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# ON THE CONVERGENCE OF AN APPROXIMATION SCHEME FOR THE VISCOSITY SOLUTIONS OF THE BELLMAN EQUATION ARISING IN A STOCHASTIC OPTIMAL CONTROL PROBLEM*

BY

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**Abstract.** The paper presents an Euler step approximation scheme for the viscosity solution of the dynamic programming equation for a stochastic control problem with infinite horizon. We established the rate of convergence of this scheme.

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**Key words:** stochastic optimal control, Bellman equations, viscosity solution.

## 1. Introduction and preliminary results

At the beginning of '80, using the viscosity solutions theory, several approximation results for the Bellman equation in the deterministic case were obtained (see [1],[2],[4],[5]). In [6] we proposed an Euler step approximation scheme (similar to those from [1],[3],[5]) for the viscosity solution of the Bellman equation of a stochastic optimal control problem and we proved its convergence. In the present paper we give a result concerning the rate of convergence for the approximation scheme presented in [6] and also we improve the convergence result there.

Let us consider the following stationary second order Bellman partial differential equation

$$\lambda u(x) + \sup_{a\in A}\left[-\nabla u(x)\cdot b(x,a) - f(x,a)\right] - \frac{\varepsilon}{2}\Delta u(x) = 0,\ x\in\mathbb{R}^n, \tag{1.1}$$

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where $u(\cdot)$ is the unknown function, $A \subset \mathbb{R}^m$ is a compact set, $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ are two given functions, $\lambda > 0$ and $\varepsilon > 0$ are two given constants, and "$\cdot$" stands for the inner product in $(\mathbb{R}^n, \|\cdot\|)$.

We suppose that the functions $b$ and $f$ satisfy the following hypotheses: There exist the positive constants $L_1 \in (0, \lambda)$, $L_2$, and $M$ such that

$$\text{(H)} \qquad \begin{array}{ll} \text{(i)} & \|b(x,a) - b(y,a)\| \leq L_1 \|x - y\|, \ x, y \in \mathbb{R}^n, \ a \in A \\ \text{(ii)} & |f(x,a) - f(y,a)| \leq L_2 \|x - y\|, \ x, y \in \mathbb{R}^n, \ a \in A \\ \text{(iiii)} & \|b(x,a)\| \leq M, \ |f(x,a)| \leq M, x \in \mathbb{R}^n, \ a \in A. \end{array}$$

In [6] we have proved that equation (1.1) has a unique viscosity solution which is the value functions of the following stochastic control problem with infinite horizon.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ be a fixed complete stochastic basis which means that $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t \geq 0}$ is an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every $P-$negligible subset of $\Omega$. Also we consider a standard $n-$dimensional $\mathcal{F}_t$-adapted Wiener process $\{w(t), t \geq 0\}$ and

$$\mathcal{A} = \begin{array}{l} \{a : [0, +\infty) \times \Omega \to A; \ a(\cdot, \cdot) \text{ is measurable,} \\ \text{and } a(t, \cdot) \text{ is } \mathcal{F}_t - \text{measurable}, t \geq 0\}. \end{array} \tag{1.2}$$

**Remark 1.1.** In the following we shall consider that the $\sigma-$algebra $\mathcal{F}_t$ is generated by $\{w(s), 0 \leq s \leq t\}$ for $t \geq 0$.

Now we present the optimal control problem:
Minimize

$$J(x,a) = E\left[\int_0^\infty f(x(t), a(t)) e^{-\lambda t} dt\right], \tag{1.3}$$

where $E$ stands for the average, $a \in \mathcal{A}$ and $(x(t), a(t))$ satisfies the state equation

$$\begin{cases} dx(t) = b(x(t), a(t))dt + \sqrt{\varepsilon} dw(t), \ \varepsilon > 0 \\ x(0) = x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

We define

$$J(x) = \inf_{a \in \mathcal{A}} J(x,a), \ x \in \mathbb{R}^n. \tag{1.5}$$

As we have mentioned before, the function given by (1.5) is the unique viscosity solution of (1.1). For the reader's convenience we shall give the definition of the viscosity solution for (1.1).

**Definition 1.1** ([7])**.** The function $u \in C(\mathbb{R}^n)$ is called viscosity subsolution (supersolution) of (1.1) iff for every $\varphi \in C^2(\mathbb{R}^n)$ for which $(u - \varphi)$ has a (strict) local maximum (minimum) in $x_0$ we have

$$\lambda u(x_0) + \sup_{a \in A}[-\nabla\varphi(x_0) \cdot b(x_0, a) - f(x_0, a)] - \frac{\varepsilon}{2}\Delta\varphi(x_0) \le 0 \quad (\ge 0).$$

A function $u(\cdot)$ which is at the same time a viscosity subsolution and a viscosity supersolution is called viscosity solution.

In the next we present the approximation scheme ([6]).

Let $h > 0$ and $t_j = jh$, $j = 0, 1, 2, \ldots$. Corresponding to each $h > 0$ we consider the set of discrete controls:

$$\begin{aligned} \mathcal{A}_h =\{&a \in \mathcal{A};\ a \text{ is constant on } [t_j, t_{j+1}),\ j = 0, 1, 2, \ldots, \text{i.e.} \\ &a(s) = a_j(\cdot) \in L^2(\Omega, A), \quad \text{for } s \in [t_j, t_{j+1}),\ j = 0, 1, 2, \ldots\}. \end{aligned} \tag{1.6}$$

Now we define recursively the sequence $\{x_j\}_{j \ge 0}$ (the approximation of (1.4))

$$\begin{cases} x_0 = x \\ x_{j+1} = x_j + hb(x_j, a_j) + \sqrt{\varepsilon}(w(t_{j+1}) - w(t_j)), j \in \mathbb{N} \end{cases} \tag{1.7}$$

where $a \in \mathcal{A}_h$ ($a_j = a(s)$, $s \in [t_j, t_{j+1})$). Also we consider the following discretization of the functional (1.3)

$$J_h(x, a) = hE\sum_{j=0}^{\infty}(1 - \lambda h)^j f(x_j, a_j), \tag{1.8}$$

were $a \in \mathcal{A}_h$ and $\{x_j\}_{j \ge 0}$ verifies (1.7).

Now we define the approximate solution of (1.1) as

$$v_h(x) = \inf_{a \in \mathcal{A}_h} J_h(x, a) = \inf_{a \in \mathcal{A}_h} E\sum_{j=0}^{\infty}(1 - \lambda h)^j f(x_j, a_j). \tag{1.9}$$

For $h \in (0; \frac{1}{\lambda}]$ we consider the following approximate equation of (1.1)

$$u_h(x) + \sup_{a \in A} E[-(1-\lambda h)u_h(x + hb(x,a) + \sqrt{\varepsilon}w(h)) + f(x,a)] = 0, \quad x \in \mathbb{R}^n. \tag{1.10}$$

In the next we shall prove a dynamic programming principle for the discrete optimal control problem (1.9), (1.7).

**Theorem 1.1.** *For each $x \in \mathbb{R}^n$ we have*

$$v_h(x) = \inf_{a_0} E[hf(x,a_0) + (1-\lambda h)v_h(x_1)], \tag{1.11}$$

*where $a_0 \in L^2(\Omega, A)$ is $\mathcal{F}_0-$measurable and $x_1$ is given by* (1.7) *with $x_0 = x$.*

**Proof.** Using (1.9) and (1.8) we obtain that for each $\nu > 0$ there exists $a^\nu \in \mathcal{A}_h$ such that

$$\begin{aligned} v_h(x) + \nu \geq J_h(x,a^\nu) &= hEf(x,a_0^\nu) + (1-\lambda h)hE \\ \sum_{j=0}^{\infty}(1-\lambda h)^j f(x_{j+1}^\nu, a_{j+1}^\nu) &\geq hEf(x,a_0^\nu) + (1-\lambda h)Ev_h(x_1^\nu), \end{aligned} \tag{1.12}$$

where $\{x_j^\nu\}_{j\in\mathbb{N}}$ and $\{a_j^\nu\}_{j\in\mathbb{N}}$ verify (1.7). Taking the infimum in (1.12) we obtain

$$v_h(x) + \nu \geq \inf_{a \in \mathcal{A}_h} E[hEf(x,a_0) + (1-\lambda h)v_h(x_1)], \tag{1.13}$$

for every $\nu > 0$. For every $\nu > 0$ there exist $\widetilde{a}^\nu, \widehat{a}^\nu \in \mathcal{A}_h$ such that

$$\begin{aligned} &\inf_{a \in \mathcal{A}_h} E\left[hf(x,a) + (1-\lambda h)v_h(x_1)\right] + \nu \\ &\geq E\left[hf(x,\widetilde{a}_0^\nu) + (1-\lambda h)v_h(\widetilde{x}_1^\nu)\right] \\ &\geq E\left[hf(x,\widetilde{a}_0^\nu) + (1-\lambda h)h\sum_{j=0}^{\infty}(1-\lambda h)^j f(\widehat{x}_j^\nu, \widehat{a}_j^\nu)\right] - \nu, \end{aligned} \tag{1.14}$$

where $\{\widetilde{x}_j^\nu\}_{j\in\mathbb{N}}$, $\{\widetilde{a}_j^\nu\}_{j\in\mathbb{N}}$ satisfy (1.7) with $\widetilde{x}_0^\nu = x$, and $\{\widehat{x}_j^\nu\}_{j\in\mathbb{N}}$, $\{\widehat{a}_j^\nu\}_{j\in\mathbb{N}}$ satisfy (1.7) with $\widehat{x}_0^\nu = \widetilde{x}_1^\nu$.

Let $\overline{a}^\nu = \{\overline{a}_j^\nu\}_{j\in\mathbb{N}}$ and $\overline{x}^\nu = \{\overline{x}_j^\nu\}_{j\in\mathbb{N}}$ defined by

$$\begin{cases} \overline{a}_0^\nu = \widetilde{a}_0^\nu, \\ \overline{a}_j^\nu = \widehat{a}_{j-1}^\nu, \quad j \geq 1 \end{cases}, \qquad \begin{cases} \overline{x}_0^\nu = \widetilde{x}_0^\nu = x, \\ \overline{x}_j^\nu = \widehat{x}_{j-1}^\nu, \qquad j \geq 1 \end{cases} \tag{1.15}$$

Taking into account (1.13), (1.14), (1.15) and (1.8) it results

$$\inf_{a\in\mathcal{A}_h} E[hf(x,a_0)+(1-\lambda h)v_h(x_1)]+2\nu \geq J_h(\overline{a}^\nu,x)\geq v_h(x). \tag{1.16}$$

Because $\nu>0$ is arbitrarily chosen, the relations (1.14) and (1.16) imply (1.11).

In the next, for the reader's convenience, we shall present some results (see also [6]) which we need in evaluation of the rate of convergence of our scheme.

**Theorem 1.2** ([6])**.** *In the hypotheses* (H) *the* $v_h$ *given by* (1.9) *is the unique bounded solution of* (1.10)*.*

*Sketch of the proof.* Using Theorem 1.1 we get that $v_h$ verifies

$$\begin{aligned} v_h(x)+\sup_{a_0} E[&-(1-\lambda h)v_h(x+hb(x,a_0)) \\ &+\sqrt{\varepsilon}w(h)-hf(x,a_0)]=0,\quad x\in\mathbb{R}^n, \end{aligned} \tag{1.17}$$

where $a_0\in L^2(\Omega,A)$ and $a_0$ is $\mathcal{F}_0-$measurable (see the definitions (1.2) and (1.6)).

Taking into account the fact that $\mathcal{F}_0$ and $\mathcal{F}_h$ are independent (see Remark 1.1) we may write the average in (1.14) as a double integral and so we obtain

$$\begin{aligned} v_h(x)+\sup_{a_0\in L^2(\Omega_1,A)}\int_{\Omega_1}\Big\{\int_\Omega[&-(1-\lambda h)v_h(x+b(x,a_0(\omega_1))) \\ &+\sqrt{\varepsilon}\omega(h)(\omega))-hf(x,a_0(\omega_1))]dP(\omega)\Big\}\,dP(\omega_1)=0, \end{aligned} \tag{1.18}$$

where $\Omega_1$ is $\Omega$ but we put subscript 1 to be more clear. Using boundness of $v_h$ (which is evident from (1.8)), from the relation (1.18) we obtain that $v_h$ is a solution of (1.10).

To prove the uniqueness, we suppose that there exist two solutions $u_1$ and $u_2$ of (1.10) and subtracting the two relations (1.10) which they verify, using their boundness and some calculus, it results

$$|u_1(x)-u_2(x)|\leq(1-\lambda h)\sup_{y\in\mathbb{R}^n}|u_1(y)-u_2(y)|,x\in\mathbb{R}^n,$$

relation which proves the uniqueness because $0<h\leq\frac{1}{\lambda}$.

**Theorem 1.3** ([6]). *In the hypotheses* (H) *the function* $v_h(\cdot)$ *converges to* $J(\cdot)$ *locally uniformly on* $\mathbb{R}^n$ *as* $h$ *tends to* $O$.

*Sketch of the proof.* Using (1.9) and the hypotheses on $b, f$, and $\lambda$, it results

$$\sup_{x\in\mathbb{R}^n} |v_h(x)| \le \frac{M}{\lambda} \tag{1.19}$$

$$|v_h(x) - v_h(x')| \le \frac{L_2}{\lambda - L_1}\|x - x'\|, \text{ for } x, x' \in \mathbb{R}^n \text{ and } h \in (0, \frac{1}{\lambda}].$$

From the Ascoli–Arzela theorem we obtain that there exists a bounded Lipschitz continuous function $u(\cdot)$ such that $v_h(\cdot)$ converges to $u(\cdot)$ locally uniformly on $\mathbb{R}^n$ as $h$ tends to 0.

In the following we shall prove that $u(\cdot)$ is a viscosity subsolution of (1.1) (in the same manner one can prove that it is also a viscosity supersolution). We shall present this proof in more details because we consider that it is interesting and useful for the reader.

Let $\varphi \in C_b^2(\mathbb{R}^n)$ (a bounded function of class $C^2$), $x_0$ a strict local maximum point of $(u - \varphi)$, $\overline{B}(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le r\}$ and $x_0^h$ a global maximum point of $(v_h - \varphi)$ on $\overline{B}(x_0, r)$. The sequence $\{x_0^h\}_{h>0}$ converges to $x_0$ as $h$ tends to 0 (see [3]) and so we can choose $h_1 > 0$ such that for every $h \in (0, h_1)$ we have

$$\|x_0^h - x_0\| \le \frac{r}{3},\ hM \le \frac{r}{3}. \tag{1.20}$$

Let

$$\Omega_r = \left\{\omega \in \Omega;\ \|w(h)(\omega)\| \le \frac{r}{3}\right\}. \tag{1.21}$$

It is well-known (see [2]) that

$$P(\Omega \setminus \Omega_r) \le \frac{E\|w(h)\|^4}{\frac{r^4}{81}} \le c\frac{h^2}{r^4}, \tag{1.22}$$

where $c > 0$ is a constant independent of $h$ and $r$.

Using now (1.10), (1.19), (1.20) and the fact that $x_0^h$ is a maximum point of $(v_h - \varphi)$ on $\overline{B}(x_0, r)$ we obtain

$$\begin{aligned}
0 = v_h(x_0^h) + \sup_{a\in A}\{\int_{\Omega_r} [-(1-\lambda h)v_h(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h))\\
- hf(x_0^h, a)]dP + \int_{\Omega\setminus\Omega_r} [-(1-\lambda h)v_h(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h))\\
- hf(x_0^h, a)]dP \geq \sup_{a\in A}\{\int_{\Omega_r} [\varphi(x_0^h) - \varphi(x_0^h + hb(x_0^h, a)\\
+ \sqrt{\varepsilon}w(h)) + \lambda h v_h(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h))]dP\\
+ \int_{\Omega\setminus\Omega_r} [v_h(x_0^h) - (1-\lambda h)v_h(x_0^h + hb(x_0^h, a)\\
+ \sqrt{\varepsilon}w(h))]dP - hf(x_0^h, a)\}.
\end{aligned}\tag{1.23}$$

Taking into account (1.21) and the boundness of $\varphi$ and $v_h$, the relation (1.22) implies

$$\begin{aligned}
0 \geq \sup_{a\in A} E[\varphi(x_0^h) - \varphi(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h))\\
+ \lambda h v_h(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h)) - hf(x_0^h, a)] - c_1\frac{h^2}{r^4},
\end{aligned}\tag{1.24}$$

where $c_1 > 0$ is a constant independent of $h$.

Because $\varphi \in C^2(\mathbb{R}^n)$, using Itô formula, the relation (1.23) gives

$$\begin{aligned}
0 \geq \sup_{a\in A} E\{-\int_0^h [\nabla\varphi(x_0^h + sb(x_0^h, a) + \sqrt{\varepsilon}w(s)) \cdot b(x_0^h, a)\\
- \frac{\varepsilon}{2}\Delta\varphi(x_0^h + sb(x_0^h, a) + \sqrt{\varepsilon}w(s))]ds - \int_0^h \nabla\varphi(x_0^h + sb(x_0^h, a)\\
+ \sqrt{\varepsilon}w(s))dw(s) + \lambda h v_h(x_0^h + hb(x_0^h, a) + \sqrt{\varepsilon}w(h))\\
- hf(x_0^h, a)\} - c_1\frac{h^2}{r^4}.
\end{aligned}\tag{1.25}$$

Dividing (1.24) by $h$ and using the fact that $\varphi \in C^2(\mathbb{R}^n)$, $v_h \to u$ (locally uniformly as $h \to 0$), and $x_0^h \to x_0$ as $h \to 0$, it results

$$\lambda u(x_0) + \sup_{a\in A}[-\nabla\varphi(x_0) \cdot b(x_0, a) - f(x_0, a)] - \frac{\varepsilon}{2}\Delta\varphi(t_0) \leq 0,$$

i.e. $u(\cdot)$ is a viscosity subsolution of (1.1) (see Definition (1.1)).

In a similar way one can prove that $u(\cdot)$ is also a viscosity supersolution of (1.1). In our hypotheses on $\lambda, b, f$ the equation (1.1) has an unique viscosity solution given by $J(\cdot)$ (see [7]). So $u(\cdot) = J(\cdot)$ and Theorem 1.3 is proved.

## 2. The rate of convergence of the approximation scheme

In this section we shall prove a result with respect to the rate of convergence of the approximate solutions $v_h(\cdot)$ to the viscosity solution $J(\cdot)$ of equation (1.1) and also we shall improve the result in Theorem 1.3. Such results for the deterministic case were given in [1], [3], [5].

**Theorem 2.1.** *In the hypotheses* (H) *we have*

$$\sup_{x \in \mathbb{R}^n} |J(x) - v_h(x)| \leq ch, \ 0 < h \leq \frac{1}{\lambda},$$

*where $c > 0$ is a constant independent of $h$.*

**Proof.** Let $\delta, \rho \in (0, 1)$ be fixed. We define the auxiliary function

$$\varphi(x, y) = v_h(x) - J(y) - \frac{\delta \|x - y\|^2}{\rho^2}, \ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, 0 < h \leq \frac{1}{\lambda}. \tag{2.1}$$

Using the fact that $v_h$ and $J$ are bounded (see (1.19) and (1.3)) it results that for each $\gamma \in (0, 1)$ there exists $(x_1, y_1) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\varphi(x_1, y_1) > \sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} \varphi(x, y) - \gamma. \tag{2.2}$$

Let's consider a smooth function $\xi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfies

$$\xi(x_1, y_1) = 1, \ 0 \leq \xi \leq 1, \ \|\nabla \xi\| \leq 1. \tag{2.3}$$

Define

$$\psi(x, y) = \varphi(x, y) + \gamma \xi(x, y), \ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2.4}$$

From (2.2) and (2.4) we obtain that there exists $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\psi(x_0, y_0) \geq \psi(x, y), \ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2.5}$$

So, the function $-\psi(x_0, y) = J(y) - (v_h(x_0) - \delta\frac{\|x_0-y\|^2}{\rho^2} + \gamma\xi(x_0, y))$ has a global minimum point in $y_0$. Using the fact that $J(\cdot)$ is a viscosity supersolution of (1.1) it results (see Definition 1.1)

$$
\begin{aligned}
&\lambda J(y_0) + \sup_{a\in A}[-\frac{2\delta}{\rho^2}(x_0 - y_0)\cdot b(y_0, a) - \gamma(\nabla_y \xi(x_0, y_0))\cdot b(y_0, a)\\
&- f(y_0, a)] + \varepsilon\frac{\delta}{\rho^2} - \frac{\varepsilon}{2}\gamma\Delta_y\xi(x_0, y_0) \geq 0.
\end{aligned}
\tag{2.6}
$$

The set $A$ being compact it results that there exists $a_0 \in A$ such that (see (2.6))

$$
\begin{aligned}
&\lambda J(y_0) - \frac{2\delta}{\rho^2}(x_0 - y_0)\cdot b(y_0, a_0) - \gamma(\nabla_y \xi(x_0, y_0))\cdot b(y_0, a_0)\\
&- f(y_0, a_0) + \varepsilon\frac{\delta}{\rho^2} - \frac{\varepsilon}{2}\gamma\Delta_y\xi(x_0, y_0) \geq 0.
\end{aligned}
\tag{2.7}
$$

The function $v_h(\cdot)$ verifies the relation (1.17) and so we have

$$
v_h(x_0) - (1-\lambda h)Ev_h(x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h)) - hf(x_0, a_0) \leq 0. \tag{2.8}
$$

Using (2.5) we obtain

$$
\begin{aligned}
v_h(x_0) - J(y_0) - \frac{\delta\|x_0 - y_0\|^2}{\rho^2} &+ \gamma\xi(x_0, y_0)\\
&\geq E[v_h(x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h))\\
&- \frac{\delta\|x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h) - y_0\|^2}{\rho^2}\\
&+ \gamma\xi(x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}(h), y_0)] - J(y_0),
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&Ev_h(x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h))\\
&\qquad \leq v_h(x_0) + \gamma\xi(x_0, y_0) - \frac{\delta\|x_0 - y_0\|^2}{\rho^2}\\
&\qquad + \frac{\delta}{\rho^2}E\|x_0 - y_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h)\|^2\\
&\qquad - \gamma E\xi(x_0 + hb(x_0, a_0) + \sqrt{\varepsilon}w(h), y_0).
\end{aligned}
\tag{2.9}
$$

The relations (2.8) and (2.9) imply

$$
\begin{aligned}
&\lambda h v_h(x_0) - (1-\lambda h)\gamma\xi(x_0,y_0) + \frac{(1-\lambda h)\delta}{\rho^2}\|x_0-y_0\|^2\\
&-\frac{(1-\lambda h)\delta}{\rho^2}E\|x_0-y_0+hb(x_0,a_0)+\sqrt{\varepsilon}w(h)\|^2\\
&+(1-\lambda h)\delta E\xi(x_0+hb(x_0,a_0)+\sqrt{\varepsilon}w(h),y_0) - hf(x_0,a_0)\le 0.
\end{aligned}
\tag{2.10}
$$

Dividing (2.10) by $h$ we obtain

$$
\begin{aligned}
&\lambda v_h(x_0) - \frac{1-\lambda h}{h}\gamma\xi(x_0,y_0) + \frac{(1-\lambda h)\delta}{h\rho^2}\|x_0-y_0\|^2\\
&-\frac{(1-\lambda h)\delta}{h\rho^2}E\|x_0-y_0+hb(x_0,a_0)+\sqrt{\varepsilon}\omega(h)\|^2\\
&+\frac{(1-\lambda h)\delta}{h}E\xi(x_0+hb(x_0,a_0)+\sqrt{\varepsilon}w(h),y_0) - f(x_0,a_0)\le 0.
\end{aligned}
\tag{2.11}
$$

Using the hypotheses (H) and also the properties of $\xi$ and $w(h)$ ($Ew(h)=0$, $E\|w(h)\|^2=h$), the inequality (2.11) implies

$$
\begin{aligned}
\lambda v_h(x_0) - \frac{\gamma}{h} - \frac{(1-\lambda h)\delta h M^2}{\rho^2}&\\
-\frac{(2-\lambda h)\delta}{\rho^2}\varepsilon - \frac{2\delta}{\rho^2}(x_0-y_0)\cdot b(x_0,a_0)&\\
-\frac{2\lambda h\delta}{\rho^2}M\|x_0-y_0\| - \frac{h\delta}{\rho^2} - f(x_0,a_0)\le 0.&
\end{aligned}
\tag{2.12}
$$

Subtracting (2.7) from (2.12) it results

$$
\begin{aligned}
&\lambda(v_h(x_0)-J(y_0)) + \gamma(\nabla_y\xi(x_0,y_0))b(y_0,a_0) - \frac{\gamma}{h}\\
&-\frac{(1-\lambda h)}{\rho^2}\delta h M^2 - \frac{(3-\lambda h)\delta}{\rho^2}\varepsilon - \frac{2\lambda h\delta}{\rho^2}M\|x_0-y_0\|\\
&-\frac{h\delta}{\rho^2} + \frac{\varepsilon}{2}\gamma\Delta_y\xi(x_0,y_0)\le 0.
\end{aligned}
\tag{2.13}
$$

Making $x=y=x_0$ in relation (2.5) and using the properties of $v_h$ and $\xi$ (see (1.19) and (2.3)) it results

$$
\delta\|x_0-y_0\| \le \left(\gamma + \frac{L_2}{\lambda - L_1}\right)\rho^2. \tag{2.14}
$$

Taking again into account hypotheses (H) and (2.3), the relations (2.13) and (2.14) imply

$$v_h(x_0) - J(y_0) \le c\left(\gamma + \frac{\gamma}{h} + \frac{h\delta}{\rho^2} + \frac{\delta}{\rho^2} + h\right), \tag{2.15}$$

where $c > 0$ is a constant independent of $h, \rho, \gamma$, and $\delta$.

Now taking $\gamma = h^2, \delta = h^3, \rho = h$, the relation (2.15) gives

$$v_h(x_0) - J(y_0) \le ch \text{ for } 0 < h \le \frac{1}{\lambda}. \tag{2.16}$$

Putting in the relation (2.5) $y = x$, it results

$$v_h(x) - J(x) + \gamma\xi(x,x) \le v_h(x_0) - J(y_0) - \frac{\delta\|x_0 - y_0\|^2}{\rho^2} + \gamma\xi(x_0, y_0)$$

from where, using (2.16), we obtain

$$v_h(x) - J(x) \le ch,\ x \in \mathbb{R}^n, 0 < h \le \frac{1}{\lambda}, \tag{2.17}$$

where $c > 0$ is a constant independent of $h$.

To prove the opposite inequality we consider the auxiliary function

$$\widetilde{\varphi}(x,y) = J(y) - v_h(x) + \frac{\delta\|x-y\|^2}{\rho^2}$$

which is bounded from below.

Using, this time, the fact that $J(\cdot)$ is a viscosity subsolution of (1.1) and making a similar calculus as before we obtain the desired result.

**Remark 2.1.** Theorem 2.1 established that the approximate solution $v_h(\cdot)$ converges globally uniformly on $\mathbb{R}^n$ to the viscosity solution $J(\cdot)$ and so we have obtained an improvement of the result in Theorem 1.3 (see [6]).

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