpha^n \leq p$.*

Proof. $(i) \Rightarrow (ii)$. By Theorem 33, D is maximal relative to an element $a \notin D$; then, if denote $p = a/D \in A/D$, $p \neq \mathbf{1}$ (since $a \notin D$) and for every $\alpha = b/D$, $\alpha \neq \mathbf{1}$ (that is $b \notin D$) by Theorem 34 there is $n \geq 1$ such that $b^n \to a \in D$, that is, $\alpha^n \leq p$.

$(ii) \Rightarrow (i)$. Let $p = a/D \in A/D \backslash \{\mathbf{1}\}$, (that is, $a \notin D$) and $\alpha = b/D \in A/D \backslash \{\mathbf{1}\}$, (that is, $b \notin D$). By hypothesis there is $n \geq 1$ such that $\alpha^n \leq p$ equivalent with $b^n \to a \in D$. Then, by Theorem 34, we deduce that $D \in Irc(A)$. □

Definition 8. A deductive system P of A is a *minimal prime* if $P \in Spec(A)$ and, whenever $Q \in Spec(A)$ and $Q \subseteq P$, we have $P = Q$.

Proposition 36. *If P is a minimal prime deductive system, then for any $a \in P$ there is $b \in A \backslash P$ such that $a \vee b = 1$.*

Proof. As in the case of BL-algebras (see [6]). □

4. Archimedean and hyperarchimedean residuated lattices

In this section we introduce the notions of *archimedean* and *hyperarchimedean* residuated lattice and will prove two theorems of Nachbin type for residuated lattices.

Definition 9. ([17]) A deductive system M of A is *maximal* if $M \neq A$ and M is a maximal element in the lattice $(Ds(A), \subseteq)$ (that is, it is proper and it is not contained in any other proper deductive system).

The following result is an immediate consequence of Zorn's lemma:

Proposition 37. *In a nontrivial residuated lattice A, every proper deductive system can be extended to a maximal deductive system.*

We shall denote by $Max(A)$ the set of all maximal deductive systems of A; clearly, $Max(A) \subseteq Spec(A)$.

We have:

Theorem 38. *If D is a proper deductive system of A, then the following conditions are equivalent:*

(i) *$D \in Max(A)$;*

(ii) *For any $x \notin D$ there exist $d \in D, n \geq 1$ such that $d \odot x^n = 0$.*

Proof. $(i) \Rightarrow (ii)$. If $x \notin D$, then $< D \cup \{x\} >= A$, hence $0 \in < D \cup \{x\} >$. By Proposition 16, (iii), there exist $n \geq 1$ and $d \in D$ such that $d \odot x^n \leq 0$. Thus $d \odot x^n = 0$.

$(ii) \Rightarrow (i)$. Assume there is a proper deductive system D' such that $D \subset D'$. Then there exists $x \in D'$ such that $x \notin D$. By hypothesis there exist $d \in D$ and $n \geq 1$ such that $d \odot x^n = 0$. But $x, d \in D'$ hence we obtain $0 \in D'$, a contradiction. □

Corollary 39. *If M is a proper deductive system of A, then the following conditions are equivalent:*

(i) *$M \in Max(A)$;*

(ii) *For any $x \in A, x \notin M$ iff $(x^n)^* \in M$, for some $n \geq 1$.*

Theorem 40. *If M is a proper deductive system of A, then the following conditions are equivalent:*

(i) *$M \in Max(A)$;*

(ii) *A/M is locally finite.*

Proof. It follows by observing that the condition (ii) can be reformulated in the following way: for any $x \in A, x/M \neq 1/M$ (that is, $x \notin M$), $(x/M)^n = 0/M$, for some $n \geq 1 \Leftrightarrow x^n/M = 0/M \Leftrightarrow (x^n)^* \notin M$. □

Definition 10 ([17])**.** The intersection of the maximal deductive systems of A is called the *radical* of A and will be denoted by $Rad(A)$; clearly, $Rad(A) \in Ds(A)$.

Example 13. Let A be the 5−element residuated lattice from Example 6. It is easy to see that A has two maximal deductive systems: $\{1, a, c\}$ and $\{1, b, c\}$, hence $Rad(A) = \{1, c\}$.

For any $n \geq 1$ and $a \in A$ we denote $\widetilde{n}a = [(a^*)^n]^*$.

Theorem 41 ([11],[17])**.** (i) *$Rad(A) = \{x \in A :$ for any $n \geq 1$ there exists $m \geq 1$ such that $\widetilde{m}(x^n) = 1\} = \{x \in A :$ for any $n \geq 1$ there is $k_n \geq 1$ such that $[(x^n)^*]^{k_n} = 0\}$;*
(ii) *$D(A) \in Ds(A)$ and $D(A) \subseteq Rad(A)$.*

For a residuated lattice A we make the following notation:

$$Rad_{BL}(A) = \{a \in A : (a^n)^* \leq a, \text{ for every } n \geq 1\}.$$

Proposition 42. *For a residuated lattice A, $Rad_{BL}(A) \subseteq Rad(A)$.*

Proof. Let $a \notin Rad(A)$, hence there is a maximal deductive system M with $a \notin M$. Then there is n such that $(a^n)^* \in M$, (by Corollary 39). If we suppose $a \in Rad_{BL}(A)$ then in particular for this n we have $(a^n)^* \leq a$, hence $a \in M$, by (D_2), a contradiction. Hence $(a^n)^* \nleq a$, i.e. $a \notin Rad_{BL}(A)$, that is, $Rad_{BL}(A) \subseteq Rad(A)$. □

Remark 12. If A is a BL-algebra, then $Rad(A) = Rad_{BL}(A)$.

Proposition 43. *If A is a residuated lattice, then $B(A) \cap Rad(A) = \{1\}$.*

Proof. Obviously, $1 \in B(A) \cap Rad(A)$. Let $e \in B(A) \cap Rad(A), e \neq 1$. By Theorem 23, there is a prime deductive system P of A such that $e \notin P$. By Proposition 12, (ii), we have $e \vee e^* = 1 \in P$, so $e^* \in P$ (since P is prime and $e \notin P$). By Proposition 37, there is a maximal deductive system M such that $P \subseteq M$. It follows that $e^* \in M$, so $e \notin M$. Thus, $e \notin Rad(A)$. □

Definition 11. An element a of a residuated lattice A is called *infinitesimal* if $a \neq 1$ and $a^n \geq a^*$ for any $n \geq 1$.

We denote by $Inf(A)$ the set of all infinitesimals of A.

Example 14. If $A = \{0, a, b, c, 1\}$ is the 5−element residuated lattice from Example 6, then a is not infinitesimal (since $a^* = b$ and $a \ngeq b$); analogously, we deduce that b is not infinitesimal. Since $c^* = 0$, then $c^n = c \geq 0 = c^*$, for every natural number n, hence c is an infinitesimal element of A. So, $Inf(A) = \{c\}$.

Proposition 44. *For every element $a \in A, a \neq 1$, a is infinitesimal implies $a \in Rad(A)$.*

Proof. Let $a \neq 1$ be an infinitesimal and suppose $a \notin Rad(A)$. Thus, there is a maximal deductive system M of A such that $a \notin M$. By Corollary 39, there is $n \geq 1$ such that $(a^n)^* \in M$. By hypothesis $a^n \geq a^*$ hence $(a^n)^* \leq a^{**}$, so $a^{**} \in M$. By c_{21} we deduce that $(a^{**})^n \leq (a^n)^{**}$, hence $(a^n)^{**} \in M$. If we denote $b = (a^n)^*$ we conclude that $b, b^* \in M$, hence $0 = b^* \odot b \in M$, that is, $M = A$, a contradiction. $\square$

Corollary 45. $Inf(A) \subseteq Rad(A)$.

Remark 13. 1. If A is the residuated lattice from Example 6, then $Inf(A) \subset Rad(A)$, since $Inf(A) = \{c\}$ and $Rad(A) = \{1, c\}$ (see Examples 13 and 14).

2. In general, $Rad(A)\backslash\{1\} \nsubseteq Inf(A)$. Indeed, let A be the residuated lattice from Example 7. Then the deductive systems of A are $\{1\}$, $\{1, a, b, c\}$ and A. It is easy to see that A has two prime deductive systems: $\{1\}$, $\{1, a, b, c\}$ and a unique maximal deductive system $\{1, a, b, c\}$; hence $Rad(A) = \{1, a, b, c\}$. Obviously, a is an infinitesimal element of A ($a^n = c$, for every $n \geq 1$, $a^* = d$ and $c \geq d$). But ($b^2 = c, b^* = e$ and c, e are incomparable), ($c^2 = c, c^* = f$ and c, f are incomparable), ($d^2 = 0, d^* = a$ and $a > 0$), ($e^2 = d, e^* = b$ and $d < b$), ($f^2 = d, f^* = c$ and $d < c$), ($0^2 = 0, 0^* = 1$ and $0 < 1$), so we conclude that $b, c, d, e, f, 0 \notin Inf(A)$. It follows that $Inf(A) = \{a\}$. Thus $Inf(A) \subseteq Rad(A)$ and $Rad(A)\backslash\{1\} \nsubseteq Inf(A)$.

Remark 14 ([6])**.** If A is a BL algebra, then $Rad(A)\backslash\{1\} = Inf(A)$.

Lemma 46. *In any residuated lattice A the following are equivalent:*

(*i*) *For every $a \in A, a^n \geq a^*$ for any $n \geq 1$ implies $a = 1$;*

(*ii*) *For every $a, b \in A, a^n \geq b^*$ for any $n \geq 1$ implies $a \vee b = 1$;*

(*iii*) *For every* $a, b \in A, a^n \geq b^*$ *for any* $n \geq 1$ *implies* $a \to b = b$ *and* $b \to a = a$.

Proof. $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $a^n \geq b^*$ for any $n \geq 1$. We get $(a \vee b)^* \overset{c_{12}}{=} a^* \wedge b^* \leq b^* \leq a^n \leq (a \vee b)^n$, hence $(a \vee b)^n \geq (a \vee b)^*$, for any $n \geq 1$. Then, $a \vee b = 1$.

$(ii) \Rightarrow (iii)$. Since $1 = a \vee b \leq [(b \to a) \to a] \wedge [(a \to b) \to b]$ we deduce that $(b \to a) \to a = (a \to b) \to b = 1$, that is, $a \to b = b$ and $b \to a = a$.

$(iii) \Rightarrow (i)$. Let $a \in A$ such that $a^n \geq a^*$ for any $n \geq 1$. If we consider $b = a$ we obtain $a \to a = a$, hence $a = 1$. □

Definition 12. A residuated lattice A is called *archimedean* if the equivalent conditions from Lemma 46 are satisfied.

One can easily remark that a residuated lattice is archimedean iff it has no infinitesimals.

Example 15. 1. Consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 9. We have $a^n = a$ for every $n \geq 1$ and $a^* = d$ hence $a^n \ngeq a^*$ for every $n \geq 1; b^n = 0$ for every $n \geq 1$ and $b^* = c$ hence $b^n \ngeq b^*$ for every $n \geq 1$; $c^2 = a \ngeq c^* = b$, $d^n = d$ for every $n \geq 1$ and $d^* = a$, hence $d^n \ngeq d^* = a$, for every $n \geq 1$. Hence if $x \in A$ and $x^n \geq x^*$, for every $n \geq 1$, then $x = 1$, that is, A is archimedean;

2. Consider $A = \{0, a, b, c, d, e, f, 1\}$ the residuated lattice from Example 7. We have $a^* = d, b^* = e, c^* = f, d^* = a, e^* = b$ and $f^* = c$. Since $a \geq d = a^*$ and $a^n = c$ for every $n \geq 2$ and $c \geq d = a^*$ we deduce that $a^n \geq a^*$, for every $n \geq 1$, hence A is not archimedean;

3. Consider $A = \{0, a, b, c, 1\}$ the residuated lattice from Example 6. Since $c^n = c$ for every natural number n, and $c^* = 0$ we deduce that $c^n \geq c^*$ for every $n \geq 1$ but $c \neq 1$, hence A is not archimedean.

Definition 13. Let A be a residuated lattice. An element $a \in A$ is called *archimedean* if it satisfy the condition: there is $n \geq 1$ such that $a^n \in B(A)$, (equivalent by Proposition 13 with $a \vee (a^n)^* = 1$). A residuated lattice A is called *hyperarchimedean* if all its elements are archimedean.

Example 16. 1. Consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 9; by Example 15 we deduce that A is archimedean. We have $B(A) = \{0, a, d, 1\}$. Since $a^2 = a \in B(A)$, $b^2 = 0 \in B(A), c^2 = a \in B(A)$ and $d^2 = d \in B(A)$ we deduce that A is even hyperarchimedean.

2. Consider $A = \{0, a, b, c, d, e, f, g, 1\}$ the residuated lattice from Example 10; we have $B(A) = \{0, b, f, 1\}$. Since $a^2 = 0 \in B(A), b^2 = b \in B(A), c^2 = 0 \in B(A)$, $d^2 = 0 \in B(A), e^2 = b \in B(A), f^2 = f \in B(A)$ and $g^2 = f \in B(A)$ we deduce that A is hyperarchimedean.

3. If consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 8 we deduce that $B(A) = \{0, 1\}$. Since $a^n = a \notin B(A)$, for every $n \geq 1$, we deduce that A is not hyperarchimedean; since $a^* = 0$, then $a^n = a \geq 0 = a^*$, for every $n \geq 1$, but $a \neq 1$, so A is not even archimedean.

From Lemma 14 we deduce:

Corollary 47. *Every hyperarchimedean residuated lattice is archimedean.*

Remark 15. For an example of archimedeean lattice that is not hyperarchimedean see [8], Example 3.42.

Theorem 48. *For a residuated lattice A, if A is hyperarchimedean, then for any deductive system D, the quotient residuated lattice A/D is archimedean.*

Proof. To prove A/D is archimedean, let $x = a/D \in A/D$ such that $x^n \geq x^*$ for any $n \geq 1$. By hypothesis, there is $m \geq 1$ such that $a \vee (a^m)^* = 1$, i.e. $a^m \in B(A)$. It follows that $x \vee (x^m)^* = 1$ (in A/D), i.e. $x^m \in B(A/D)$. In particular we have $x^m \geq x^*$, so by Lemma 14 we deduce that $x = 1$, that is, A/D is archimedean. □

The following result (suggested by the referee) is immediate:

Lemma 49. *If $(S, \leq)$ is a partially ordered set and $B \subset S$ then the following are equivalent:*

(*i*) *Every $b \in B$ is maximal in B;*

(*ii*) *Every $b \in B$ is minimal in B;*

(*iii*) *$(B, \leq)$ is unordered (or, maybe better said, the relation $\leq$ restricted to B is trivial).*

Theorem 50. *For a residuated lattice A the following conditions are equivalent:*

(*i*) *A is hyperarchimedean;*

(*ii*) $Spec(A) = Max(A)$;

(*iii*) $(Spec(A), \subseteq)$ *is unordered;*

(*iv*) *Any prime deductive system is minimal prime.*

Proof. $(i) \Rightarrow (ii)$. Since $Max(A) \subseteq Spec(A)$, we only have to prove that any prime deductive system of A is maximal. Let $P \in Spec(A)$. To prove $P \in Max(A)$, let $x \notin P$. Since A is hyperarchimedean there is $n \geq 1$ such that $x^n \in B(A)$, hence $x \vee (x^n)^* = 1$, (by Proposition 13). Since $1 \in P$ we deduce that $x \vee (x^n)^* \in P$. Since $x \notin P$, by Corollary 21 we deduce that $(x^n)^* \in P$, that is, $P \in Max(A)$ (see Corollary 39).

The equivalence $(ii) \Leftrightarrow (iii)$ is an immediate consequence of Lemma 49 (with $(S, \leq) = (Ds(A), \subseteq)$ and $B = Spec(A)$).

$(ii) \Rightarrow (iv)$. Let P, Q prime deductive systems such that $P \subseteq Q$. By hypothesis, P is maximal, so $P = Q$. Thus Q is minimal prime.

$(iv) \Rightarrow (i)$. Let $a \in A, a \neq 1$. We shall prove that a is an archimedean element. If we denote $D = < a >^* = \{x \in A : a \vee x = 1\}$ (by Corollary 19), then $D \in Ds(A)$. Since $a \neq 1$, then $a \notin D$ and we consider $D' = D(a) = \{x \in A : x \geq d \odot a^n$ for some $d \in D$ and $n \geq 1\}$. If we suppose that D' is a proper deductive system of A, then by Corollary 23, there is a prime deductive system P such that $D' \subseteq P$. By hypothesis, P is a minimal prime. Since $a \in P$, using Proposition 36 , we infer that there is $x \in A \backslash P$ such that $a \vee x = 1$. It follows that $x \in D \subseteq D' \subseteq P$, hence $x \in P$, so we get a contradiction.

Thus D' is not proper, so $0 \in D'$, hence there are $n \geq 1$ and $d \in D$ such that $d \odot a^n = 0$. Thus $d \leq (a^n)^*$. We get $a \vee d \leq a \vee (a^n)^*$. But $a \vee d = 1$ (since $d \in D$), so we obtain that $a \vee (a^n)^* = 1$, that is, a is an archimedean element. □

Remark 16. 1. For the case of lattices we have the following result of Nachbin (see [1], p.73): *A distributive lattice is relatively complemented iff every prime ideal is maximal*;

2. For the case of BL-algebras we have ([6]) the following result: If A is a BL algebra, the following conditions are equivalent:

(i) *A is hyperarchimedean*;

(ii) *For any deductive system D, the quotient BL algebra A/D is an archimedean BL algebra*;

(iii) $Spec(A) = Max(A)$;

(iv) *Any prime deductive system is minimal prime.*

3. For the case of lattices we have the following result of Nachbin (see [13], p. 76): *If L is a distributive lattice with* 0 *and* 1, *then L is a Boolean lattice iff $P(L)$ is unordered* (*where $P(L)$ is the set of all prime ideals of L*).

Acknowledgments. The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referee.

REFERENCES

1. Balbes, R.; Dwinger, P. – *Distributive Lattices,* University of Missouri Press, Columbia, Mo., 1974.

2. Birkhoff, G. – *Lattice Theory,* Third edition, American Mathematical Society Colloquium Publications, vol. XXV, American Mathematical Society, Providence, R.I., 1967.

3. Blyth, T.S.; Janowitz, M.F. – *Residuation Theory,* International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, Oxford-New York-Toronto, Ont., 1972.

4. Blok, W.J.; Pigozzi, D. – *Algebraizable logics,* Mem. Amer. Math. Soc., 77 (1989).

5. Blount, K.; Tsinakis, C. – *The structure of residuated lattices,* Internat. J. Algebra Comput., 13 (2003), 437–461.

6. Buşneag, D.; Piciu, D. – *On the lattice of deductive systems of a BL-algebra,* Cent. Eur. J. Math., 1 (2003), 221–237.

7. Buşneag, D.; Piciu, D. – *Residuated lattice of fractions relative to a $\wedge$-closed system,* Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 49 (2006), 13–24.

8. Busneag, D.; Rudeanu, S. – *A gimpse of deductive systems in algebra,* to appear in Cent. Eur. J. Math., vol. 8, 4 (2010).

9. Creţan, R.; Jeflea, A. – *On the lattice of congruence filters of a residuated lattice*, An. Univ. Craiova Ser. Mat. Inform., 33 (2006), 174–188.

10. Dilworth, R.P. – *Non-commutative residuated lattices,* Trans. Amer. Math. Soc., 46 (1939), 426–444.

11. Freytes, H. – *Injectives in residuated algebras,* Algebra Universalis, 51 (2004), 373–393.

12. Georgescu, G.; Ploščica, M. – *Values and minimal spectrum of an algebraic lattice,* Math. Slovaca, 52 (2002), 247–253.

13. Grätzer, G. – *Lattice theory*, W.H. Freeman and Company, San Francisco, 1979.

14. Höhle, U. – *Commutative, residuated l-monoids,* In: U. Hohle and E.P. Klement (Eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Boston, Dordrecht, 1995.

15. Idziak, P.M. – *Lattice operation in BCK-algebras,* Math. Japon., 29 (1984), 839–846.

16. Iorgulescu, A. – *Algebras of logic as BCK algebras*, Ed. ASE, Bucharest, 2008.

17. Kowalski, T.; Ono, H. – *Residuated lattices: an algebraic glimpse at logic without contraction*, Japan Advanced Institute of Science and Technology, 2001.

18. Krull, W. – *Axiomatische Begründung der allgemeinen Ideal theorie*, Sitzungsberichte der physikalisch medizinischen Societäd der Erlangen, 56 (1924), 47–63.

19. Okada, M.; Terui, K. – *The finite model property for various fragments of intuitionistic linear logic,* J. Symbolic Logic, 64 (1999), 790–802.

20. Ono, H.; Komori, Y. – *Logics without the contraction rule,* J. Symbolic Logic, 50 (1985), 169–201.

21. Pavelka, J. – *On fuzzy logic II. Enriched residuated lattices and semantics of propositional calculi,* Z. Math. Logik Grundlag. Math., 25 (1979), 119–134.

22. Turunen, E. – *Mathematics Behind Fuzzy Logic,* Advances in Soft Computing. Physica-Verlag, Heidelberg, 1999.

23. Ward, M. – *Residuated distributive lattices,* Duke Math. J., 6 (1940), 641–651.

24. Ward, M.; Dilworth, R.P. – *Residuated lattices,* Trans. Amer. Math. Soc., 45 (1939), 335–354.

Received: 8.IV.2009
Revised: 18.II.2010

Faculty of Mathematics and Informatics,
University of Craiova,
13, Al.I. Cuza st., 200585, Craiova,
ROMANIA
busneag@central.ucv.ro
danap@central.ucv.ro

Faculty of Bookkeeping Financial Management,
University Spiru Haret,
32-34, Unirii st., Constantza,
ROMANIA
antojeflea@yahoo.com