ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVII, 2011, Supliment
DOI: 10.2478/v10157-011-0012-5

LEBESGUE POINTS FOR ORLICZ-SOBOLEV FUNCTIONS ON METRIC MEASURE SPACES

BY

MARCELINA MOCANU

Abstract. In this paper we deal with a metric measure space equipped with a doubling measure and supporting an Orlicz-Poincaré inequality, namely a weak $(1,\Phi)$-Poincaré inequality, that is more general than the $(1,1)$-Poincaré inequality. For a wide class of Orlicz spaces, we prove that the corresponding Orlicz-Sobolev functions have Lebesgue points outside a set of zero Orlicz-Sobolev capacity. This results extends a theorem of Tuominen (2009) from the case where Φ is the identity of $[0,\infty)$ to the case where Φ is a doubling Young function. Our main tools are the Hardy-Littlewood maximal operator and a discrete maximal operator introduced by Kinnunen and Latvala (2002).
Mathematics Subject Classification 2000: 46E30, 46E35.
Key words: metric measure space, weak upper gradient, Orlicz-Sobolev space, capacity, Lebesgue point, maximal operator.

1. Introduction and preliminaries

During the last decade, advances of analysis on metric measure spaces with no smooth structure included the study of several Sobolev-type spaces: Hajłasz spaces [8], Newtonian spaces [18], Cheeger spaces [4]. These spaces are essential in order to extend quasiconformal theory and nonlinear potential theory to the metric setting. A recent direction of research in analysis on metric spaces deals with Orlicz-Sobolev spaces [1], [19] and some of their generalizations [15].

In this paper we generalize a recent result from [20] on the pointwise behaviour of Orlicz-Sobolev functions on metric measure spaces. We deal with a metric measure space equipped with a doubling measure and supporting an Orlicz-Poincaré inequality, namely a weak $(1,\Phi)$-Poincaré inequality, that is more general than the $(1,1)$-Poincaré inequality. For a wide class of Orlicz spaces, we show that the corresponding Orlicz-Sobolev functions have Lebesgue points outside a set of zero Orlicz-Sobolev capacity. This result extends [20, Theorem 3] from the case where Φ is the identity of $[0,\infty)$ to the case where Φ is a doubling Young function. Our main tools are the Hardy-Littlewood maximal operator and a discrete maximal operator introduced in [13].

For the notions from the theory of Orlicz spaces we refer to [17]. We deal with the growth rates of Young functions given by Δ_2-, ∇_2- and Δ'-conditions. A Young function satisfying a Δ_2-condition is called *doubling*. Let $(X,\mathcal{A},\mu)$ be a measure space with a complete and σ-finite measure μ and let $\Phi:[0,\infty]\to[0,\infty]$ be a Young function. The Orlicz space associated to Φ, denoted by $L^\Phi(X)$, is a Banach space with the Luxemburg norm $\|\cdot\|_{L^\Phi(X)}$. For every measurable function $u: X\to[-\infty,+\infty]$, denote $I_\Phi(u)=\int_X \Phi(|u|)d\mu$. If $I_\Phi(u)<\infty$, then $u\in L^\Phi(X)$ and the converse is true provided that Φ is doubling.

Remark 1. By [20, Lemma 4], for every doubling Young function Φ and all $u\in L^\Phi(X)$, the following inequalities hold

$$\|u\|_{L^\Phi(X)} \le f_\Phi(I_\Phi(u)) \text{ and } I_\Phi(u)\le h_\Phi(\|u\|_{L^\Phi(X)}),$$

where we denoted $f_\Phi(t)=\max\{t, 2t^{1/\log_2 C_\Phi}\}$ and $h_\Phi(t)=\max\{t, C_\Phi t^{\log_2 C_\Phi}\}$.

Throughout this paper we deal with a metric measure space (X,d,μ), which is a metric space (X,d) equipped with a Borel regular outer measure μ. Assume that μ is finite and positive on balls. Recall that a metric space is called *proper* if every closed ball is compact.

Remark 2. Since μ is finite on balls, for every doubling N-function $\Phi:[0,\infty)\to[0,\infty)$ we have $L^\Phi(X)\subset L^1_{loc}(X)$, by [17, Proposition 3.1.7].

Definition 1. The measure μ on the metric space (X,d,μ) is said to be doubling if there is a constant $C_d\ge 1$ such that

$$\mu(B(x,2r))\le C_d\mu(B(x,r)) \tag{1.1}$$

for every ball $B(x,r)\subset X$.

For every doubling measure μ there are some positive constants C_b and Q so that $\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))}\ge C_b(\frac{r}{r_0})^Q$, for all $0<r\le r_0$ and $x\in B(x_0,r_0)$. Here Q is called a *homogeneous dimension* of the metric measure space X.

The Hardy-Littlewood maximal function of $f \in L^1_{loc}(X)$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu.$$

The classical Vitali covering theorem, Lebesgue's differentiation theorem and the maximal function theorem have natural extensions to the setting of doubling metric measure spaces [12], [5].

In the following it is assumed that the measure μ is doubling.

If Φ is an N-function satisfying the ∇_2-condition, then $\mathcal{M}$ is bounded as an operator from $L^\Phi(X)$ into itself [16]. Under this assumptions on Φ it follows by [7, Theorem 2.2] that there exist some positive constants A and b such that

$$I_\Phi(\mathcal{M}f) \le A I_\Phi(bf) \tag{1.2}$$

for every $f \in L^\Phi(X)$, a property stronger in general than the boundedness of $\mathcal{M}$ in $L^\Phi(X)$.

Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is said to be an *upper gradient* of u in X if

$$|u(\gamma(1)) - u(\gamma(0))| \le \int_\gamma g \; ds, \tag{1.3}$$

for every rectifiable path $\gamma : [0,1] \to X$.

Since upper gradients are unstable under changes μ-a.e. and under limits, the more general notion of weak upper gradient has been introduced [11].

The notion of modulus of a path family, an important tool in geometric function theory, is indispensable in the definition of Sobolev-type spaces based on upper gradients.

Definition 2 ([19])**.** Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function. The Φ-modulus of a family Γ of paths in X is $M_\Phi(\Gamma) = \inf \|\rho\|_{L^\Phi(X)}$, where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ satisfying $\int_\gamma \rho ds \ge 1$ for all locally rectifiable paths $\gamma \in \Gamma$.

Definition 3 ([19])**.** Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is called a $\Phi-$weak upper gradient of u if (1.3) holds for all compact rectifiable paths $\gamma : [0,1] \to X$ except for a path family Γ_0 with $M_\Phi(\Gamma_0) = 0$ in X.

The collection $\widetilde{N}^{1,\Phi}(X)$ of all functions $u \in L^{\Phi}(X)$ having a Φ-weak upper gradient $g \in L^{\Phi}(X)$ is a vector space. For $u \in \widetilde{N}^{1,\Phi}(X)$ define $\|u\|_{1,\Phi} = \|u\|_{L^{\Phi}(X)} + \inf \|g\|_{L^{\Phi}(X)}$, where the infimum is taken over all Φ-weak upper gradients $g \in_{L^{\Phi}(X)}$ of u. Consider the equivalence relation $u \sim v \Leftrightarrow \|u - v\|_{1,\Phi} = 0$. Then $N^{1,\Phi}(X) = \widetilde{N}^{1,\Phi}(X)/\sim$ is a Banach space with the norm $\|u\|_{N^{1,\Phi}} := \|u\|_{1,\Phi}$ [19].

If $X = \Omega \subset \mathbb{R}^n$ is a domain and Φ is a doubling Young function, then $N^{1,\Phi}(X) = W^{1,\Phi}(\Omega)$ as Banach spaces and the norms are equivalent [19, Theorem 6.19].

A *capacity* with respect to the space $N^{1,\Phi}(X)$, called Φ-capacity, is defined by $Cap_{\Phi}(E) = \inf\{\|u\|_{N^{1,\Phi}} : u \in N^{1,\Phi}(X) : u \geq 1 \text{ on } E\}$ [19].

In the classical theory of Sobolev spaces on $\mathbb{R}^n$, Poincaré inequality is a result that allows one to obtain integral bounds on the oscillation of a function in terms of integral bounds on its derivatives.

Denote the mean value of a function $u \in L^1(A)$ over A by $u_A := \frac{1}{\mu(A)} \int_B u d\mu$, where $0 < \mu(A) < \infty$.

Definition 4 ([19], Definition 5.2)**.** Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing Young function and $\Omega \subset X$ an open set. We say that a function $u \in L^1_{loc}(\Omega)$ and a Borel measurable non-negative function g on Ω satisfy a weak $(1, \Phi)$- Poincaré inequality in Ω if there exist some constants $C_{P,\Phi} > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C_{P,\Phi} r \Phi^{-1} \left(\frac{1}{\mu(\tau B)} \int_{\tau B} \Phi(g) d\mu \right) \tag{1.4}$$

for each ball $B = B(x, r)$ satisfying $\tau B := B\,(x, \tau r) \subset \Omega$. It is said that Ω supports a weak $(1, \Phi)$-Poincaré inequality if the above inequality holds for each function $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants.

For $\Phi(t) = t^p$, where $p \geq 1$, the weak $(1, \Phi)$- Poincaré inequality is the weak $(1, p)$- Poincaré inequality introduced in [10] and investigated in [9].

Remark 3. Every Φ-weak upper gradient can be approximated in L^{Φ}-norm by a sequence of upper gradients [19, Lemma 4.3]. If Φ is doubling, we may replace in the above definition upper gradients by Φ-weak upper gradients.

2. Lebesgue points of Orlicz-Sobolev functions

Recall that $x \in X$ is said to be a Lebesgue point of a locally integrable function $u : X \to \mathbb{R}$ if

$$\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u(x)|\, d\mu = 0.$$

In doubling metric measure spaces Lebesgue differentiation theorem holds, i.e. almost every point is a Lebesgue point for a locally integrable function [12]. As the regularity of the function increases, the size of the complement of the set of Lebesgue point decreases.

For Sobolev functions in Euclidean spaces the exceptional set from the Lebesgue differentiation theorem has zero capacity. Moreover, if $u \in W^{1,p}_{loc}(\mathbb{R}^n)$, where $1 < p \leq n$, then there exists a set $E \subset \mathbb{R}^n$ with zero p-capacity such that

$$\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u(x)|^s\, d\mu = 0$$

for every $x \in X \setminus E$, whenever $0 < s < \frac{np}{n-p}$ [6].

The proofs of the refinements of Lebesgue differentiation theorem for Sobolev functions on Euclidean spaces are based on weak capacitary estimates for Hardy-Littlewood maximal function and use tools that are not available in metric spaces.

Kinnunen and Latvala [13] extended the above result to Hajlasz-Sobolev spaces $M^{1,p}(X)$ on doubling metric measure spaces, using as tool a *discrete maximal operator* $\mathcal{M}^* : L^1_{loc}(X) \to \mathbb{R}$. The operator $\mathcal{M}^*$ is pointwise equivalent to the Hardy-Littlewood maximal operator:

$$c_e^{-1}\mathcal{M}u \leq \mathcal{M}^*u \leq c_e\mathcal{M}u \tag{2.1}$$

for every $u \in L^1_{loc}(X)$, where $c_e \geq 1$ is a constant depending only on the doubling constant C_d of μ. [13]

Let Q be a homogeneous dimension of the doubling metric measure space (X, d, μ). If $1 < p \leq Q$, then for every $u \in M^{1,p}(X)$ there is $E \subset X$ with $Cap_p(E) = 0$ such that

$$\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u^*(x)|^s\, d\mu = 0,$$

for every $x \in X \setminus E$, whenever $1 < s \leq \frac{Qp}{Q-p}$ [13, Theorem 4.5].

Kinnunen and Latvala [13] proved that $\mathcal{M}^*$ is bounded in $M^{1,p}(X)$ for $1 < p < \infty$, then Tuominen [20, Theorem 2] obtained the following counterpart of this result in Orlicz-Sobolev spaces. If X supports a $(1,1)$-Poincaré inequality, Ψ is doubling and the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded in $L^{\Psi}(X)$, then the discrete maximal operator $\mathcal{M}^*$ is bounded in $N^{1,\Psi}(X)$, i.e. for every $u \in N^{1,\Psi}(X)$ we have $\mathcal{M}^*u \in N^{1,\Psi}(X)$and $\|\mathcal{M}^*u\|_{N^{1,\Psi}(X)} \leq C\|u\|_{N^{1,\Psi}(X)}$ for some positive constant C not depending on u.

Under the assumptions of the above theorem in a proper metric space, Tuominen [20, Theorem 3] proved that every Orlicz-Sobolev function has Lebesgue points outside a set with the corresponding Orlicz-Sobolev capacity zero. If X is proper and supports a $(1,1)$-Poincaré inequality, Ψ is doubling and the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded in $L^{\Psi}(X)$, then for every $u \in N^{1,\Psi}(X)$ there is $E \subset X$ with $Cap_{\Psi}(E) = 0$ such that every $x \in X \setminus E$ is a Lebesgue point of u.

We generalize the above results of Tuominen to spaces supporting a $(1,\Phi)$-Poincaré inequality. By Jensen's inequality in integral form, a weak $(1,1)$- Poincaré inequality implies a weak $(1,\Phi)$- Poincaré inequality, for every Young function Φ.

In all the following lemmas we assume that X supports a weak $(1,\Phi)$-Poincaré inequality, for some doubling Young function Φ.

Lemma 1. *Let Ψ be a doubling N-function. If $u \in L^1_{loc}(X)$ and $g \in L^{\Psi}(X)$ is a Ψ-weak upper gradient of u, then the function $C_1\Phi^{-1}(\mathcal{M}(\Phi \circ g))$ is a Ψ-weak upper gradient of $\mathcal{M}^*u$, where $C_1 > 0$ is a constant depending only on the doubling constant C_d and on $C_{P,\Phi}$ from* (1.4).

Proof. For an arbitrary $r > 0$ we consider an open cover $\{B(x_i,r) : i \geq 1\}$ with bounded overlap: $X = \bigcup_{i=1}^{\infty} B(x_i,r)$ and $\sum_{i=1}^{\infty} \chi_{B(x_i,6r)} \leq N < \infty$. Here N is a constant depending on the doubling constant C_d of μ, but independent of r [5].

The proof is similar to that of [13, Lemma 3.3] and of [20, Lemma 8]. The difference appears in the telescoping argument needed to estimate the oscillation of u on $B(x_i,6r)$, around the average of u on $B(x_i,3r)$, since the $(1,1)$-Poincaré inequality is replaced by the weak $(1,\Phi)$- Poincaré. As in [14, Proof of Theorem 2, Step 1], we get the following estimate, using the weak $(1,\Phi)$- Poincaré inequality in the doubling space X:

$$\left|u(x) - u_{B(x_i,3r)}\right| \leq C_1 r \Phi^{-1}(\mathcal{M}(\Phi \circ g))(x) \tag{2.2}$$

for almost all $x \in B(x_i, 6r)$ and for every $i \geq 1$, where $C_1 > 0$ is a constant depending only on C_d and $C_{P,\Phi}$.

As in the proof of [13, Lemma 3.3] and of [20, Lemma 8], it turns out that $C_1\Phi^{-1}(\mathcal{M}(\Phi \circ g))$ is a Ψ-weak upper gradient of $|u|_r$ for each $r > 0$, hence is a Ψ-weak upper gradient of $\mathcal{M}^*u$, by [20, Lemma 6]. $\square$

Lemma 2. *Let Ψ be a doubling Young N-function such that $\Psi \circ \Phi^{-1}$ is a Young function and the Hardy-Littlewood maximal operator is bounded in $L^{\Psi\circ\Phi^{-1}}(X)$. If $g \in L^\Psi(X)$, then $\Phi^{-1}(\mathcal{M}(\Phi \circ g)) \in L^\Psi(X)$. Moreover, there exists a strictly increasing continuous function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ such that $\left\|\Phi^{-1}(\mathcal{M}(\Phi \circ g))\right\|_{L^\Psi(X)} \leq F(\|g\|_{L^\Psi(X)})$ for every $g \in L^\Psi(X)$. Assuming in addition that Φ and Φ^{-1} satisfy the Δ'-condition and $\Psi\circ\Phi^{-1}$ is a N-function satisfying the ∇_2-condition, we may take in the previous estimate $F(t) = C't$ for all $t \geq 0$, where C' is a positive constant.*

Proof. Let $g \in L^\Psi(X)$. Since $\Phi \circ g \in L^{\Psi\circ\Phi^{-1}}(X)$ and $\mathcal{M}$ is bounded in $L^{\Psi\circ\Phi^{-1}}(X)$, $\mathcal{M}(\Phi \circ g) \in L^{\Psi\circ\Phi^{-1}}(X)$ and

$$\|\mathcal{M}(\Phi \circ g)\|_{L^{\Psi\circ\Phi^{-1}}(X)} \leq C_{HL} \|\Phi \circ g\|_{L^{\Psi\circ\Phi^{-1}}(X)}$$

for some constant $C_{HL} > 0$. Since Ψ is doubling, $\mathcal{M}(\Phi \circ g) \in L^{\Psi\circ\Phi^{-1}}(X)$ implies $\Phi^{-1}(\mathcal{M}(\Phi \circ g)) \in L^\Psi(X)$. According to Remark 1,

$$\left\|\Phi^{-1}(\mathcal{M}(\Phi \circ g))\right\|_{L^\Psi(X)} \leq f_\Psi(I_{\Psi\circ\Phi^{-1}}(\mathcal{M}(\Phi \circ g)))$$

and $I_{\Psi\circ\Phi^{-1}}(\mathcal{M}(\Phi\circ g)) \leq h_{\Psi\circ\Phi^{-1}}(\|\mathcal{M}(\Phi \circ g)\|_{L^{\Psi\circ\Phi^{-1}}(X)})$, but f_Ψ and $h_{\Psi\circ\Phi^{-1}}$ are increasing, hence

$$\left\|\Phi^{-1}(\mathcal{M}(\Phi \circ g))\right\|_{L^\Psi(X)} \leq (f_\Psi \circ h_{\Psi\circ\Phi^{-1}})(C_{HL} \|\Phi \circ g\|_{L^{\Psi\circ\Phi^{-1}}(X)}).$$

Using again Remark 1, $\|\Phi \circ g\|_{L^{\Psi\circ\Phi^{-1}}(X)} \leq f_{\Psi\circ\Phi^{-1}}(I_\Psi(g)) \leq (f_{\Psi\circ\Phi^{-1}} \circ h_\Psi)(\|g\|_{L^\Psi(X)})$. We obtain

$$\left\|\Phi^{-1}(\mathcal{M}(\Phi \circ g))\right\|_{L^\Psi(X)} \leq F(\|g\|_{L^\Psi(X)}),$$

where $F(t) = (f_\Psi \circ h_{\Psi\circ\Phi^{-1}})(C_{HL}(f_{\Psi\circ\Phi^{-1}} \circ h_\Psi)(t))$, for $t \geq 0$. By the subadditivity of Φ^{-1}, we may take $C_{\Psi\circ\Phi^{-1}} = C_\Psi$, hence $f_{\Psi\circ\Phi^{-1}} = f_\Psi$ and $h_{\Psi\circ\Phi^{-1}} = h_\Psi$.

Assume in addition that Φ and Φ^{-1} satisfy the Δ'-condition and $\Psi\circ\Phi^{-1}$ is a N-function satisfying the ∇_2-condition. There is a constant $C \geq 1$ such that

$$\Phi(ts) \leq C\Phi(t)\Phi(s) \text{ and } \Phi^{-1}(ts) \leq C\Phi^{-1}(t)\Phi^{-1}(s) \tag{2.3}$$

for every $t, s \in [0, \infty)$. Since $\Psi \circ \Phi^{-1}$ is a N-function satisfying the ∇_2-condition, there exist some positive constants A and b such that

$$I_{\Psi\circ\Phi^{-1}}(\mathcal{M}f) \leq AI_{\Psi\circ\Phi^{-1}}(bf)$$

for every $f \in L^{\Psi\circ\Phi^{-1}}(X)$. Assuming without loss of generality that $A \geq 1$, it follows that there is some constant $C' > 0$ such that $I_\Psi(\frac{\Phi^{-1}(\mathcal{M}(\Phi\circ g))}{C'\|g\|_{L^\Psi(X)}}) \leq 1$ for every $g \in L^\Psi(X)$. Then, by the definition of the Luxemburg norm, it follows that $\|\Phi^{-1}(\mathcal{M}(\Phi \circ g))\|_{L^\Psi(X)} \leq C'\|g\|_{L^\Psi(X)}$ for every $g \in L^\Psi(X)$. □

Lemma 3. *If Ψ is a doubling Young N-function, such that the Hardy-Littlewood maximal operator is bounded both in $L^\Psi(X)$ and in $L^{\Psi\circ\Phi^{-1}}(X)$, then the discrete maximal operator $\mathcal{M}^*$ maps $N^{1,\Psi}(X)$ into $N^{1,\Psi}(X)$. Moreover, there is a continuous strictly increasing function $H : [0,\infty) \to [0,\infty)$ with $H(0) = 0$ such that $\|\mathcal{M}^*u\|_{N^{1,\Psi}(X)} \leq H(\|u\|_{N^{1,\Psi}(X)})$ for every $u \in N^{1,\Psi}(X)$. If in addition Φ and Φ^{-1} satisfy the Δ'-condition and $\Psi\circ\Phi^{-1}$ is a N-function satisfying the ∇_2-condition, then $\mathcal{M}^* : N^{1,\Psi}(X) \to N^{1,\Psi}(X)$ is a bounded operator.*

Proof. Since the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded in $L^\Psi(X)$,there exists C'_{HL} such that $\|\mathcal{M}u\|_{L^\Psi(X)} \leq C'_{HL}\|u\|_{L^\Psi(X)}$ for every $u \in L^\Psi(X)$. It follows by (2.1) that $\mathcal{M}^*$ maps $L^\Psi(X)$ into $L^\Psi(X)$ and $\|\mathcal{M}^*u\|_{L^\Psi(X)} \leq c_eC'_{HL}\|u\|_{L^\Psi(X)}$ for every $u \in L^\Psi(X)$. Let $C_1 > 0$ be as in Lemma 1 and let F be as in Lemma 2. Fix an arbitrary $u \in N^{1,\Psi}(X)$. Let $g \in L^\Psi(X)$ be a Ψ-weak upper gradient of u. By Lemma 1 and Lemma 2, the function $C_1\Phi^{-1}(\mathcal{M}(\Phi \circ g)) \in L^\Psi(X)$ is a Ψ-weak upper gradient of M^*u. Then $\mathcal{M}^*u \in N^{1,\Psi}(X)$ and $\|\mathcal{M}^*u\|_{N^{1,\Psi}(X)} \leq c_eC'_{HL}\|u\|_{L^\Psi(X)} + C_1F(\|g\|_{L^\Psi(X)})$. We consider a sequence $g_n \in L^\Psi(X)$, $n \geq 1$, of Ψ-weak upper gradients of u, such that $\lim\limits_{n\to\infty}\|g_n\|_{L^\Psi(X)} = \|u\|_{N^{1,\Psi}(X)} - \|u\|_{L^\Psi(X)}$. For every $n \geq 1$, $\|\mathcal{M}^*u\|_{N^{1,\Psi}(X)} \leq c_eC'_{HL}\|u\|_{L^\Psi(X)} + C_1F(\|g_n\|_{L^\Psi(X)})$. Letting $n \to \infty$ and using the continuity and monotonicity of F we get $\|\mathcal{M}^*u\|_{N^{1,\Psi}(X)} \leq H(\|u\|_{N^{1,\Psi}(X)})$. We denoted $H(t) = c_eC'_{HL}t + C_1F(t)$, $t \in [0,\infty)$.

Assume that Ψ and Ψ^{-1} satisfy the Δ'-condition and $\Psi \circ \Phi^{-1}$ is a N-function satisfying the ∇_2-condition. By Lemma 2, we may take $F(t) = C't$, hence $\|\mathcal{M}^* u\|_{N^{1,\Psi}(X)} \leq C^* \|u\|_{N^{1,\Psi}(X)}$ for every $u \in N^{1,\Psi}(X)$, where $C^* = c_e C'_{HL} + C_1 C'$, therefore $\mathcal{M}^*$ is a bounded operator. □

For $u \in L^1_{loc}(X)$ and $x \in X$, denote

$$A_u(x) := \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u| \, d\mu.$$

Recall that a real function u on X is said to be Ψ-quasicontinuous if for every $\varepsilon > 0$ there exists a set $E \subset X$ with $Cap_\Psi(E) < \varepsilon$ such that the restriction of u to $X \setminus E$ is continuous with respect to the relative topology.

Lemma 4. *Let Ψ be a doubling N-function. Assume that continuous functions are dense in $N^{1,\Psi}(X)$, every function in $N^{1,\Psi}(X)$ is Ψ-quasicontinuous and there exist a constant $k > 0$ and a function $G : [0, \infty) \to [0, \infty)$ with $\lim_{t \to \infty} G(t) = 0$ such that, for every $f \in N^{1,\Psi}(X)$*

$$Cap_\Psi(\{x \in X : A_f(x) > \lambda\}) \leq k\lambda^{-1} G\left(\|f\|_{N^{1,\Psi}(X)}\right). \tag{2.4}$$

Then for every $u \in N^{1,\Psi}(X)$ there is $E \subset X$ with $Cap_\Psi(E) = 0$ such that every $x \in X \setminus E$ is a Lebesgue point of u.

Proof. We modify the proof of [20, Lemma 4], where in the estimate (2.4) the special case $G(t) \equiv t$ was considered. Denote

$$D_u(x) := \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u(x)| d\mu.$$

The complement of the set of all Lebesgue point of u is $E = \{x \in X : D_u(x) > 0\}$. Let $E_\lambda = \{x \in X : D_u(x) > \lambda\}$ for each $\lambda > 0$. We prove that $Cap_\Psi(E_\lambda) = 0$ for all $\lambda > 0$. Since $E = \cup_{i=1}^{\infty} E_{1/i}$ and the Ψ-capacity is countably subadditive, the claim $Cap_\Psi(E) = 0$ follows.

Let $u \in N^{1,\Psi}(X)$ and $\lambda > 0$. Let $\varepsilon > 0$. Using the density of $C(X)$ in $N^{1,\Psi}(X)$ we find $v \in C(X) \cap N^{1,\Psi}(X)$ such that $\|u - v\|_{N^{1,\Psi}(X)} < \varepsilon$. The function $w = u - v$ is quasicontinuous. For every $x \in X$, using the triangle inequality and the continuity of v at x, we get $D_u(x) \leq D_w(x) \leq A_w(x) + |w(x)|$. The inequality $D_u(x) \leq A_w(x) + |w(x)|$, $x \in X$, shows that $E_\lambda \subset$

$\{x \in X : A_w(x) > \lambda/2\} \cup \{x \in X : |w(x)| > \lambda/2\}$. By (2.4), $Cap_\Psi(\{x \in X : A_w(x) > \lambda/2\}) \leq 2k\lambda^{-1}G(\|w\|_{N^{1,\Psi}(X)}) \leq 2k\lambda^{-1}G(\varepsilon)$. By the definition of Ψ-capacity, we have $Cap_\Psi(\{x \in X : |w(x)| > \lambda/2\}) \leq \|2\lambda^{-1}|w|\|_{N^{1,\Psi}(X)}$, hence $Cap_\Psi(\{x \in X : |w(x)| > \lambda/2\}) \leq 2\lambda^{-1}\|w\|_{N^{1,\Psi}(X)} < 2\lambda^{-1}\varepsilon$. The subadditivity of Ψ-capacity yields $Cap_\Psi(E_\lambda) \leq 2\lambda^{-1}(kG(\varepsilon) + \varepsilon)$. Letting $\varepsilon \to 0$ we get $Cap_\Psi(E_\lambda) = 0$. □

Theorem 1. *Assume that the metric space X is equipped with a doubling measure and supports a weak $(1, \Phi)$-Poincaré inequality for some doubling Young function Φ. Let Ψ be a doubling N-function such that $\Psi \circ \Phi^{-1}$ is a Young function and the Hardy-Littlewood maximal operator is bounded both in $L^\Psi(X)$ and $L^{\Psi\circ\Phi^{-1}}(X)$. Assume that continuous functions are dense in $N^{1,\Psi}(X)$ and every function in $N^{1,\Psi}(X)$ is Ψ-quasicontinuous. Then for every $u \in N^{1,\Psi}(X)$ there is $E \subset X$ with $Cap_\Psi(E) = 0$ such that every $x \in X \setminus E$ is a Lebesgue point of u.*

Proof. Let $u \in N^{1,\Psi}(X)$. Since $A_u(x) \leq \mathcal{M}u(x) \leq c_e\mathcal{M}^*u(x)$ for every $x \in X$,

$$Cap_\Psi\left(\{x \in X : A_u(x) > \lambda\}\right) \leq Cap_\Psi\left(\{x \in X : c_e\mathcal{M}^*u(x) > \lambda\}\right).$$

Using the definition of Ψ-capacity and Lemma 3, we get

$$\begin{aligned} Cap_\Psi\left(\{x \in X : c_e\mathcal{M}^*u(x) > \lambda\}\right) &\leq c_e\lambda^{-1}\left\|\mathcal{M}^*u\right\|_{N^{1,\Psi}(X)} \\ &\leq c_e\lambda^{-1}H(\|u\|_{N^{1,\Psi}(X)}). \end{aligned}$$

Then $Cap_\Psi(\{x \in X : A_u(x) > \lambda\}) \leq c_e\lambda^{-1}H(\|u\|_{N^{1,\Psi}(X)})$. The proof is completed by Lemma 4. □

Remark 4. If Ψ is a doubling Young function and X supports a weak $(1, \Psi)$-Poincaré inequality, then Lipschitz continuous functions are dense in $N^{1,\Psi}(X)$, both in norm and in Lusin's sense [19, Theorem 6.17]. If X is proper, Ψ is a doubling Young function and continuous functions are dense in $N^{1,\Psi}(X)$, then each function $u \in N^{1,\Psi}(X)$ is Ψ- quasicontinuous [20, Theorem 1].

Remark 5. If Φ is the identity, Lemma 1, Lemma 3, Lemma 4 and Theorem 1 give Lemma 8, Theorem 2, Lemma 9 and Theorem 3 from [20].

REFERENCES

1. Aïssaoui, N. – *Another extension of Orlicz-Sobolev spaces to metric spaces*, Abstr. Appl. Anal., 1 (2004), 1–26.
2. Aïssaoui, N. – *Strongly nonlinear potential theory on metric spaces*, Abstr. Appl. Anal., 7 (2002), 357–374.
3. Bennett, C.; Sharpley, R. – *Interpolation of Operators,* Pure and Applied Mathematics, 129, Academic Press, Inc., Boston, MA, 1988.
4. Cheeger, J. – *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), 428–517.
5. Coifman, R.R.; Weiss, G. – *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes,* Lecture Notes in Mathematics, 242, Springer-Verlag, Berlin-New York, 1971.
6. Federer, H.; Ziemer, W.P. – *The Lebesgue set of a function whose distribution derivatives are p-th power summable,* Indiana Univ. Math. J., 22 (1972), 139–158.
7. Gallardo, D. – *Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded*, Publ. Mat., 32 (1988), 261–266.
8. Hajłasz, P. – *Sobolev spaces on an arbitrary metric space*, Potential Anal., 5 (1996), 403–415.
9. Hajłasz, P.; Koskela, P. – *Sobolev met Poincaré*, Mem. Amer. Math. Soc., 145 (2000), 101 pp.
10. Hajłasz, P.; Koskela, P. – *Sobolev meets Poincaré,* C.R. Acad. Sci. Paris Sér. I Math., 320 (1995), 1211–1215.
11. Koskela, P.; MacManus, P. – *Quasiconformal mappings and Sobolev spaces*, Studia Math., 131 (1998), 1–17.
12. Heinonen, J. – *Lectures on Analysis on Metric Spaces*, Springer Verlag, New York, 2001.
13. Kinnunen, J.; Latvala, V. – *Lebesgue points for Sobolev functions on metric spaces*, Rev. Mat. Iberoamericana, 18 (2002), 685–700.
14. Mocanu, M. – *A Poincaré inequality for Orlicz-Sobolev functions with zero boundary values on metric spaces,* Complex Anal. Oper. Theory, to appear (DOI 10.1007/s11785-010-0068-3).
15. Mocanu, M. – *A generalization of Orlicz-Sobolev spaces on metric measure spaces via Banach function spaces,* Complex Var. Elliptic Equ., 55 (2010), 253–267.

16. Mocanu, M. – *Maximal operators and Orlicz-Sobolev functions on metric measure spaces*, Proceedings of the Sixth Congress of Romanian Mathematicians, 169178, Ed. Acad. Române, Bucharest, 2009.

17. Rao, M.M.; Ren, Z.D. – *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker Inc., New York, 1991.

18. Shanmugalingam, N. – *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, 16 (2000), 243–279.

19. Tuominen, H. – *Orlicz-Sobolev spaces on metric measure spaces*, Ann. Acad. Sci. Fenn. Math. Diss., 135 (2004), 86 pp.

20. Tuominen, H. – *Pointwise behaviour of Orlicz-Sobolev functions*, Ann. Mat. Pura Appl., 188 (2009), 35–59.

Received: 18.X.2010
Revised: 10.12.2010

"Vasile Alecsandri" University of Bacău,
Department of Mathematics and Informatics,
Calea Mărăşeşti 157, Bacău 600115,
ROMANIA
mmocanu@ub.ro