ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVII, 2011, Supliment
DOI: 10.2478/v10157-011-0003-6

ULAM-HYERS STABILITY FOR OPERATORIAL EQUATIONS

BY

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Abstract. Using the weakly Picard operator technique, we present some Ulam-Hyers stability results for coincidence point problems for multivalued operators.

Mathematics Subject Classification 2000: 47H10, 54H25, 54C60.

Key words: Ulam-Hyers stability, generalized Ulam-Hyers stability, multivalued operator, weakly Picard operator, c-weakly Picard operator, fixed point, coincidence point.

1. Introduction

Let (X,d) be a metric space and consider the following families of subsets of X:

$$P(X):=\{Y\in\mathcal{P}(X)|\ Y\neq\emptyset\},\ P_b(X):=\{Y\in P(X)|\ Y \text{ is bounded}\},$$
$$P_{cl}(X):=\{Y\in P(X)|\ Y \text{ is closed}\},\ P_{cp}(X):=\{Y\in P(X)|\ Y \text{ is compact}\}.$$

If (X,d) is a metric space, then the gap functional in $P(X)$ is defined as

$$D_d:P(X)\times P(X)\to\mathbb{R}_+,\ D_d(A,B)=\inf\{d(a,b)\mid a\in A,\ b\in B\}.$$

In particular, if $x_0\in X$, we put $D_d(x_0,B)$ in place of $D_d(\{x_0\},B)$.

We will denote by H the generalized Pompeiu-Hausdorff functional on $P(X)$, defined as

$$H_d:P(X)\times P(X)\to\mathbb{R}_+\cup\{+\infty\},$$
$$H_d(A,B)=\max\left\{\sup_{a\in A}D_d(a,B),\sup_{b\in B}D_d(b,A)\right\}.$$

Let (X, d) be a metric space. If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set $Fix(F) := \{x \in X|\ x \in F(x)\}$ is called the fixed point set of F.

Let Y be a nonempty set and $T, S : X \to P(Y)$ be two multivalued operators. An element $x^* \in X$ is a coincidence point for T and S if $T(x^*) \cap S(x^*) \neq \emptyset$. We denote by $C(T, S)$ the set of all coincidence points for T and S.

For a multivalued operator $F : X \to P(Y)$ we will denote by

$$\text{Graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

the graph of F and by

$$F^{-1} : Y \to \mathcal{P}(X),\ F^{-1}(y) := \{x \in X : y \in F(x)\}$$

the inverse operator of F. We say that $f : X \to Y$ is a selection for $F : X \to P(Y)$ if $f(x) \in F(x)$, for each $x \in X$. Also, $F : X \to P(Y)$ is said to be onto if and only if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$.

In particular, when F (or T and S) is a singlevalued operator, we obtain the similar well-known concepts in fixed point theory and coincidence point theory, see [2], [4], [7], [14], [16].

For the following notions see Rus [13], Rus-Petruşel-Sîntămărian [18], Petruşel [12] and Rus-Petruşel-Petruşel [17]. See also [1], [5], [6], [8], [10], [19], [20].

Definition 1.1. Let (X, d) be a metric space, and $F : X \to P_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard (briefly MWP) operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Remark 1.2. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition (i) and (ii), in the Definition 1.1 is called a sequence of successive approximations of F starting from $(x, y) \in \text{Graph}(F)$.

If $F : X \to P(X)$ is a MWP operator, then we define $F^\infty : \text{Graph}(F) \to P(\text{Fix}\, F)$ by the formula $F^\infty(x, y) := \{ z \in \text{Fix}(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z \}$.

Definition 1.3. Let (X, d) be a metric space and $F : X \to P(X)$ be a MWP operator. Then, F is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq c\, d(x, y), \text{ for all } (x, y) \in \text{Graph}(F).$$

For the theory of multivalued weakly Picard operators see [12].

The purpose of this paper is to present some Ulam-Hyers stability results for coincidence point equation and inclusion. The approach is based on the weakly Picard operator technique. Our results are connected to some recent papers of Castro-Ramos [3], Jung [9] and Rus [15], [16] (where integral and differential equations are considered), Rus [13] and Petru-Petruşel-Yao [11] (where the Ulam-Hyers stability of the fixed point problem are discussed).

2. Ulam-Hyers stability for fixed point problem with multivalued operators

We start this section by presenting the Ulam-Hyers stability concepts for the fixed point problem associated to a multivalued operator.

Definition 2.1. Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. Then $F : X \to P(X)$ is said to be a multivalued ψ-weakly Picard operator if it is a multivalued weakly Picard operator and there exists a selection $f^\infty : \text{Graph}(F) \to Fix(F)$ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in \text{Graph}(F).$$

If there exists $c > 0$ such that $\psi(t) = ct$, for each $t \in \mathbb{R}_+$, then we obtain the notion of multivalued c-weakly Picard operator given above.

Definition 2.2. Let (X, d) be a metric space and $F : X \to P(X)$ be a multivalued operator. The fixed point inclusion

$$x \in F(x),\ x \in X \tag{2.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$D_d(y, F(y)) \leq \varepsilon \tag{2.2}$$

there exists a solution x^* of the fixed point inclusion (2.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (2.1) is said to be Ulam-Hyers stable.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the fixed point inclusion (2.1) for multivalued operators with compact values. For the sake of completeness we sketch the proof of this result.

Theorem 2.3 (Rus [13])**.** *Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be a multivalued ψ-weakly Picard operator. Then, the fixed point inclusion* (2.1) *is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y^* \in X$ be a solution of (2.2), i.e., $D_d(y^*, F(y^*)) \leq \varepsilon$. Let $u^* \in F(y^*)$ such that $d(y^*, u^*) = D_d(y^*, F(y^*))$. Since F is a multivalued ψ-weakly Picard operator, for each $(x, y) \in \text{Graph}(F)$ we have

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)).$$

Hence, taking into account that $(y^*, u^*) \in \text{Graph}(F)$, we can choose $x^* := f^\infty(y^*, u^*)$ and thus we get that x^* is a solution of the fixed point inclusion (2.1) and $d(y^*, x^*) = d(y^*, f^\infty(y^*, u^*)) \leq \psi(d(y^*, u^*)) \leq \psi(\varepsilon)$. □

3. Ulam-Hyers stability for coincidence point problem with multivalued operators

Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P(Y)$ be two multivalued operators. Let us consider now the following coincidence point problem with multivalued operators.

$$T(x) \cap S(x) \neq \emptyset. \tag{3.3}$$

Remark 3.1. Let (X, d) and $A, B \in P_{cl}(X)$. Then:

i) if $A \cap B \neq \emptyset$, then $D_d(A, B) = 0$;

ii) if A (or B) is compact, such that $D_d(A, B) = 0$, then $A \cap B \neq \emptyset$.

Definition 3.2. Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued operators. The coincidence problem (3.3) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution u^* of the inequation

$$D_\rho(T(u), S(u)) \leq \varepsilon \tag{3.4}$$

there exists a solution x^* of (3.3) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the coincidence point equation (3.3) is said to be Ulam-Hyers stable.

The following concept is important for our further considerations.

Definition 3.3. Let (X, d) and (Y, ρ) be two metric spaces. Then, the operators $T, S : X \to P_{cl}(Y)$ form a ψ-weakly Picard pair of multivalued operators, denoted by $[T, S]$, if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi(0) = 0$ and there exists a multivalued operator $F : X \to P(X)$ such that:

(i) F is a multivalued ψ-weakly Picard operator;

(ii) $Fix(F) = C(T, S)$;

(iii) for each $x \in X$ there exists $y \in F(x)$ such that $d(x, y) \leq D_\rho(T(x), S(x))$.

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the operators $T, S : X \to P_{cl}(Y)$ form a c-weakly Picard pair of multivalued operators.

A result on the stability of a coincidence point problem is the following.

Theorem 3.4. *Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued operators such that $[T, S]$ forms a ψ-weakly Picard pair of multivalued operators (respectively a c-weakly Picard pair of multivalued operators). Then the coincidence point problem* (3.3) *is generalized Ulam-Hyers stable (respectively Ulam-Hyers stable).*

Proof. Let $\varepsilon > 0$ and $u^* \in X$ be a solution of (3.4), i.e., $D_\rho(T(u^*), S(u^*)) \leq \varepsilon$. Then, by (iii), for $u^* \in X$ there exists $y^* \in F(u^*)$ such that

$$d(u^*, y^*) \leq D_\rho(T(u^*), S(u^*)).$$

Since F is a ψ-multivalued Picard operator, we get that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)) \text{ for each } (x, y) \in \text{Graph}(F).$$

If we consider $x^* := f^\infty(u^*, y^*)$, then, by (ii) and (iii) in Definition 3.3, we obtain that $x^* \in C(T, S)$ and

$$d(u^*, x^*) = d(u^*, f^\infty(u^*, y^*)) \leq \psi(d(u^*, y^*)) \leq \psi(D_\rho(T(u^*), S(u^*))) \leq \psi(\varepsilon).$$

□

We will present now a consequence of the above abstract result.
The following lemma is quite obvious.

Lemma 3.5. *Let X, Y be two nonempty sets and let $T : X \to P(Y)$ and $S : X \to P(Y)$ be two multivalued operators. Suppose that T (respectively S) is onto. Then $C(T, S) = Fix(F)$, where $F := T^{-1} \circ S$ (respectively $F := S^{-1} \circ T$).*

By Lemma 3.5 and the above theorem we get the following consequences.

Theorem 3.6. *Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued operators such that:*

(i) T is onto;

(ii) $T^{-1} \circ S$ is a multivalued a-contraction with compact values;

(iii) for each $x \in X$ there exists $y \in X$ such that $T(y) \cap S(x) \neq \emptyset$ and $d(x, y) \leq D_\rho(T(x), S(x))$.

Then the coincidence point problem (3.3) is Ulam-Hyers stable.

Proof. By (i) and (ii) we get that $F := T^{-1} \circ S$ is a multivalued c-weakly Picard operator with $c := \frac{1}{1-a}$. By (iii) it follows that the condition (iii) in Definition 3.3 holds. The conclusion follows now by Theorem 3.4.□

Example 3.7. Let $T, S : [0,1] \to P([0,3])$ be given by

$$T(x) = [2x, 3x],\ S(x) = [0, \frac{x}{3}],\ \text{for each } x \in [0,1],$$

then T is onto, $T^{-1} \circ S : [0,1] \to P([0,1])$ given by $(T^{-1} \circ S)(x) = [0, \frac{x}{6}]$ is $\frac{1}{6}$-contraction and $C(T,S) = \{0\}$. Notice also that for each $x \in [0,1]$ there exists $y \in [0,1]$ such that $T(y) \cap S(x) \neq \emptyset$ and $|x - y| \leq D_{|\cdot|}(T(x), S(x))$. Thus, in this case, the coincidence point problem (3.3) is Ulam-Hyers stable.

Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P(Y)$ be two multivalued operators. Let us consider now the following coincidence point equation with multivalued operators:

$$T(x) = S(x), \quad x \in X. \tag{3.5}$$

We need two more concepts. The former of these is similar to the one of Definition 3.3.

Definition 3.8. Let (X, d) and (Y, ρ) be two metric spaces. Then, the operators $T, S : X \to P(Y)$ form a ψ-Picard pair of multivalued operators, denoted by $|T, S|$, if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi(0) = 0$ and there exists an operator $f : X \to X$ such that:

(i) f is weakly Picard operator;

(ii) $Fix(f) = C(T, S)$;

(iii) $d(x, f^\infty(x)) \leq \psi(H_\rho(T(x), S(x)))$, for each $x \in X$.

Definition 3.9. Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued operators. The coincidence point equation (3.5) is generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution u^* of the inequality

$$H_\rho(T(u), S(u)) \leq \varepsilon \tag{3.6}$$

there exists a solution x^* of (3.5) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the coincidence point equation (3.5) is said to be Ulam-Hyers stable.

Theorem 3.10. *Let (X, d) and (Y, ρ) be two metric spaces and $T, S : X \to P_{cl}(Y)$ be two multivalued operators such that $|T, S|$ forms a ψ-Picard pair of multivalued operators. Then the coincidence point equation* (3.5) *is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $u^* \in X$ be a solution of (3.6), i.e., $H_\rho(T(u^*), S(u^*)) \leq \varepsilon$. By the fact that $|T, S|$ forms a ψ-Picard pair, we have that $x^* := f^\infty(u^*) \in C(T, S)$ and

$$d(u^*, f^\infty(u^*)) \leq \psi(H_\rho(T(u^*), S(u^*))).$$

Thus,

$$d(u^*, x^*) \leq \psi(H_\rho(T(u^*), S(u^*))) \leq \psi(\varepsilon).$$

□

Example 3.11. Let $T, S : [0, 1] \to P([0, \frac{1}{2}])$ be given by $T(x) = [0, \frac{x}{2}]$ and

$$S(x) = \begin{cases} [0, \frac{x}{2}], & \text{for } x \in [0, \frac{1}{2}] \\ [\frac{-x}{2} + \frac{1}{2}, \frac{x}{2}], & \text{for } x \in]\frac{1}{2}, 1]. \end{cases}$$

Then, $|T, S|$ forms a ψ-Picard pair of multivalued operators (with $\psi(t) = 2t,\ t \in \mathbb{R}_+$), $C(T, S) = [0, \frac{1}{2}]$ and hence the coincidence point equation (3.5) is Ulam-Hyers stable.

Notice, that here $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{2}[\\ \frac{x}{2} + \frac{1}{4}, & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

is a weakly Picard operator with $Fix(f) = C(T, S)$.

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Received: 18.X.2010
Revised: 19.01.2011

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