RELATIONSHIPS BETWEEN FARTHEST POINT PROBLEM AND BEST APPROXIMATION PROBLEM

BY

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Abstract. The aim of this paper is to establish characterizations of remotal and antiremotal sets by a property of closedness, respectively openess, of some associated sets. We also examine the connection between farthest point problem and best approximation problem associated to a given set in a linear normed space.

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1. Introduction

Let X be a linear normed space and let A be a given nonvoid set A in X. Let us consider the *farthest point problem*

$$(F_x) \qquad\qquad \max_{y \in A} \|x - y\|, \;\; x \in X.$$

This problem is similar to the well known *best approximation problem*

$$(A_x) \qquad\qquad \min_{y \in A} \|x - y\|, \;\; x \in X,$$

but the most properties of this two problems are different.

First, we remark that the problem (F_x) can be considered only for convex sets A because it has solutions if and only if there exist solutions in its convex hull, conv A (see, for instance, [16]). Even in the case of convex sets A when problem (A_x) is a *convex optimization problem*, the problem (F_x) is not convex being a typical *d.c. optimization problem* (see [15], [28], [30]).

According to the *classification of* HIRRIART-URRUTY [15], the farthest point problem (F_x) can be given in one of the following three types: P_1 (the maximization of a convex function on a convex set) P_2 (the minimization of a difference of two convex functions on a convex set) or P_3 (the minimization of a convex function on the complement of a convex set). Indeed, the problem (F_x) can be equivalently written as:

$$(F'_x) \qquad \min_{y \in A}\{I_A(x) - \|x - y\|\},$$

or

$$(F''_x) \qquad \min_{\|x-y\| \geq t}\{I_A(x) - t\},$$

where I_A is the *indicator function* of A ([5], [17]).

Consequently, we obtain some optimality conditions using the normal cone of A and the ε-subdifferential of the norm.

In the sequel, we recall some concepts associated to the farthest point problem which are similar to some known concepts of the best approximation theory.

We denote

$$(1.1) \qquad \Delta_A(x) = \sup_{y \in A} \|x - y\|, \quad x \in X,$$

called the *farthest distance function* of the set A,

$$(1.2) \qquad Q_A(x) = \{\overline{x} \in A; \|x - \overline{x}\| = \Delta_A(x)\},$$

called the *farthest point mapping* (or *antiprojection* by several authors, see for example, [2], [34]) with respect to A. The elements of $Q_A(x)$ are called *farthest points* of x through elements of the set A. The mappings $\Delta_A(x)$ and $Q_A(x)$ correspond to $d(A; x)$ and $P_A(x)$, respectively the distance from x to the set A and the projection of x in the set A. The set A is called *remotal* if $Q_A(x) \neq \emptyset$ for all $x \in X$, and *antiremotal* if $Q_A(x) = \emptyset$ for all $x \in X$. Special properties of remotal and antiremotal sets are established by ASPLUND [1], [2], BALAGANSKII [3], [4], BARONTI and PAPINI [6], BLATTER [7], COBZAŞ [9], EDELSTEIN [10], HIRIART-URRUTY [16], KLEE [21], KA-SING LAU [19], PANDA and KAPOOR [23], VLASOV [33], ZHIVKOV [34] (see also the literature cited therein). Detailed and important bibliographics comments can be found in [9].

All the elements of nonlinear analysis used in this paper are in accordance with [5] and [17].

It is obvious that the mapping Δ_A is a continuous convex function. Moreover

$$(1.3) \qquad |\Delta_A(x_1) - \Delta_A(x_2)| \le \|x_1 - x_2\|, \ \ x_1, x_2 \in X.$$

Consequently, Δ_A is subdifferentiable and $\partial \Delta_A(x) \subset \overline{S}_{X^*}(0;1)$ for every $x \in X$ [15].

On the other hand we remark that by TOLAND duality [30], [31] we have the following equality

$$\Delta_A(x) = \sup\{x^*(x) - s_A(x^*); \|x^*\| \le 1\},$$

where s_A is the support functional of A, i.e. $s_A(x^*) = \sup\{x^*(u); u \in A\}$. Other dual forms for $\Delta_A(x)$ are established, for example, in [22], [28], [32].

We also recall some simple convexity properties.

$$(1.4) \qquad \Delta_A(x) = \Delta_{\overline{conv\,A}}(x),$$

$$(1.5) \qquad \Delta_A(\lambda x + (1-\lambda)y) = \lambda \Delta_A(x), \ \text{ for all } y \in Q_A(x).$$

$$(1.6) \qquad y \in Q_A(\lambda x + (1-\lambda)y) \text{ if } y \in Q_A(x) \text{ and } \lambda > 1.$$

(See, for instance, [16].)

In this paper we establish characterizations of remotal and antiremotal sets by a closedness, respectively openess, condition for an associated set of the set A. These results hold for an arbitrary nonvoid set of a linear normed space being at the same type as the earlier characterizations of proximal and antiproximinal sets obtained even in nonconvex case in [25]. Therefore, the remotability and antiremotability are strong dependent with two important properties in functional analysis, namely the sum of two special set is closed or open. While one of sets is closed and convex the other is a complement of an open convex set. Moreover, we remark that the remotability (antiremotability) of a set A having the origin in its interior is equivalent to the proximinality (antiproximinality) of the set $cS(0;1)$ (the complement of the open unit ball) with respect to *Minkowski functional p_A* associated to the set A.

In the last part of this paper, we establish a pointwise result which says that the farthest point problem (F_x) is equivalent to a best approximation problem of the type (A_x), where the set A and the norm $\|\cdot\|$ are replaced by $cS(0;\alpha)$, respectively by p_A for a certain $\alpha > 0$.

2. Characterizations of remotal and antiremotal sets

In an earlier paper [27] we established a characterization of optimality of a family of optimization problems indexed in an abstract topological space Y, by a closedness property in the associated affine space $Y \times \mathbb{R}$ (see also [5]).

In [24], [25] this result was adapted to characterize the gloabal nonoptimality of a family of optimization problems by an openess property. Moreover, in some special cases the closedness, respectively the openess property can be required only for the all sections in Y for a fixed element in $\mathbb{R}$. This is the case of problems $(F_x)_{x \in X}$ and $(A_x)_{x \in X}$, when we obtain characterizations of remotal and antiremotal sets by closedness and openess properties in X.

Theorem 1. *A nonvoid bounded set A in a linear normed space X is remotal if and only if the following associated set*

$$(2.1) \qquad\qquad K_d = A + cS(0; d),$$

is closed for every $d > 0$.

Proof. Let x be an adherent element of $A + cS(0; d)$, i.e. there exist a sequence $(x_n)_{n \in \mathbb{N}}$ convergent to x and a sequence $(u_n)_{n \in \mathbb{N}} \subset A$ such that $\|x_n - u_n\| \geq d$ for all $n \in \mathbb{N}$. Thus, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x - u_n\| > d - \varepsilon$ for all $n \geq n_\varepsilon$. Now, if A is remotal, taking an element $\overline{x} \in Q_A(x)$ we obtain that $\|x - \overline{x}\| \geq \|x - u_n\|, n \in \mathbb{N}$, and so $\|x - \overline{x}\| > d - \varepsilon$, for every $\varepsilon > 0$. Consequently, $\|x - \overline{x}\| \geq d$, i.e. $x \in A + cS(0; d)$.

Conversely, for an arbitrary element $x \in X$ we take $d = \Delta_A(x)$. We can suppose $d > 0$ since $\Delta_A(x) = 0$ if and only if $A = \{x\}$ when A is obviously remotal. For every $n \in \mathbb{N}^*$ there exits $u_n \in A$ such that $\|x - u_n\| \geq d - 1/n$. But, we have

$$\frac{1}{n}(d - 1/n)^{-1}(x - u_n) + x \in u_n + cS(0; d) \subset A + cS(0; d)$$

for all $n \in \mathbb{N}^*$ such that $n > 1/d$. Since $(u_n)_{n \in \mathbb{N}}$ is bounded passing to the limit we get $x \in \overline{A + cS(0; d)}$. Therefore, if $A + cS(0; d)$ is closed there exists $\overline{x} \in A$ such that $\|x - \overline{x}\| \geq d$, i.e. $\overline{x} \in Q_A(x)$. Hence the set A is remotal. $\qquad \square$

Remark 1. It is easy to see that

$$(2.2) \qquad\qquad cK_d = \bigcap_{a \in A} S(a; d)$$

and so, the set K_d is always the complement of a convex bounded set. Consequently, a nonvoid bounded set A is remotal if and only if the convex set $\bigcap_{a \in A} S(a; d)$ is open for any $d > 0$.

Corollary 1. *Any closed ball in a linear normed space is remotal.*

Remark 2. We denote $r(A) = \inf\{\Delta_A(x); x \in X\}$, usually called the *radius* of A. If $d < r(A)$, then $K_d = X$, and so in Theorem 1 it suffices to consider only the case $d \geq r(A)$. Obviously, the set K_d is nonvoid complement of a convex set for any $d \geq 0$ and $K_d \neq X$ if $d > r(A)$.

Remark 3. We say that a set A is *d-remotal* (*d-proximinal*) if $Q_A(x) \neq \emptyset$, $P_A(x) \neq \emptyset$ whenever $\Delta_A(x) = d$ ($d(A; x) = d$). From the proof of Theorem 1 it follows that a set A is *d*-remotal if K_d is closed. Generally, the converse statement is not true. But, if a set is *d*-remotal for any $d \geq d_0$, then the sets K_d are closed for all $d \geq d_0$. Therefore, the property of *d*-remotability is different of the property of the set K_d to be closed. Similar statements for *d*-proximinality hold. A relationship between *d*-proximinality and *d*-remotability we will given in the last section (see Remark 9).

The above characterization of remotal sets is similar to the one of proximinal sets given as a consequences of the same general optimality criterium for an arbitrary family of optimization problems established in [5], [25], [26].

Theorem 2 ([25])**.** *A nonvoid set A in a linear normed space is proximinal if and only if the set*

$$(2.3) \qquad\qquad H_d = A + \overline{S}(0; d)$$

is closed for every $d \geq 0$.

Remark 4. In the case of linear closed subspaces Theorem 2 was obtained by GODINI [14] in the following equivalent form: *a closed linear subspace A is proximinal if and only if the image of the closed unit ball by the quotient operator with respect to A is closed* (see also [17], p.98).

Equivalently, *the image of the unit ball is just the unit ball of quotient space.*

In the case of convex bounded sets the associated sets H_d defined by (2.3) was considered by EDELSTEIN [11] to prove an elegant characterization of the best approximation elements.

Theorem 3 ([11], Lemma 1). *An element $x \in X$ has at least best approximation element in a closed, bounded and convex set A with $d(x; A) > 0$, if and only if $x \in H_d \backslash (int H_d)$, where $d = d(x; A)$.*

This result has already been used by EDELSTEIN and THOMPSON in [12] to obtain the well known characterization of antiproximinal sets.

Theorem 4 ([12]). *A closed bounded convex set A is antiproximinal if and only if the set H_d is open for any $d > 0$.*

Remark 5. This result is also true for any nonvoid set A ([24], [25]).

Now, we establish a similar characterization of antiremotal set where the closed ball is replaced by the complement of an open ball. Therefore, the set K_d has the role of the set H_d.

Theorem 5. *A bounded set A is antiremotal if and only if the set K_d is open for any $d > 0$.*

Proof. Let A be an antiremotal set and let us consider an arbitrary element $x \in K_d$, i.e. $x = u + v$ where $u \in A$ and $\|v\| \geq d$. Therefore $\Delta_A(x) > d$ and so there exist $\overline{u} \in A$ and $\varepsilon_0 > 0$ such that $\|x - \overline{u}\| = d + \varepsilon_0$. Thus, if $y \in S(x; \varepsilon_0)$ we have

$$\|y - \overline{u}\| \geq \|x - \overline{u}\| - \|y - x\| > d + \varepsilon_0 - \varepsilon_0 = d,$$

that is $y \in K_d$. Hence K_d is an open set.

Now, let us suppose by contradiction that the sets K_d are open for every $d > 0$ and there exists $x \in X$ and $\overline{x} \in A$ such that $\|x - \overline{x}\| = \Delta_A(x)$. Obviously, $x \in K_d$ for $d = \Delta_A(x)$ and so, there exists $\varepsilon_0 > 0$ such that $S(x; \varepsilon_0) \subset K_d$. Therefore, for every $y \in S(x; \varepsilon_0)$ there exists $u_y \in A$ such that $\|y - u_y\| \geq d$. Hence $\Delta_A(y) \geq d = \Delta_A(x)$ for all $y \in S(x; \varepsilon_0)$ which prove that the element x is a local minimum element of the function Δ_A. Since Δ_A is convex (an upper hull of a family of convex functions) it follows that x is also an absolute minimum element (see for example [5], [17]) and so, the functions Δ_A is constant on dom Q_A. On the other hand, if $y \in Q_A(x)$, then taking $x_\lambda = \lambda x + (1 - \lambda)y, \lambda > 1$, by (1.5), (1.6) we also have $y \in Q_A(x_\lambda)$ and $\Delta_A(x_\lambda) = \lambda \Delta_A(x)$. $\square$

Consequently, there exists elements $x_\lambda \in \text{dom} Q_A$ such that $\Delta_A(x_\lambda) > \Delta_A(x)$ which is a contradiction. Hence the set A is necessary antiremotal.

Remark 6. We remark that all results of this paper are conserved if we consider asymmetric norms.

The problem of existence of some sets with bounded complement which have different approximation properties was studied by many authors under various conditions on the norm of the liniar normed space X. Thus BALAGANSKI [3] proved that there are no antiproximinal sets with smooth complement in any reflexive Banach spaces. Other similar results can be found, for example, in [4], [6], [9], [13], [21], [29], [33].

Taking into account that the associated sets H_d and K_d have the properties of symmetry with respect to the sets A and $\overline{S}(0;1)$, respectively $cS(0;1)$ we obtain some properties of duality between proximinality (antiproximinality) and remotability (antiremotability).

Theorem 6 ([13])**.** *Let A be a closed bounded convex set such that $0 \in intA$. Then:*

(i) *A is proximinal (antiproximinal) if and only $\overline{S}(0;1)$ is proximinal (antiproximinal) with respct to p_A;*

(ii) *A is remotal (antiremotal) if and only if $cS(0;d)$ is proximinal (antiproximinal), for any $d > 0$, with respect to p_A,*

where p_A is the Minkowski functional associated to the set A.

Proof. (i) By hypothesis the Minkowski functional p_A is an equivalent norm in X, generally asymmetric and

$$\overline{S}_{p_A}(0;d) = dA, \quad S_{p_A}(0;1) = intA.$$

Therefore, according to Theorem 2, in the linear (asymmetric) normed space (X, p_A) the closed set $\overline{S}(0;1)$ is proximinal with respect to p_A if and only if $\overline{S}(0;1) + \overline{S}_{p_A}(0;d) = d(A + \overline{S}(0;1/d))$ is closed for all $d > 0$, i.e., A is proximinal in X. The other assertions can be proved using the corresponding above theorems.

Corollary 2. *If in a linear normed space X there exists a remotal (antiremotal) set, then X can be endowed with an equivalent norm, generally asymmetric, such that there exists a bounded, symmetric, convex body whose complement is proximinal (antiproximinal).*

Proof. By Theorem 6 the set $S(0;1)$ has the required properties.

Remark 7. The Theorems 3, 5 and 6 are also true in any asymmetric normed space.

Therefore, the four approximation properties: proximinality, antiproximinality, remotability and antiremotability, are dependent of the topological properties of a pair of sets (A, B) such that $A + B, A + cB$ is closed, respectively open. If A, B are convex sets then $A + B$ is also convex, while $A + cB$ is not convex being the complement of a convex set. Thus, the case $A + cB$ is little difficult. An important problem studied by many authors is to find some types of spaces which contain antiproximinal convex sets whose complements are bounded convex bodies, or equivalently are antiremotal with respect to an equivalent norm. Moreover, for example, Balaganskii proved that any linear normed space which does not satisfy the Radon-Nikodym condition contains bounded, closed symmetric antiremotal sets [4]. Moreover, he found an antiremotal set in $L_1[0;1]$ whose the closure of its complement is antiproximinal.

If A, B are bounded closed convex bodies the above four topological properties of the sets $A + B$ and $A + cB$ can be reformulated in terms of equivalent norms, generally asymmetric. Thus, a pair two equivalent norms $\|\cdot\|_1, \|\cdot\|_2$ in a linear space X is called *proximinal* (*antiproximinal*) if $\overline{S}_{\|\cdot\|_1}(0;1) + \overline{S}_{\|\cdot\|_2}(0;d)$ is closed (open) for every $d > 0$. Therefore, in a linear space any pair of equivalent complete norms is proximinal only in the reflexive case (see [18], p.322). Also, some sufficient conditions for proximinality can be obtained using the closedness criterium established by KA-SING LAU in [20]. The antiproximinal norms was studied by BORWEIN, JIMENEZ-SEVILLA and MORENO [8]. They obtained many existence results in some special spaces.

Similarly, two equivalent norms $\|\cdot\|_1, \|\cdot\|_2$ can be called *remotal-proximinal* (*antiremotal-antiproximinal*) if $\overline{S}_{\|\cdot\|_1}(0;1) + cS_{\|\cdot\|_2}(0;d)$ is closed (open) for every $d > 0$.

3. An equivalent best approximation problem associated to a farthest point problem

The property (ii) in Theorem 6 can be also presented in a pointwise form which, at the same time, establishes a relationship between two d.c.

optimization problems. Let us consider the farthest point problem

$$(P_x) \qquad \max\{\|x - y\|_1; y \in \overline{S}_{\|\cdot\|_2}(0; 1)\}, \ x \in X,$$

and the following associated best approximation problem

$$(D_x) \qquad \min\{\|x - \alpha y\|_2; y \in cS_{\|\cdot\|_1}(0; 1)\|\}, \ x \in X,$$

where $\|\cdot\|_1, \|\cdot\|_2$ are two equivalent norms in X and

$$(3.1) \qquad \alpha = \sup\{\|x - y\|_1; y \in \overline{S}_{\|\cdot\|_2}(0; 1)\}.$$

Theorem 7. *The problem (P_x) has an optimal solution if and only if the problem (D_x) has an optimal solution.*

Proof. Let $\overline{y}$ be an optimal solution of (P_x), i.e. $\alpha = \|x - \overline{y}\|_1$ and $\|\overline{y}\|_2 \leq 1$. Obviously, $\alpha > 0$ and $\|\overline{y}\|_2 = 1$. Taking $x - \overline{y} = \alpha\overline{z}$ we have $\|\overline{z}\|_1 = 1$ and $\|x - \alpha y\|_2 \geq 1$ for any $y \in cS_{\|\cdot\|_1}(0; 1)$. Indeed, in the contrary case it follows that there exists $y_1 \in X$ such that $\|y_1\|_1 \geq 1$ and $\|x - \alpha y_1\|_2 < 1$. Since $\|x - y\|_1 < \alpha$ for any $y \in S_{\|\cdot\|_2}(0; 1)$ (the solutions of (P_x) are boundary elements of $S_{\|\cdot\|_2}(0; 1)$) it follows $\alpha\|y_1\|_1 = \|x - (x - \alpha y_1)\|_1 < \alpha$, that is $\|y_1\| < 1$ which is a contradiction.

Conversely, if $\overline{z}$ is an optimal solution of (D_x) we denote $x - \alpha\overline{z} = \overline{y}$. But, necessarily it follows $\|\overline{z}\|_1 = 1$ and so $\|x - \overline{y}\|_1 = \alpha$. On the other hand, for every $\varepsilon > 0$ there exits $y_\varepsilon \in \overline{S}_{\|\cdot\|_2}(0; 1)$ such that $\|x - y_\varepsilon\|_1 > \alpha - \varepsilon$ and so

$$\|x - \frac{\alpha}{\alpha - \varepsilon}(x - y_\varepsilon)\|_2 \geq \|x - \alpha\overline{z}\|_2$$

which implies

$$\|y_\varepsilon - \frac{\varepsilon}{\alpha - \varepsilon}(x - y_\varepsilon)\|_2 \geq \|x - \alpha\overline{z}\|_2 = \|\overline{y}\|_2.$$

But

$$\|y_\varepsilon - \frac{\varepsilon}{\alpha - \varepsilon}(x - y_\varepsilon)\|_2 \leq \|y_\varepsilon\|_2 + \frac{\varepsilon}{\alpha - \varepsilon}\|x - y_\varepsilon\|_2 \leq 1 + \frac{\varepsilon}{\alpha - \varepsilon}(\|x\|_2 + 1).$$

Therefore

$$\|\overline{y}\|_2 \leq 1 + \frac{\varepsilon}{\alpha - \varepsilon}(1 + \|x\|_2)$$

for any $\varepsilon > 0$. Consequently, for $\varepsilon \searrow 0$ we obtain $\|\overline{y}\|_2 \leq 1$. Since $\|x - \overline{y}\|_1 = \alpha$ it follows that $\overline{y}$ is an optimal solution of (P_x). $\qquad\square$

Remark 8. In fact we have

$$val(P_x) = val(D_x) = \alpha$$

and booth problems are d.c. optimization problems of type P_2, respectively P_3, according to classification of HIRTART-URRUTY [15].

Remark 9. According to Remark 3, we obtain that Theorem 7 can be reformulated as follows: if $\overline{S}_{\|\cdot\|_2}(0;1)$ is d-remotal with respect to $\|\cdot\|_1$, then $cS_{\|\cdot\|_1}(0;d)$ is d-proximinal for $\|\cdot\|_2$ (see also (2.1) in this special case).

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