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CONNECTIONS IN THE HOLOMORPHIC JETS BUNDLE OF ORDER TWO

BY

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Abstract. In a previous paper, the first author made a general study of the geometry of $J^{(2,0)}M$ jets bundle, which has a holomorphic structure.

In the present paper we define the complex second order Lagrange space (M, L) and we prove the existence of a special complex nonlinear connection, provided by a complex spray deduced from the variational problem. With respect to adapted frames of this (c.n.c.) we emphasize the existence of a N-linear connection, named the Chern-Lagrange connection on (M, L), which is of $(1, 0)$-type and will play a fundamental role in the study of the complex second order Lagrange spaces.

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1. Introduction

Let M be a complex manifold, $dim_{\mathbf{C}} M = n$, (z^i) be complex coordinates in a local chart. The complexified tangent bundle $T_{\mathbf{C}}M$ admits the classical decomposition $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is a holomorphic vector bundle over M and its conjugate $T''M$ is the anti-holomorphic tangent bundle.

The holomorphic bundle of k-th *order jets differential* was introduced by Green and Griffiths in [6] as the sheaf of germs of holomorphic curves $\{f : \Delta_r \to M,\ f \in \mathcal{H}_{z_0},\ f(0) = z_0\}$ depending on a complex parameter θ.

By denoting $f^i = z^i \circ f$, $\forall i = \overline{1, n}$, $f \in \mathcal{H}_{z_0}$, according to [14], [15], $f, g \in \mathcal{H}_{z_0}$ are said to *be k-equivalent*, $f \overset{k}{\sim} g$, iff $f^i(0) = g^i(0)$ and $\frac{d^p f^i}{d\theta^p}(0) = \frac{d^p g^i}{d\theta^p}(0)$, $\forall i = \overline{1, n},\ p = \overline{1, k}$. The class of f is $[f]_{\underset{\sim}{k}}$ and the set of all classes is $J^{(k,0)}M = \cup_{z_0 \in M}\mathcal{H}_{z_0}/_{\underset{\sim}{k}}$. By $j^k f(0) = (f(0), \frac{df}{d\theta}(0), ..., \frac{d^p f}{d\theta^p}(0))$ we denote the $k-$jet of $f \in [f]_{\underset{\sim}{k}}$.

Let $\pi^{(k,0)} : J^{(k,0)}M \to M$ be the canonical projection. Then we check immediately that $(J^{(k,0)}M, \pi^{(k,0)}, M)$ has a fibre bundle structure, named in [15] the restricted k-jet bundle, and in [5] the parametrized k-jet bundle. Further on we call it simply *the* $J^{(k,0)}M$ *jets bundle*. Note that $J^{(k,0)}M$ does not have a vector bundle structure, aside from $k = 1$, when it is identified with $T'M$, the holomorphic tangent bundle.

$J^{(k,0)}M$ has a structure of complex differentiable manifold, whose geometry was discussed in [16].

We note that the rank of the fibre bundle $J^{(k,0)}M$ is kn, while the dimension of complex manifold structure is $(k+1)n$.

More generally, a (p, q)-jet on M could be spanned by $\frac{\partial f}{\partial \theta}(0), \frac{\partial f}{\partial \bar\theta}(0), \frac{\partial^2 f}{\partial \theta^2}(0), \frac{\partial^2 f}{\partial \theta \partial \bar\theta}(0), \frac{\partial^2 f}{\partial \bar\theta^2}(0), ...$, where $f \in \mathcal{F}(M)$, not necessarily holomorphic in $z_0 = f(0)$. In this position $J^{(p,q)}M$ is not always holomorphic ([7]). Certainly, if f is in $\mathcal{H}_{z_0}$ then $\frac{\partial f}{\partial \bar\theta}(0) = 0$, and it shows that $J^{(p,0)}M$ is a subbundle (holomorphic) of $J^{(p,q)}M$.

Further on in this paper we will resume our study to the second order jets manifold $J^{(2,0)}(M)$. We have the decomposition of $J^{(2,2)}(M) = J^{(2,0)}(M) \oplus J^{(1,1)}(M) \oplus J^{(0,2)}(M)$, where the terms are fiber bundles over the complex manifold M, the first being a holomorphic bundle which contains the holomorphic second order jets on M.

In the previous paper [16], the first author studied the geometric structure of the holomorphic bundle $J^{(k,0)}M$ over the complex manifold M, such as complex distributions, nonlinear and N-linear connections. Subsequently, we resume in brief the framework for the complex manifold $J^{(2,0)}M$. In a local chart, the coordinates are denoted by $Z = (z^i, \eta^i, \zeta^i)$, $i = \overline{1, n}$, and at changes of local charts on M will transform as follow:

$$
\begin{aligned}
z'^i &= z'^i(z); \\
\eta'^i &= \frac{\partial z'^i}{\partial z^j}\eta^j; \\
2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j
\end{aligned}
\tag{1.1}
$$

and that $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$; $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$. A local base in the holomorphic bundle $T'(J^{(2,0)}M)$ is $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\}$ and in $T''(J^{(2,0)}M)$ it is obtained by conjugation. The changes of the local basis are made according to the following rules:

$$(1.2)\qquad \begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i};\\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i};\\ \frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i} \end{aligned}$$

and similarly for the conjugate basis that corresponds in $T''_z(J^{(2,0)}M)$.

Two structures play a special role in defining the linear and nonlinear connection on $J^{(2,0)}M$: the natural complex structure J and the almost second order tangent structure F, see [11], [16].

A complex nonlinear connection, (c.n.c.) in brief, is given by $H(J^{(2,0)}M)$ which is supplementary to $W(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $W_z(J^{(2,0)}M)$ is spanned by $\{\frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j}\}$ in a local chart. With $V(J^{(2,0)}M)$ we denote the vertical bundle spanned by $\{\frac{\partial}{\partial \zeta^j}\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A local base in $H_z(J^{(2,0)}M)$ is called adapted base of the (c.n.c.), and it is written as $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \eta^i} - \overset{(2)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}$, iff $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$. Then $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta \eta^j} = \frac{\partial}{\partial \eta^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}$ span a local adapted base in $W_z(J^{(2,0)}M)$. The changes (1.1) of coordinates on $J^{(2,0)}M$ produce the changes of the coefficients $\overset{(1)}{N}{}^i_j$ and $\overset{(2)}{N}{}^i_j$ of the (c.n.c.) in the form:

$$(1.3)\qquad \begin{aligned} \overset{(1)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(1)}{N}{}^k_j - \frac{\partial \eta'^i}{\partial z^j};\\ \overset{(2)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(2)}{N}{}^k_j + \frac{\partial \eta'^i}{\partial z^k}\overset{(1)}{N}{}^k_j - \frac{\partial \zeta'^i}{\partial z^j}. \end{aligned}$$

The adapted basis will change as follows: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$ and $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \eta'^i}$. Obviously, $\frac{\delta}{\delta \zeta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \zeta'^i}$ and so these fields are changing as those on the base manifold M. Generally, the geometrical objects which are changed by $\frac{\partial z'^i}{\partial z^j}$ or by their conjugates $\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}$, are called *d-tensor fields*. The corresponding adapted basis on $T''(J^{(2,0)}M)$ are obtained by conjugation everywhere. The relation between the dual cobasis $\{dz^i, \delta\eta^i = d\eta^i + \overset{(1)}{M^i_j} dz^j, \delta\zeta^i = d\zeta^i + \overset{(1)}{M^i_j} d\eta^j + \overset{(2)}{M^i_j} dz^j\}$ and the adapted basis is given by the rules:

$$\overset{(1)}{M^i_j} = \overset{(1)}{N^i_j} \;\; ; \;\; \overset{(2)}{M^i_j} = \overset{(2)}{N^i_j} + \overset{(1)}{N^i_k}\overset{(1)}{N^k_j}$$

where $\overset{(1)}{M^i_j}$ and $\overset{(2)}{M^i_j}$ are changing by the following rules (see [16]):

$$\begin{aligned} \frac{\partial z'^i}{\partial z^k} \overset{(1)}{M^k_j} &= \overset{(1)}{M'^i_k} \frac{\partial z'^k}{\partial z^j} + \frac{\partial \eta'^i}{\partial z^j}; \\ \frac{\partial z'^i}{\partial z^k} \overset{(2)}{M^k_j} &= \overset{(2)}{M'^i_k} \frac{\partial z'^k}{\partial z^j} + \overset{(1)}{M'^i_k} \frac{\partial \eta'^k}{\partial z^j} + \frac{\partial \zeta'^i}{\partial z^j} \end{aligned} \tag{1.4}$$

The formulas which make the connection between $\overset{(1)}{N^i_j}, \overset{(2)}{N^i_j}$ and $\overset{(1)}{M^i_j}, \overset{(2)}{M^i_j}$ are: $\overset{(1)}{M^i_j} = \overset{(1)}{N^i_j}$ and $\overset{(2)}{M^i_j} = \overset{(2)}{N^i_j} + \overset{(1)}{N^i_k}\overset{(1)}{N^k_j}$. The notion of complex nonlinear connection is connected with the *complex spray* notion, which is defined as a field $S \in T'(J^{(2,0)}M)$ with property $F \circ S = \overset{(2)}{\mathcal{L}}$, where $\overset{(2)}{\mathcal{L}} = \eta^i \frac{\partial}{\partial \eta^i} + 2\zeta^i \frac{\partial}{\partial \zeta^i}$ is the Liouville field. The spray S has the coefficients G^i, thus $S = \eta^i \frac{\partial}{\partial z^i} + 2\zeta^i \frac{\partial}{\partial \eta^i} - 3G^i(z, \eta, \zeta) \frac{\partial}{\partial \zeta^i}$, and they are transformed by the rule:

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j} G^j - \left(\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j} \right). \tag{1.5}$$

In short, a normal complex nonlinear connection, N-(*c.l.c.*), is a derivative law which acts on $T_{\mathbf{C}}(J^{(2,0)}M)$ with respect to adapted frames, preserves the distributions and is well defined by the set of coefficients $D\Gamma = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^i_{jk}, F^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$ which are changing as follows:

$$L'^i_{jk} = \frac{\partial z'^i}{\partial z^r} \frac{\partial z^p}{\partial z'^j} \frac{\partial z^q}{\partial z'^k} L^r_{pq} + \frac{\partial z'^i}{\partial z^p} \frac{\partial^2 z^p}{\partial z'^j \partial z'^k} \tag{1.6}$$

and the others are d-tensors. For details see [16].

2. The complex Chern-Lagrange connection

In this section we will highlight two (c.n.c.) on $T_{\mathbf{C}}(J^{(2,0)}M)$ which will be very important in the geometry of the $J^{(2,0)}$- holomorphic bundle.

Proposition 2.1. *If* $\overset{(1)}{M}{}^i_j$ *and* $\overset{(2)}{M}{}^i_j$ *are the dual coefficients of a* (*c.n.c.*) *on* $J^{(2,0)}M$, *then a complex spray is given by:*

$$3G^i = \overset{(2)}{M}{}^i_j\, \eta^j + 2\, \overset{(1)}{M}{}^i_j\, \zeta^j. \tag{2.1}$$

Proof. We have to verify the changes (1.6), using (1.2) and (1.5).

$$\frac{\partial z'^i}{\partial z^k}(\overset{(2)}{M}{}^k_j\, \eta^j + 2\, \overset{(1)}{M}{}^k_j\, \zeta^j) - \left(\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j}\right)$$
$$= \overset{(2)}{M'}{}^i_k\, \frac{\partial z'^k}{\partial z^j}\eta^j + \overset{(1)}{M'}{}^i_k\, \frac{\partial \eta'^k}{\partial z^j}\eta^j + \frac{\partial \zeta'^i}{\partial z^j}\eta^j$$
$$+ 2\, \overset{(1)}{M'}{}^i_k\, \frac{\partial z'^k}{\partial z^j}\zeta^j + 2\frac{\partial \eta'^i}{\partial z^j}\zeta^j - \frac{\partial \zeta'^i}{\partial z^j}\eta^j - 2\frac{\partial \zeta'^i}{\partial \eta^j}\zeta^j = \overset{(2)}{M'}{}^i_k\, \eta'^k + 2\, \overset{(1)}{M'}{}^i_k\, \zeta^k,$$

which is just (1.6).

We used here $\frac{\partial \zeta'^i}{\partial \eta^j} = \frac{\partial \eta'^i}{\partial z^j}$ and $\frac{\partial \eta'^k}{\partial z^j}\eta^j = \frac{\partial \eta'^k}{\partial z^j}\frac{\partial z^j}{\partial z'^h}\eta'^h = \frac{\partial \eta'^k}{\partial z'^h}\eta'^h = 0.$ □

Conversely, any complex spray determines a (c.n.c):

Proposition 2.2. *If S is a complex spray with coefficients G^i which are changing by the rule* (1.5), *then*

$$\overset{(1)}{M}{}^i_j = \frac{\partial G^i}{\partial \zeta^j} \quad , \quad \overset{(2)}{M}{}^i_j = \frac{\partial G^i}{\partial \eta^j} \tag{2.2}$$

determine a (*c.n.c*) *with the dual coefficients* $\overset{(1)}{M}{}^i_j$ *and* $\overset{(2)}{M}{}^i_j$.

Proof. By differentiating (1.6) with respect to $\frac{\partial}{\partial \zeta^j} = \frac{\partial z'^k}{\partial z^j}\frac{\partial}{\partial \zeta'^k}$ we obtain $3\frac{\partial G'^i}{\partial \zeta'^k}\frac{\partial z'^k}{\partial z^j} = 3\frac{\partial z'^i}{\partial z^j}\frac{\partial G^k}{\partial \zeta^j} - \eta^k \frac{\partial^2 \zeta'^i}{\partial z^k \partial \zeta^j} - 2\frac{\partial \zeta'^i}{\partial \eta^j} - 2\zeta^k \frac{\partial^2 z'^i}{\partial \eta^k \zeta^j}$. If we take into account relations (1.2) and relations $\eta^k \frac{\partial^2 \zeta'^i}{\partial \zeta^j \partial z^k} = \eta^k \frac{\partial^2 z'^i}{\partial z^j \partial z^k}$, $\frac{\partial \zeta'^i}{\partial \eta^k} = \frac{\partial \eta'^i}{\partial z^k}$, it follows that $\frac{\partial^2 {\zeta'}^i}{\partial \eta^k \partial \zeta^j} = \frac{\partial}{\partial \eta^k}(\frac{\partial {z'}^i}{\partial z^j}) = 0$. Hence, the first (1.5) rule is fulfilled. Similarly, by differentiating (1.6) with respect to $\frac{\partial}{\partial \eta^j} = \frac{\partial {\eta'}^l}{\partial \eta^j}\frac{\partial}{\partial {\eta'}^l} + \frac{\partial {\zeta'}^l}{\partial \eta^j}\frac{\partial}{\partial {\zeta'}^l}$ we obtain the second rule from formulas (1.5), taking into account that $2\zeta^k \frac{\partial^2 {\zeta'}^i}{\partial \eta^k \partial \eta^j} = 2\zeta^k \frac{\partial}{\partial \eta^k}(\frac{\partial {\eta'}^i}{\partial z^j}) = 2\zeta^k \frac{\partial}{\partial z^j}(\frac{\partial {\eta'}^i}{\partial \eta^k}) = 2\frac{\partial}{\partial z^j}(\zeta^k \frac{\partial {z'}^i}{\partial z^k}) = 2\frac{\partial {\zeta'}^i}{\partial z^j}$ and $\eta^k \frac{\partial^2 {\zeta'}^i}{\partial z^k \partial \eta^j} = \frac{\partial z^k}{\partial {z'}^h}{\eta'}^h \frac{\partial^2 {\zeta'}^i}{\partial z^k \partial \eta^j} = {\eta'}^h \frac{\partial}{\partial {z'}^h}(\frac{\partial {\zeta'}^i}{\partial \eta^j}) = {\eta'}^h \frac{\partial}{\partial \eta^j}(\frac{\partial {\zeta'}^i}{\partial {z'}^h}) = 0$. □

Therefore, the problem of determining a (c.n.c.) on $J^{(2,0)}M$ is closely related with the problem of determining a complex spray. In the real case, [3], [4], [8], [9], [12], and in the complex Lagrange space (of first order), [10], one method to determine a spray to use a variational problem. A similar technique will be followed in this paper.

Definition 2.3. A complex second order Lagrange space is a pair (M, L), where $L : J^{(2,0)}M \to \mathbf{R}$ is a smooth function of order at least two, with the Hermitian matrix

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial \zeta^i \partial \bar{\zeta}^j} \tag{2.3}$$

non-degenerated.

Let $c(t)$ be a differentiable curve of class C^∞ on M and $\widetilde{c}(t)$ its extension at $J^{(2,0)}M$ defined by $t \in \mathbf{R} \to (z^i(t), \eta^i(t) = \frac{dz^i}{dt}, \zeta^i(t) = \frac{1}{2}\frac{d^2 z^i}{dt^2})$. Because t is a real parameter, the variational problem for the complex second order Lagrangian L leads us to very similar calculations as in the real case, i.e. to Euler-Lagrange equation:

$$\frac{\partial L}{\partial z^i} - \frac{d}{dt}\frac{\partial L}{\partial \eta^i} + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \zeta^i}\right) = 0. \tag{2.4}$$

Actually, $L(z, \eta, \zeta)$ depends implicitly on the conjugates of these variable. In this way, along the curve $\widetilde{c}$ we have

$$\frac{d}{dt} = \frac{dz^j}{dt}\frac{\partial}{\partial z^j} + \frac{d\bar{z}^j}{dt}\frac{\partial}{\partial \bar{z}^j} + \frac{d\eta^j}{dt}\frac{\partial}{\partial \eta^j} + \frac{d\bar{\eta}^j}{dt}\frac{\partial}{\partial \bar{\eta}^j} + \frac{d\zeta^j}{dt}\frac{\partial}{\partial \zeta^j} + \frac{d\bar{\zeta}^j}{dt}\frac{\partial}{\partial \bar{\zeta}^j}$$

or, taking into account that $\eta^j = \frac{dz^j}{dt}$ and $2\zeta^j = \frac{d^2 z^j}{dt^2} = \frac{d\eta^j}{dt}$, using (as in the real case) the operator $\Gamma = \eta^j \frac{\partial}{\partial z^j} + 2\zeta^j \frac{\partial}{\partial \eta^j}$ along the curve $\widetilde{c}$, we have:

$\frac{d}{dt} = \Gamma + \overline{\Gamma} + 3\frac{d^3 z^j}{dt^3}\frac{\partial}{\partial \zeta^j} + 3\frac{d^3 \overline{z}^j}{dt^3}\frac{\partial}{\partial \overline{\zeta}^j}$. As a result, the Euler-Lagrange equation (2.4) is rewritten:

$$\frac{\partial L}{\partial z^i} - \frac{d}{dt}\left\{\frac{\partial L}{\partial \eta^i} - \Gamma(\frac{\partial L}{\partial \zeta^i}) - \overline{\Gamma}(\frac{\partial L}{\partial \zeta^i}) - 3g_{ij}\frac{d^3 z^j}{dt^3} - 3g_{i\overline{j}}\frac{d^3 \overline{z}^j}{dt^3}\right\} = 0 \tag{2.5}$$

where $g_{ij} := \frac{\partial^2 L}{\partial \zeta^i \partial \zeta^j}$. As we will see below, the vanishing of the bracket in (2.5), named as in the real case, the complex Craig-Synge covector:

$$E_i(L) = -\frac{\partial L}{\partial \eta^i} + \frac{d}{dt}\left(\frac{\partial L}{\partial \zeta^i}\right) \tag{2.6}$$

will play a fundamental role in determining the (c.n.c.). If we ignore the general expression of the $\frac{d}{dt}$ along the $\widetilde{c}$, the equations $E_i(L) = 0$ are a consequence of the first order variational problem for the curve restriction $\widetilde{c}$ at the distributions $W(J^{(2,0)}M)$. In the complex Finsler spaces ([13]), Royden studied the problem of complex geodesics which are holomorphic curves $c : \Delta_r \to M$ with the property that $\gamma(t) = c(e^{i\theta}t)$ is tangent at all lines from z, $\forall \theta \in \mathbf{R}$. This leads to simultaneous cancellation of the Hermitian and nonhermitian terms in the Euler-Lagrange equation. If we use the same reasoning for the equations $E_i(L) = 0$ from (2.5), then we have the system of equations:

$$3g_{ij}\frac{d^3 z^j}{dt^3} + \Gamma\left(\frac{\partial L}{\partial \zeta^i}\right) - \frac{\partial L}{\partial \eta^i} = 0; \tag{2.7}$$

$$3g_{i\overline{j}}\frac{d^3 \overline{z}^j}{dt^3} + \overline{\Gamma}\left(\frac{\partial L}{\partial \zeta^i}\right) = 0. \tag{2.8}$$

For the moment, we leave the equations (2.7) as an algebraic requirement. In a complex Finsler space, an analogous condition to the first requirement from the formulas (2.7) is equivalent with the weakly Kahler metrics, [1], [2], [10], [17]. By conjugation, the second condition from the formulas (2.8) gives:

$$\frac{d^2 \eta^i}{dt^2} + 2G^i(z(t), \eta(t), \zeta(t)) = 0 \text{ , where } 3G^i = g^{\overline{m}i}\Gamma\left(\frac{\partial L}{\partial \overline{\zeta}^m}\right) \tag{2.9}$$

that is $3G^i = g^{\overline{m}i}\frac{\partial^2 L}{\partial z^j \partial \overline{\zeta}^m}\eta^j + 2g^{\overline{m}i}\frac{\partial^2 L}{\partial \eta^j \partial \overline{\zeta}^m}\zeta^j$.

Theorem 2.4. *The pair* $\overset{(1)}{M}{}^i_j$, $\overset{(2)}{M}{}^i_j$ *determines the dual coefficients of a* (*c.n.c.*), *named Chern-Lagrange connection, where*

$$\overset{(1)}{M}{}^i_j = g^{\overline{m}i}\frac{\partial^2 L}{\partial\eta^j\partial\overline{\zeta}^m} \;\; ; \;\; \overset{(2)}{M}{}^i_j = g^{\overline{m}i}\frac{\partial^2 L}{\partial z^j\partial\overline{\zeta}^m}. \tag{2.10}$$

Proof. By using (1.2) and direct calculus, we verify the (1.5) rules of transformation for the coefficients of the (c.n.c.). We have:

$$\begin{aligned}\frac{\partial z'^i}{\partial z^k}\overset{(1)}{M}{}^k_j &= \frac{\partial z'^i}{\partial z^k}g^{\overline{m}k}\frac{\partial^2 L}{\partial\overline{\zeta}^m\partial\eta^j}\\ &= \frac{\partial z'^i}{\partial z^k}\frac{\partial\overline{z}'^p}{\partial\overline{z}^m}g^{\overline{m}k}\frac{\partial}{\partial\overline{\zeta}'^p}\left(\frac{\partial\eta'^h}{\partial\eta^j}\frac{\partial L}{\partial\eta'^h}+\frac{\partial\zeta'^h}{\partial\eta^j}\frac{\partial L}{\partial\zeta'^h}\right)\\ &= g'^{\overline{p}i}\frac{\partial\eta'^h}{\partial\eta^j}\frac{\partial^2 L}{\partial\eta'^h\partial\overline{\zeta}'^p}+g'^{\overline{p}i}\frac{\partial\zeta'^h}{\partial\eta^j}g'_{h\overline{p}} = \overset{(1)}{M}{}'^i_h\frac{\partial\eta'^h}{\partial\eta^j}+\frac{\partial\zeta'^i}{\partial\eta^j}\end{aligned}$$

which is the first condition from (1.5). Analogously we have:

$$\begin{aligned}\frac{\partial z'^i}{\partial z^k}\overset{(2)}{M}{}^k_j &= \frac{\partial z'^i}{\partial z^k}g^{\overline{m}k}\frac{\partial^2 L}{\partial\overline{\zeta}^m\partial z^j}\\ &= g'^{\overline{p}i}\frac{\partial}{\partial\overline{\zeta}'^p}\left(\frac{\partial z'^h}{\partial z^j}\frac{\partial L}{\partial z'^h}+\frac{\partial\eta'^h}{\partial z^j}\frac{\partial L}{\partial\eta'^h}+\frac{\partial\zeta'^h}{\partial z^j}\frac{\partial L}{\partial\zeta'^h}\right)\\ &= \overset{(2)}{M}{}'^i_h\frac{\partial z'^h}{\partial z^j}+\overset{(1)}{M}{}'^i_h\frac{\partial\eta'^h}{\partial z^j}+g'^{\overline{p}i}\frac{\partial\zeta'^h}{\partial z^j}g'_{h\overline{p}}\end{aligned}$$

i.e. the second condition from (1.5). □

From Proposition 2.1 and the previous Theorem, we deduce that:

Corollary 2.5. *The functions* G^i *given by* (2.10) *define a complex spray on* $J^{(2,0)}M$, *called the canonical spray and denoted by* $\overset{c}{G}{}^i$. *Following the Proposition* 2.2, *we can obtain a sequence of* (*c.n.c.*). *The functions* $\overset{(1)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial\zeta^j}$ *and* $\overset{(2)c}{M}{}^i_j = \frac{\partial \overset{c}{G}{}^i}{\partial\eta^j}$ *will be called the coefficients of the canonical* (*c.n.c.*).

The terminology of the complex Chern-Lagrange nonlinear connection and the canonical one, used here, is purely formal and it was introduced by analogy with that from complex Lagrange spaces (of first order), [10].

Proposition 2.6. *If* $\frac{\delta}{\delta\eta^j} = \frac{\partial}{\partial\eta^j} - \overset{(1)}{N_j^i}\frac{\partial}{\partial\zeta^i}$ *is the adapted base of the complex Chern-Lagrange nonlinear connection, then*

$$\left[\frac{\delta}{\delta\eta^j}, \frac{\delta}{\delta\eta^k}\right] = 0. \tag{2.11}$$

Proof. We have

$$\left[\frac{\delta}{\delta\eta^j}, \frac{\delta}{\delta\zeta^k}\right] = \left(\frac{\delta \overset{(1)}{M_j^i}}{\delta\eta^k} - \frac{\delta \overset{(1)}{M_k^i}}{\delta\eta^j}\right)\frac{\delta}{\delta\zeta^i} =: \overset{(1)}{B}{}^i_{(jk)}\frac{\partial}{\partial\zeta^i}.$$

By direct calculus, using the relations (2.9), we find after the reduction of the terms, that: $\overset{(1)}{B}{}^i_{(jk)} = g^{\overline{m}i}(\overset{(1)}{N_k^h}\overset{(1)}{N_j^l} - \overset{(1)}{N_j^h}\overset{(1)}{N_k^l})\frac{\partial g_{l\overline{m}}}{\partial\zeta^h}$. Because $\frac{\partial g_{l\overline{m}}}{\partial\zeta^h}$ is symmetric in indices l and h and the bracket is anti-symmetric, changing l with h, we obtain that $\overset{(1)}{B}{}^i_{(jk)} = -\overset{(1)}{B}{}^i_{(jk)}$, so it cancels. □

For other brackets of the adapted frames, the computations are quite complicated.

Further on, we want to determine a derivation law which we wish to be a complex N-linear connection, with respect to the adapted frames of the Chern-Lagrange (c.n.c.). Let (M, L) be a complex second order Lagrange space with the metric tensor $g_{i\overline{j}}(z, \eta, \zeta)$ and $(\overset{(1)}{N_j^i} = \overset{(1)}{M_j^i}, \overset{(2)}{N_j^i} = \overset{(2)}{M_j^i} - \overset{(1)}{M_k^i}\overset{(1)}{M_j^k})$ the Chern-Lagrange (c.n.c.) given by (2.10). We set $\delta_{0i} := \frac{\delta}{\delta z^i}, \delta_{1i} := \frac{\delta}{\delta\eta^i}, \delta_{2i} := \frac{\delta}{\delta\zeta^i}$, and $\{dz^i, \delta\eta^i, \delta\zeta^i\}$ the dual basis adapted to the (c.n.c.) C-L. Then

$$G = g_{i\overline{j}}dz^i \otimes d\overline{z}^j + g_{i\overline{j}}\delta\eta^i \otimes \delta\overline{\eta}^j + g_{i\overline{j}}\delta\zeta^i \otimes \delta\overline{\zeta}^j \tag{2.12}$$

defines a metric structure on $T_{\mathbf{C}}(J^{(2,0)}M)$.

Theorem 2.7. *The set of the coefficients:*

$$L^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}\ ;\ F^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta\eta^k}\ ;\ C^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta\zeta^k} \tag{2.13}$$

and $L^{\overline{i}}_{\overline{j}k} = F^{\overline{i}}_{\overline{j}k} = C^{\overline{i}}_{\overline{j}k} = 0$, *define a* N *-(c.l.c.) on* $J^{(2,0)}M$, *which is a metric one by extension at* $T_{\mathbf{C}}(J^{(2,0)}M)$ *and it is of* $(1,0)$*-type, named the Chern-Lagrange connection.*

Proof. Let D be a derivation law on $J^{(2,0)}M$ which acts on the adapted fields from $T_{\mathbf{C}}(J^{(2,0)}M)$. D is metric if $DG=0$, i.e. $XG(Y,Z)=G(D_XY,Z)$ $+G(Y,D_XZ)$, $\forall X,Y,Z\in\Gamma T_{\mathbf{C}}(J^{(2,0)}M)$. If we consider D of (1,0)-type, i.e. $L^{\bar{i}}_{\bar{j}k}=F^{\bar{i}}_{\bar{j}k}=C^{\bar{i}}_{\bar{j}k}=0$, and if we take in turns $X=\delta_{0k},Y=\delta_{0j},Z=\delta_{0\overline{m}}$, then $X=\delta_{0k},Y=\delta_{1j},Z=\delta_{1\overline{m}}$ and $X=\delta_{0k},Y=\delta_{2j},Z=\delta_{2\overline{m}}$, respectively, it results that $\overset{(0)}{L}{}^i_{jk}=\overset{(1)}{L}{}^i_{jk}=\overset{(2)}{L}{}^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}$. Analogously, for the choice $X=\delta_{1k},Y=\delta_{1j},Z=\delta_{1\overline{m}}$ we find $\overset{(0)}{F}{}^i_{jk}=\overset{(1)}{F}{}^i_{jk}=\overset{(2)}{F}{}^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta\eta^k}$. Respectively, for the choice $X=\delta_{2k},Y=\delta_{2j},Z=\delta_{2\overline{m}}$ (and other choices) we obtain $\overset{(0)}{C}{}^i_{jk}=\overset{(1)}{C}{}^i_{jk}=\overset{(2)}{C}{}^i_{jk}=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta\zeta^k}$. To achieve the rules of a N-(c.l.c.), we must check that (2.12) satisfies (1.7) and the other components are complex d-tensors. This is a direct computation. For example:

$$\begin{aligned}L'^i_{jk}&=g'^{\overline{m}i}\frac{\delta g'_{j\overline{m}}}{\delta z'^k}=\frac{\partial\overline{z}'^m}{\partial\overline{z}^p}\frac{\partial z'^i}{\partial z^q}g^{\overline{p}q}\frac{\partial z^l}{\partial z'^k}\frac{\delta}{\delta z^l}\left(\frac{\partial z^h}{\partial z'^j}\frac{\partial\overline{z}^r}{\partial\overline{z}'^m}g_{h\overline{r}}\right)\\&=\frac{\partial z'^i}{\partial z^q}\frac{\partial z^l}{\partial z'^k}\frac{\partial z^h}{\partial z'^j}L^q_{hl}+\frac{\partial z^l}{\partial z'^k}\frac{\delta}{\delta z^l}\left(\frac{\partial z^h}{\partial z'^j}\right)\frac{\partial\overline{z}^r}{\partial\overline{z}'^m}g_{h\overline{r}}\frac{\partial\overline{z}'^m}{\partial\overline{z}^p}\frac{\partial z'^i}{\partial z^q}g^{\overline{p}q}\\&=\frac{\partial z'^i}{\partial z^q}\frac{\partial z^l}{\partial z'^k}\frac{\partial z^h}{\partial z'^j}L^q_{hl}+\frac{\partial z^l}{\partial z'^k}\frac{\delta}{\delta z^l}\left(\frac{\partial z^h}{\partial z'^j}\right)\frac{\partial z'^i}{\partial z^h}.\end{aligned}$$

But $\frac{\partial z'^i}{\partial z^h}\frac{\partial z^l}{\partial z'^k}\frac{\delta}{\delta z^l}(\frac{\partial z^h}{\partial z'^j})=\frac{\partial z'^i}{\partial z^h}\frac{\partial^2 z^h}{\partial z'^k\partial z'^j}$, knowing that $\frac{\delta}{\delta z^l}=\frac{\partial}{\partial z^l}-\overset{(1)}{N}{}^i_l\frac{\partial}{\partial\eta^i}-\overset{(2)}{N}{}^i_l\frac{\partial}{\partial\zeta^i}$. In this way, we verify (1.7). Analogously it is proved that all the others entities are complex d -tensors, i.e. $F'^i_{jk}=\frac{\partial z'^i}{\partial z^q}\frac{\partial z^l}{\partial z'^j}\frac{\partial z^h}{\partial z'^k}F^q_{lh}$, e.t.c. □

Proposition 2.8. *We have* $F^i_{jk}=\frac{\partial\overset{(1)}{M}{}^i_k}{\partial\zeta^j}$ *and* $L^i_{jk}=\frac{\partial\overset{(2)}{M}{}^i_k}{\partial\zeta^j}-\overset{(1)}{M}{}^l_k F^i_{jl}-\overset{(2)}{M}{}^l_k C^i_{jl}$.

Proof. First, we have

$$\begin{aligned}\frac{\partial\overset{(1)}{M}{}^i_k}{\partial\zeta^j}&=\frac{\partial}{\partial\zeta^j}\left(g^{\overline{m}i}\frac{\partial^2 L}{\partial\eta^k\partial\overline{\zeta}^m}\right)=g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial\eta^k}-g^{\overline{m}p}g^{\overline{q}i}\frac{\partial g_{p\overline{q}}}{\partial\zeta^j}\frac{\partial^2 L}{\partial\eta^k\partial\overline{\zeta}^m}\\&=g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial\eta^k}-g^{\overline{q}i}\frac{\partial g_{j\overline{q}}}{\partial\zeta^p}\overset{(1)}{N}{}^p_k=g^{\overline{m}i}\left(\frac{\partial g_{j\overline{m}}}{\partial\eta^k}-\overset{(1)}{N}{}^p_k\frac{\partial g_{j\overline{m}}}{\partial\zeta^p}\right)=g^{\overline{m}i}\left(\frac{\delta g_{j\overline{m}}}{\delta\eta^k}\right)=F^i_{jk}.\end{aligned}$$

For the second formula the steps are similar as above and use the fact that

$$g^{\overline{m}i}\left(\frac{\partial g_{j\overline{m}}}{\partial \eta^p}\right) = g^{\overline{m}i}\frac{\partial}{\partial \zeta^j}\left(\frac{\partial^2 L}{\partial \eta^p \partial \overline{\zeta}^m}\right)$$

$$= \frac{\partial}{\partial \zeta^j}\overset{(1)}{N_p^i} - g^{\overline{m}l}g^{\overline{q}i}\left(\frac{\partial g_{l\overline{q}}}{\partial \zeta^j}\frac{\partial^2 L}{\partial \eta^p \partial \overline{\zeta}^m}\right) = F^i_{jp} + \overset{(1)}{N_p^l}\, C^i_{lj}.$$

□

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