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NEW SHARP BOUNDS FOR GAMMA AND DIGAMMA FUNCTIONS

BY

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Abstract. Motivated by Sandor and Debnath, Batir, we prove that a function involving gamma function is completely monotonic. As applications, we establish new upper and lower bounds for the gamma and digamma functions, with sharp constants.

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1. Introduction

We discuss here the approximations of the factorial function of the form

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \le n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}, \tag{1.1}$$

where α, β are real parameters. The bounds (1.1) were stated in Sandor and Debnath [9] with $\alpha = 0$ and $\beta = 1$. Their result was rediscovered by Guo [5]. Very recently, Batir [3] determined the largest number $\alpha = 1 - 2\pi e^{-2}$ and the smallest number $\beta = 1/6$ such that the inequalities (1.1) hold for all $n = 1, 2, 3, \ldots$.

Numerical computations made in [3] show that the upper approximation

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} \tag{1.2}$$

is better than the other lower approximation from (1.1) (with $\alpha = 1-2\pi e^{-2}$) and also it is more accurate than other known formulas as Stirling's formula, or Burnside's formula. These facts entitled us to consider the following function associated with the approximation (1.2):

$$f(x) = \ln \Gamma(x+1) - (x+1)\ln x + x - \ln \sqrt{2\pi} + \frac{1}{2}\ln\left(x - \frac{1}{6}\right).$$

We prove that $-f$ is strictly completely monotonic and as a direct consequence, we establish new double inequalities for $x \geq 1$:

$$\omega \cdot \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x-1/6}} \leq \Gamma(x+1) < \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x-1/6}},$$

where $\omega = e\sqrt{\frac{5}{12\pi}} = 0.989\,95\ldots$ is best possible. Moreover, the following double inequality for $x \geq 1$ is established $\frac{1}{x} - \frac{1}{2\left(x-\frac{1}{6}\right)} < \psi(x) - \left(\ln x - \frac{1}{x}\right) < \frac{1}{x} - \frac{1}{2\left(x-\frac{1}{6}\right)} + \zeta$, where $\zeta = -\gamma + \frac{3}{5} = 0.022785\ldots$ ($\gamma = 0.577215\ldots$ is the Euler constant).

2. The results

The gamma Γ and digamma ψ functions are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad , \quad \psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for all complex numbers x with $\operatorname{Re} x > 0$, but here we restrict them to positive real numbers x. We also have $\psi(x+1) = \psi(x) + \frac{1}{x}$, for all $x > 0$. The gamma function is an extension of the factorial function, since $\Gamma(n+1) = n!$, for $n = 0, 1, 2, 3\ldots$. The derivatives ψ', ψ'', ..., known as polygamma functions, have the following integral representations:

$$\psi^{(n)}(x) = (-1)^{n-1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt \tag{2.1}$$

for $n = 1, 2, 3, \ldots$. For proofs and other details, see for example, [2]. We also use the following integral representation

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt \ , \quad n \geq 1. \tag{2.2}$$

Recall that a function f is completely monotonic in an interval I if f has derivatives of all orders in I such that $(-1)^n f^{(n)}(x) \geq 0$, for all $x \in I$

and $n = 0, 1, 2, 3 \ldots$. If this inequality is strict for all $x \in I$ and all non-negative integers n, then f is said to be strictly completely monotonic. Completely monotonic functions involving $\ln \Gamma(x)$ are important because they produce bounds for the polygamma functions. A consequence of the famous Hausdorff-Bernstein-Widder theorem states that f is completely monotonic on $[0, \infty)$ if and only if$f(x) = \int_0^\infty e^{-xt}\varphi(t)\,dt$, where φ is a non-negative function on $[0, \infty)$ such that the integral converges for all $x > 0$, see [10, p. 161].

Lemma 2.1. *For the sequence* $x_n = \frac{1}{2}(\frac{7^{n-1}-1}{6^{n-1}}) + \frac{1}{n} - 1$ *we have* $x_n > 0$, *for all* $n \geq 4$.

Proof. First note that $x_4 = \frac{1}{24}$ and $x_5 = \frac{17}{135}$, so we are concentrated to show that $x_n > 0$, for all $n \geq 6$.

The function $g(x) = (7^x - 1)/6^x$ is strictly increasing, since $g'(x) = \frac{1}{6^x}\left(\ln 6 + 7^x \ln \frac{7}{6}\right) > 0$. Then for all $n \geq 6$, we have $x_n > \frac{1}{2}(\frac{7^{n-1}-1}{6^{n-1}}) - 1 \geq \frac{1}{2}(\frac{7^5-1}{6^5}) - 1 > 0$ and the conclusion follows. □

Now we are in position to prove the following

Theorem 2.1. *Let* $f : (1/6, \infty) \to \mathbb{R}$, *given by* $f(x) = \ln \Gamma(x+1) - (x+1)\ln x + x - \ln\sqrt{2\pi} + \frac{1}{2}\ln\left(x - \frac{1}{6}\right)$. *Then* $-f$ *is strictly completely monotonic.*

Proof. We have $f'(x) = \psi(x) - \ln x + \frac{1}{2\left(x-\frac{1}{6}\right)}$ and $f''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2\left(x-\frac{1}{6}\right)^2}$. Using the representations (2.1)-(2.2), we obtain $f''(x) = \int_0^\infty \frac{e^{-xt}}{e^t-1}\varphi(t)\,dt$, where $\varphi(t) = te^t - (e^t - 1) - \frac{1}{2}t(e^{\frac{7}{6}t} - e^{\frac{1}{6}t})$, or $\varphi(t) = -\sum_{n=4}^\infty \frac{x_n}{(n-1)!}t^n$, where $(x_n)_{n\geq 4}$ is defined in Lemma 2.1. According to Lemma 2.1, we have $\varphi < 0$ and then, $-f''$ is strictly completely monotonic.

Now, f' is strictly decreasing, since $f'' < 0$. But we have $\lim_{x\to\infty} f'(x) = 0$, so $f'(x) > 0$ and consequently, f is strictly increasing. Using the fact that $\lim_{x\to\infty} f(x) = 0$, we deduce that $f < 0$. Finally, $-f$ is strictly completely monotonic. □

As a direct consequence of the fact that f is strictly increasing, we have $f(1) \leq f(x) < \lim_{x\to\infty} f(x) = 0$, for all $x \geq 1$. As $f(1) = 1 + \ln\sqrt{\frac{5}{12\pi}}$, we derive

$$\omega \cdot \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x - 1/6}} \leq \Gamma(x+1) < \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x - 1/6}},$$

where $\omega = e\sqrt{\frac{5}{12\pi}} = 0.98995\ldots$ is best possible.

Using the fact that f' is strictly decreasing, we have $\lim_{x\to\infty} f'(x) = 0 < f'(x) \leq f'(1)$, for all $x \geq 1$. As we have $f'(1) = -\gamma + \frac{3}{5} = 0.022785\ldots$, we obtain $-\frac{1}{2\left(x-\frac{1}{6}\right)} < \psi(x) - \ln x < -\frac{1}{2\left(x-\frac{1}{6}\right)} + \zeta$, with best possible constant $\zeta = -\gamma + \frac{3}{5} = 0.022785\ldots$, which improve other results of the form $\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}$, $x > 1$, see [1, 4, 6, 7, 8].

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