

NONEXPANSIVE FIXED POINT TECHNIQUE USED TO SOLVE BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

BY

MONICA LAURAN and VASILE BERINDE

Abstract. In this paper, by using the technique of nonexpansive operators we shall establish sufficient conditions for the existence of solutions for a class of boundary value problems for fractional differential equations involving the Caputo fractional derivative and nonlinear integral boundary value conditions.

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Key words: boundary value problem, fractional differential equation, Caputo fractional derivative, nonexpansive operator, fixed point.

1. Introduction

This paper deals with the existence of solutions of boundary value problems (BVP for short), for fractional order differential equations with nonlinear integral conditions of the form:

$$(1) \quad {}^c D^\alpha y(t) = f(t, y), \text{ for each } t \in J = [0, T], \quad 1 < \alpha \leq 2;$$

$$(2) \quad y(0) = \int_0^T g(s, y) ds;$$

$$(3) \quad y(T) = \int_0^T h(s, y) ds,$$

where ${}^c D^\alpha$ is the Caputo fractional derivative, and $f, g, h : J \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science

and engineering. Numerous applications in viscoelasticity, electrochemistry, electromagnetic etc., can be found in the papers [8], [9], [10].

Using the technique of Picard operators, BENCHOHRA and HAMANI [3], established the following existence theorem for the BVP (1)-(3).

Theorem 1.1 ([3]). *Assume:*

(H₁) *There exists a constant $k > 0$ such that*

$$|f(t, u) - f(t, \bar{u})| \leq k \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₂) *There exists a constant $k^* > 0$ such that*

$$|g(t, u) - g(t, \bar{u})| \leq k^* \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₃) *There exists a constant $k^{**} > 0$ such that*

$$|h(t, u) - h(t, \bar{u})| \leq k^{**} \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₄) *If*

$$\frac{2kT}{\Gamma(\alpha + 1)} + T(k^* + k^{**}) < 1$$

then the BVP (1)-(3) has at least one solution on $[0, T]$.

The proof of this result in [3] is essentially based on Banach's contraction principle. Starting from Theorem 1.1, the main aim of this paper is to obtain a more general result, by using the technique of nonexpansive mappings, see [1], instead of the Banach fixed point theorem.

The same technique has been applied in [1] and also used in the paper [10]. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems, see [5], [6].

They include two, three, multipoint and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamics and cellular systems [9] and [10].

2. Preliminaries

In this section we introduce notations, definitions and preliminary facts which will be used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J to \mathbb{R} with the norm:

$$\|y\|_{\infty} = \sup \{|y(t)| : t \in J\}.$$

Definition 2.1 ([3]). For a function h given on the interval $[a, b]$, the *fractional order integral* of $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by:

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot h(s) ds,$$

where Γ is the gamma function, $h \in L^1([a, b], \mathbb{R}_+)$ and $L^1([a, b], \mathbb{R}_+)$ is space of integrable functions $h : [a, b] \rightarrow \mathbb{R}_+$. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$ where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$.

Definition 2.2 ([3]). For a function h given on the interval $[a, b]$, the α -th *Riemann-Liouville fractional-order derivative* of h is defined by

$$\left(D_{a+}^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds.$$

Here $n = [\alpha] + 1$, $[\alpha]$ denote the integer part of α .

Definition 2.3 ([3]). For a function h given on the interval $[a, b]$, the *Caputo fractional-order derivative* of h is defined by:

$$\left({}^c D_{a+}^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \cdot h^{(n)}(s) ds.$$

Here $n = [\alpha] + 1$.

Remark 2.1. For $\alpha = 0$ we have $n = 1$ and then

$$\left({}^c D_{a+}^0 h\right)(t) = \frac{1}{\Gamma(1)} \left(\frac{d}{dt}\right) \int_a^t h'(s) ds.$$

Definition 2.4 ([2]). Let K be a nonempty subset of a real normed linear space E and $T : K \rightarrow K$ be a map. A point $x \in K$ is called a fixed point of T if $Tx = x$. In this setting, T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for each } x, y \in K.$$

Nonexpansive mappings, although are generalizations of contractions do not inherit more from contraction mappings.

We can now formulate one of the most important fixed point theorems for nonexpansive mappings, due to Browder, Ghode and Kirk, see for example [1], [12] and Schauder's fixed point theorem [4].

Theorem 2.1 (Browder-Ghode-Kirk, [1]). *Let K be a nonempty closed convex and bounded subset of a uniformly Banach space E . Then any non-expansive mapping $T : K \rightarrow K$ has at least a fixed point.*

Theorem 2.2 (Schauder, [4]). *Let K be a nonempty compact and convex subset of normed space E . Then any continuous mapping $T : K \rightarrow K$ has at least a fixed point.*

Remark 2.2. As showed by Theorem 2.1, no information on the approximation of the fixed point of T is given. The most usual methods for approximation the fixed point will be defined in the following in view of their use.

Let K be a convex subset of a normed linear space E and let $T : K \rightarrow K$ be a self mapping. For $x_0 \in K$ and $\lambda \in [0, 1]$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, \dots$$

is usually called *Krasnoselskij iteration*.

For $x_0 \in K$ the sequence $\{x_n\}$ defined by:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n, \quad n = 0, 1, 2, \dots \text{ where } \{\lambda_n\} \subset [0, 1]$$

is called *Mann iteration*.

It was shown by KRASNOSELSKIJ in [7], for $\lambda = 1/2$ and by SCHAEFER [11], for an arbitrary $\lambda \in (0, 1)$, that if E is a uniformly convex Banach space and K is a nonempty, convex and compact subset of E , then the Krasnoselskij iteration converges to a fixed point of T .

Theorem 2.3 ([1]). *Let K be a subset of a Banach space E and let $T : K \rightarrow K$ be a nonexpansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process $\{x_n\}$ under the following assumptions:*

- (a) $x_n \in K$ for all positive integers;
- (b) $0 \leq \lambda_n \leq b < 1$ for all positive integers;
- (c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\{x_n\}$ is bounded, then $x - Tx_n \rightarrow 0$, as $n \rightarrow \infty$.

Corollary 2.1 ([1]). *Let K be a convex and compact subset of Banach space E and let $T : K \rightarrow K$ be a nonexpansive mapping. If the Mann iteration process $\{x_n\}$ satisfies assumptions (a)-(c) in Theorem 2.2, then $\{x_n\}$ converge strongly to a fixed point of T .*

Corollary 2.2 ([1]). *Let E be a real normed space, K a closed bounded convex subset of E and let $T : K \rightarrow K$ be a nonexpansive mapping. If $I - T$ maps closed bounded subset of E into closed subset of E and $\{x_n\}$ is the Mann iteration, with $\{\lambda_n\}$ satisfying assumptions (a)-(c) in Theorem 2.2, then $\{x_n\}$ converges strongly to a fixed point of T in K .*

3. Existence of solutions

By a solution of the BVP (1)-(3) we mean a function $y \in C^2(J, \mathbb{R})$ which satisfies equations (1)-(3).

For the existence of solutions for the problem (1)-(3), we need the following auxiliary lemmas.

Lemma 3.1 ([3]). *Let $\alpha > 0$, then*

$$I^{\alpha c} D^{\alpha} h(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

As a consequence of Lemma 3.1., BENCHOHRA and HAMANI [3] have obtained the following result.

Lemma 3.2 ([1]). *Let $1 < \alpha \leq 2$ and let $\sigma, \rho_1, \rho_2 : J \rightarrow \mathbb{R}$ be continuous functions. A function y is a solution of the fractional integral equations*

$$(4) \quad y(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^t (t-s)^{\alpha-1} \sigma(s) ds - \frac{t}{T \cdot \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \sigma(s) ds \\ - \left(\frac{t}{T} - 1 \right) \int_0^T \rho_1(s) ds + \frac{t}{T} \int_0^T \rho_2(s) ds$$

if and only if y is a solution of the following fractional BVP.

$$(5) \quad {}^c D^{\alpha} y(t) = \sigma(t), \quad t \in J$$

$$(6) \quad y(0) = \int_0^T \rho_1(s) ds$$

$$(7) \quad y(T) = \int_0^T \rho_2(s) ds$$

The following result, established in [3], is based on the Banach contraction mapping principle.

Theorem 3.1 ([3]). *Assume that*

(H₁) *There exists a constant $k > 0$ such that*

$$|f(t, u) - f(t, \bar{u})| \leq k \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₂) *There exists a constant $k^* > 0$ such that*

$$|g(t, u) - g(t, \bar{u})| \leq k^* \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₃) *There exists a constant $k^{**} > 0$ such that*

$$|h(t, u) - h(t, \bar{u})| \leq k^{**} \cdot |u - \bar{u}|, \forall t \in [0, T], u, \bar{u} \in \mathbb{R}.$$

(H₄) *If*

$$\frac{2kT}{\Gamma(\alpha + 1)} + T(k^* + k^{**}) < 1$$

then the BVP (1)-(3) has at least one solution on $C([0, T])$.

Starting from Theorem 3.1, the main aim of this paper is to obtain a more general result, by using the technique of nonexpansive mappings, see [1], instead of the Banach fixed point theorem.

The same technique has been used previous in [1], [9].

For a constant $L > 0$, denote

$$(*) \quad C_L = \{y \in C(J, J) : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|, \forall t_1, t_2 \in J\}.$$

As a consequence of Arzela-Ascoli theorem, the set C_L is a nonempty convex and compact subset of the Banach space $(C[a, b], \|\cdot\|)$, where $\|\cdot\|$ is the usual sup norm.

The main result of this paper is given by the next theorem. This theorem extends Theorem 3.1, by weakening (H₄) to condition (H₈).

Theorem 3.2. *Assume that for BVP (1)-(3), the hypotheses (H₁)-(H₃) are satisfied and:*

(H₄') $f, g, h \in C(J \times \mathbb{R})$;

(H₅) *If L is the Lipschitz constant involved in (*), let*

$$M = \max\{|f(t, u)| : (t, u) \in J \times \mathbb{R}\} \leq L,$$

$$N = \max\{|g(t, u)| : (t, u) \in J \times \mathbb{R}\} \leq L,$$

$$Q = \max\{|h(t, u)| : (t, u) \in J \times \mathbb{R}\} \leq L$$

$$(H_6) \frac{M \cdot T^{\alpha-1}}{\Gamma(\alpha+1)} + N + Q \leq L;$$

$$(H_7) \frac{2 \cdot T^{\alpha-1}}{\Gamma(\alpha+1)} + 1 \leq \frac{1}{L};$$

$$(H_8) \frac{2 \cdot k \cdot T^\alpha}{\Gamma(\alpha+1)} + T(k^* + k^{**}) \leq 1.$$

Then there exists at least one solution $y^* \in C_L$ of the BVP (1)-(3).

Proof. Consider the integral operator $F : C_L \rightarrow C(J, \mathbb{R})$

$$\begin{aligned} (Fy)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot f(s, y(s)) ds \\ &\quad - \frac{t}{T \cdot \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds \\ &\quad - \left(\frac{t}{T} - 1 \right) \int_0^T g(s, y(s)) ds + \frac{t}{T} \int_0^T h(s, y(s)) ds \end{aligned}$$

it is clear that $y \in C_L$ is a solution of problem (1)-(3) if and only if y is a fixed point of F , that is $y = Fy$.

We first prove that C_L is an invariant set with respect to F , i.e., we have $F(C_L) \subset C_L$.

$$\begin{aligned} 0 \leq |(Fy)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \cdot |f(s, y(s))| ds \\ &\quad + \frac{t}{T \cdot \Gamma(\alpha)} \int_0^T |T-s|^{\alpha-1} |f(s, y(s))| ds \\ &\quad + \left(\frac{t}{T} - 1 \right) \int_0^T |g(s, y(s))| ds + \frac{t}{T} \int_0^T |h(s, y(s))| ds \\ &\leq \frac{-1}{\Gamma(\alpha)} M \frac{(t-s)^\alpha}{\alpha} \Big|_0^t + \frac{tM}{T\Gamma(\alpha)} \cdot (-1) \cdot \frac{(T-s)^\alpha}{\alpha} \Big|_0^T \\ &\quad + \left(\frac{t}{T} - 1 \right) NT + \frac{t}{T} QT = \frac{Mt^\alpha}{\Gamma(\alpha-1)} + \frac{tMT^\alpha}{T\Gamma(\alpha+1)} \\ &\quad + (t-T) \cdot N + t \cdot Q = \frac{M}{\Gamma(\alpha+1)} (t^\alpha + t \cdot T^{\alpha-1}) + t(N+Q) - TN \\ &\leq \frac{L}{\Gamma(\alpha+1)} (t^\alpha + t \cdot T^{\alpha-1}) + T \cdot L \leq T \cdot L \left(\frac{2T^{\alpha-1}}{\Gamma(\alpha+1)} + 1 \right) \leq T. \end{aligned}$$

So, $Fy \in [0, T]$, $\forall y \in C_L$. Now, for $t_1, t_2 \in [0, T]$ we have

$$|(Fy)(t_1) - (Fy)(t_2)| \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |f(s, y(s))| \cdot [|t_1 - s|^{\alpha-1} + |t_2 - s|^{\alpha-1}] ds$$

$$\begin{aligned}
& + \frac{|t_1 - t_2|}{T \Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} |f(s, y(s))| ds \\
& + \frac{|t_1 - t_2|}{T} \int_0^T |g(s, y(s))| ds + \frac{|t_1 - t_2|}{T} \int_0^T |h(s, y(s))| ds \\
& \leq \frac{|t_1 - t_2| \cdot M}{\Gamma(\alpha + 1)} \cdot T^{\alpha-1} + |t_1 - t_2| \cdot (N + Q) \\
& = \left(\frac{M \cdot T^{\alpha-1}}{\Gamma(\alpha + 1)} + N + Q \right) \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2|.
\end{aligned}$$

So, $Fy \in C_L, \forall y \in C_L$.

We consider $x, y \in C_L, t \in [0, T]$

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \cdot |f(s, x(s)) - f(s, y(s))| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} \cdot |f(s, x(s)) - f(s, y(s))| ds \\
& + \int_0^T |g(s, x(s)) - g(s, y(s))| ds \\
& + \int_0^T |h(s, x(s)) - h(s, y(s))| ds \leq \frac{t^\alpha k}{\Gamma(\alpha + 1)} |x(s) - y(s)| \\
& + \frac{T^\alpha k}{\Gamma(\alpha + 1)} |x(s) - y(s)| + T(k^* + k^{**}) \cdot |x(s) - y(s)| \\
& \leq \left[\frac{2kT^\alpha}{\Gamma(\alpha + 1)} + T(k^* + k^{**}) \right] \cdot |x(s) - y(s)|.
\end{aligned}$$

Now, by letting suppreum in the last inequality, we get

$$\|Fx - Fy\|_\infty \leq \left[\frac{2kT^\alpha}{\Gamma(\alpha + 1)} + T(k^* + k^{**}) \right] \cdot \|x - y\|_\infty$$

which, in view of conditions (H₈), proves that F is a nonexpansive operator, hence continuous.

Now apply the Schauder's fixed point theorem, we obtain the conclusion. \square

Under the assumption of Theorem 3.5, see [3], it is known that the solution y^* of BVP (1)-(3) can be approximated by means of the Picard

iterations $\{y_n\}$ defined by $y_1 \in C_L$ and

$$\begin{aligned} y_{n+1}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot f(s, y_n(s)) ds \\ &\quad - \frac{t}{T \cdot \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \cdot f(s, y_n(s)) ds \\ &\quad - \left(\frac{t}{T} - 1 \right) \int_0^T g(s, y_n(s)) ds + \frac{t}{T} \int_0^T h(s, y_n(s)) ds. \end{aligned}$$

If the condition (H₄) of Theorem 3.1 is replaced by (H₈) of Theorem 3.2, we still could approximate the (non unique) solution of BVP (1)-(3) by means of Krasnoselski-Mann iteration procedure.

Theorem 3.3. *Assume all conditions of Theorem 3.2 are satisfied. Then, the solution $y^* \in C_L$ of BVP (1)-(3) can be approximated by means of the Krasnoselskij iteration*

$$\begin{aligned} y_{n+1}(t) &= (1-\lambda)y_n(t) + \lambda \left[\left(\frac{t}{T} - 1 \right) \int_0^T g(s, y_n(s)) ds \right. \\ &\quad \left. + \frac{t}{T} \int_0^T h(s, y_n(s)) ds \right] + \lambda \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot f(s, y_n(s)) ds \right. \\ &\quad \left. - \frac{t}{T \cdot \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y_n(s)) ds \right] \end{aligned}$$

where $\lambda \in (0, 1)$ and $y_1 \in C_L$ is arbitrary.

Proof. By applying Corollary 2.1 or Corollary 2.2 we get the conclusion of Theorem 3.3. \square

4. An example

The example in this section illustrates the effectiveness of our new results in this paper, i.e., Theorem 3.2 is more general than Theorem 3.1.

Let us consider the following fractional boundary value problem:

$$(8) \quad {}^c D^\infty y(t) = \frac{e^{-t} \cdot |y(t)|}{(17 + e^t)(1 + |y(t)|)}, \quad t \in [0, 1], \quad 1 < \alpha \leq 2$$

$$(9) \quad y(0) = \int_0^1 \frac{4s^2 \cdot e^{-s}}{8 + e^s} \cdot y(s) ds$$

$$(10) \quad y(1) = \int_0^1 \frac{1}{1 + e^s} \cdot y(s) ds.$$

For $x, y \in [0, \infty)$ and $t \in [0, 1]$ we have:

$$|f(t, x) - f(t, y)| = \frac{e^{-t}}{17 + e^t} \cdot \frac{|x - y|}{(1 + x)(1 + y)} \leq \frac{e^{-t}}{17 + e^t} \cdot |x - y| \leq \frac{1}{18} \cdot |x - y|.$$

Hence condition (H₁) holds with $k = \frac{1}{6}$

$$|g(t, x) - g(t, y)| = \frac{4t^2 e^{-t}}{8 + e^t} \cdot |x - y| \leq \frac{4}{9} \cdot |x - y|$$

$$|h(t, x) - h(t, y)| = \frac{1}{1 + e^t} \cdot |x - y| \leq \frac{1}{2} \cdot |x - y|.$$

The conditions (H₂) and (H₃) hold with $k^* = \frac{4}{9}$ and $k^{**} = \frac{1}{2}$. We shall check that condition (H₈) is satisfied with $T = 1$, $k^* = \frac{4}{9}$ and $k^{**} = \frac{1}{2}$.

Indeed:

$$\frac{2k \cdot T^\alpha}{\Gamma(\alpha + 1)} + T(k^* + k^{**}) = 2 \cdot \frac{1}{18} \cdot \frac{1}{\Gamma(\alpha + 1)} + \frac{17}{18} = 1 \Rightarrow \Gamma(\alpha + 1) = 2$$

and from (H₇) one obtains $\Gamma(\alpha + 1) \geq 1$, which is satisfied for $\alpha \in (1, 2]$.

The assumptions of Theorem 3.5 in [3] aren't satisfied, but the Theorem 3.2 can be applied and the problem (8)-(10) has at least one solution in C_L for $L = \frac{1}{3}$. Starting from $y_0(t) = e^t$ and $\lambda = \frac{1}{2}$ we obtain the first iteration

$$y_1(t) = \frac{1}{2}y_0(t) + \frac{1}{2} \left[\left(\frac{t}{T} - 1 \right) \int_0^t g(s, y_0(s)) ds + \frac{t}{T} \int_0^T h(s, y_0(s)) ds \right. \\ \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} - f(s, y_0(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y_0(s)) ds \right]$$

$$y_1(t) = \frac{1}{2}e^t + \frac{1}{2} \left[\left(\frac{t}{T} - 1 \right) \cdot \frac{4}{3} \cdot \frac{t}{8 + e^t} + \frac{t^2}{T} \cdot \frac{e^t}{1 + e^t} - \frac{1}{4} I^\alpha (17 + e^s)^{-1}(t) \right. \\ \left. + \frac{1}{4} I^\alpha (1 + e^s)^{-1}(t) - \frac{t}{T} \left(-\frac{1}{4} I^\alpha (17 + e^s)^{-1}(T) + \frac{1}{4} I^\alpha (1 + e^s)^{-1}(T) \right) \right].$$

By applying the Lemma 3.1, the first iteration $y_1(t)$ for $T = 1$ and $n = 3$ can be approximated as:

$$y_1(t) \approx \frac{1}{2} \cdot e^t + \frac{2}{3} \cdot \frac{(t-1)t^3}{8 + e^t} + \frac{1}{2} \cdot \frac{t^2 e^t}{1 + e^t} + t^2 - 2t + t^3.$$

The first iteration for $T = 1, n = 3$ and $I^\alpha (17 + e^s)^{-1}(t) + I^\alpha (1 + e^s)^{-1}(t) = \frac{1}{4} \cdot t^2 - \frac{3}{8} \cdot t + \frac{1}{8} \cdot t^3$ can be approximated as:

$$y_1(t) \approx \frac{1}{2} \cdot e^t + \frac{2}{3} \cdot \frac{(t-1)t^3}{8 + e^t} + \frac{1}{2} \cdot \frac{t^2 e^t}{1 + e^t} + \frac{1}{4} \cdot t^2 - \frac{3}{8} \cdot \frac{t+1}{8} \cdot t^3.$$

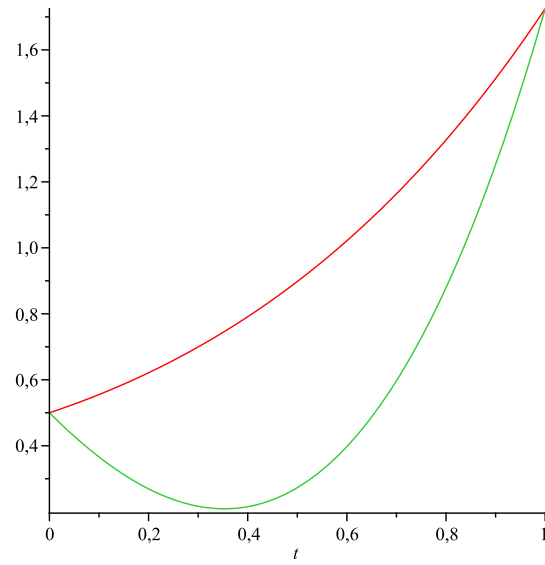


Figure 1. Graphics representation of first iteration

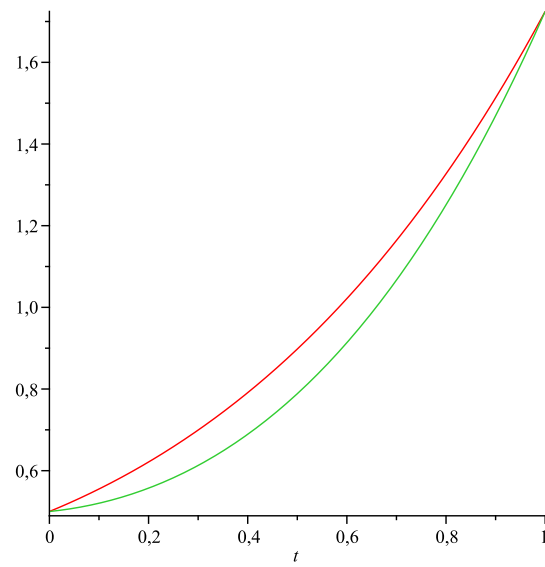


Figure 2. Graphics representation of first iteration

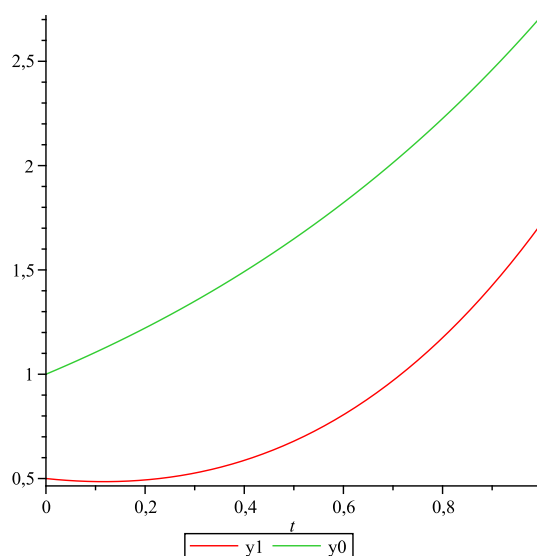


Figure 3. Graphics representation of y_0 and y_1

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North University of Baia Mare,
Department of Mathematics and Computer Science,
Victoriei 76, 430122 Baia Mare,
ROMANIA
lauranmonica@yahoo.com
vberinde@ubm.ro