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BIHARMONIC HYPERSURFACES OF LP-SASAKIAN MANIFOLDS

BY

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Abstract. In this paper the biharmonic hypersurfaces of Lorentzian para-Sasakian manifolds are studied. We firstly find the biharmonic equation for a hypersurface which admits the characteristic vector field of the Lorentzian para-Sasakian as the normal vector field. We show that a biharmonic spacelike hypersurface of a Lorentzian para-Sasakian manifold with constant mean curvature is minimal. The biharmonicity condition for a hypersurface of a Lorentzian para-Sasakian manifold is investigated when the characteristic vector field belongs to the tangent hyperplane of the hypersurface. We find some necessary and sufficient conditions for a timelike hypersurface of a Lorentzian para-Sasakian manifold to be proper biharmonic. The nonexistence of proper biharmonic timelike hypersurfaces with constant mean curvature in a Ricci flat Lorentzian para-Sasakian manifold is proved.

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1. Introduction

A smooth map $\Psi : (M, g) \to (N, h)$ between two Riemannian manifolds is called harmonic if it is a critical point of the energy functional

$$E : C^{\infty}(M, N) \to R, E(\Psi) = \frac{1}{2} \int_M |d\Psi|^2 v_g$$

and characterized by the vanishing of the tension field $\tau(\Psi) = \operatorname{trace} \nabla d\Psi$, where $C^{\infty}(M, N)$ denotes the space of smooth maps, ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^{Ψ}. As a natural extension of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by EELLS and SAMPSON in [14]. Biharmonic maps are the critical points of the bienergy functional

$$E_2(\Psi) = \frac{1}{2}\int_M |\tau(\Psi)|^2 v_g.$$

JIANG [20, 21] derived the first variation formula for the bienergy and this formula shows that the Euler-Lagrange equation for the bienergy is $\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - \operatorname{trace} R^N(d\Psi, \tau(\Psi))d\Psi = 0$, where $\Delta = -\operatorname{trace}(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator on N. From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

Biharmonic submanifolds are special cases of biharmonic maps. In general, if the inclusion map $i : (M, i^*h) \to (N, h)$ is a biharmonic isometric immersion then the submanifold M of (N, h) is said to be a biharmonic submanifold. By a different point of view, CHEN [9] defined biharmonic submanifolds $M \subset E^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian, and proposed to classify all biharmonic submanifolds. If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

Concerning the proper biharmonic map, there are several non-existence results for non-positive sectional curvature codomains [39]. These non-existence theorems and the generalized Chen's conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal, encouraged the study of proper biharmonic submanifolds in non-negatively curved spaces, especially in spheres (see [6, 7, 15, 18, 28, 30]).

Despite non-existence of proper biharmonic submanifolds in Euclidean spaces ([20], [10], [9], [17], [13]), CHEN and ISHIKAWA in [10] gave many examples of proper biharmonic spacelike surfaces with constant mean curvature in the pseudo-Euclidean spaces E^4_t, $(t = 1, 2)$ (see also [19]). But biharmonicity may still imply minimality in some specific cases of semi-Euclidean spaces. For example, the authors in [10] proved that any biharmonic surface in E^3_t, $(t = 1, 2)$, is also minimal. In [12] it is shown that a nondegenerate biharmonic hypersurface of 4-dimensional pseudo-Euclidean space with diagonalizable shape operator must be minimal.

Biharmonic submanifolds in the 3-sphere S^3 are classified by Caddeo, Montaldo and Oniciuc [6]. The authors in [4] studied biharmonic submanifolds of the Euclidean sphere that satisfy certain geometric properties. It is shown in [5] that a constant mean curvature surface is proper-biharmonic in the unit Euclidean sphere S^4 if and only if it is minimal in a hypersphere $S^3(\frac{1}{\sqrt{2}})$. A full classification of proper biharmonic hypersurfaces in 4-dimensional space forms was obtained in [8]. In [31], the author studied the biharmonic hypersurfaces in a generic Riemannian manifold and showed that Chen's conjecture is true for totally umbilical hypersurfaces in an Einstein space. Also, for conformal biharmonic submanifolds see [32].

In contact geometry, there is a well known analog of real space form, namely a Sasakian space form. In particular, a simply connected three-dimensional Sasakian space form of constant holomorphic sectional curvature 1 is isometric to S^3. So this motivated the authors to study biharmonic submanifolds in Sasakian space forms ([18], [16], [34], [1], [2]).

Pseudo-Riemannian spaces especially the constant curvature ones, namely de Sitter, Minkowski, anti de Sitter space, play important roles in the general relativity. Ouyang [33] and Sun [36] studied the spacelike biharmonic submanifolds in the pseudo-Riemannian spaces. In [40] Zhang constructed examples of proper biharmonic hypersurfaces in the anti de Sitter space.

The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto in 1989 [24]. Also he introduced the notion of Lorentzian para-Sasakian (for short LP-Sasakian) manifold. Mihai and Roşca [26] defined the same notion independently and thereafter many authors [25, 27, 37, 38] studied Lorentzian para-Sasakian manifolds and their submanifolds. Especially, in [22] the authors studied biharmonic curves in LP-Sasakian manifolds and investigated proper biharmonic curves in the Lorentzian sphere S_4^1.

In the present paper we study the biharmonic timelike and spacelike hypersurfaces of Lorentzian para-Sasakian manifolds. The first section is devoted to preliminaries. In section 2, we find the biharmonic equation for a hypersurface which admits the characteristic vector field of the Lorentzian para-Sasakian as the normal vector field. We show that a biharmonic spacelike hypersurface of a Lorentzian para-Sasakian manifold with constant mean curvature is minimal. In section 3, the biharmonicity condition for a hypersurface of a Lorentzian para-Sasakian manifold is investigated when the characteristic vector field belongs to the tangent hyperplane of the hypersurface. We find some necessary and sufficient conditions for a timelike hypersurface of a Lorentzian para-Sasakian manifold to be proper biharmonic. The nonexistence of proper biharmonic timelike hypersurfaces with constant mean curvature in a Ricci flat Lorentzian para-Sasakian manifold is proved.

2. Preliminaries

2.1. Biharmonic maps

Let $\Psi : (M, g) \to (\overline{M}, \overline{g})$ be a smooth map between Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$. The tension field of Ψ is given by $\tau(\Psi) = \operatorname{trace} \nabla d\Psi$, where $\nabla d\Psi$ is the second fundamental form of Ψ defined by

$$\nabla d\Psi(X, Y) = \nabla^{\Psi}_{X} d\Psi(Y) - d\Psi(\nabla^{M}_{X} Y), \tag{2.1.1}$$

for all X, $Y \in \Gamma(TM)$. The tension field $\tau(\Psi)$ is a section of the pull-back bundle $\Psi^{-1}T\overline{M}$. Then a smooth map Ψ is said to be a harmonic map if its tension field vanishes. Also, it is well-known that Ψ is harmonic if and only if it is a critical point of the energy (integral) which is defined by

$$E(\Psi) = \frac{1}{2} \int_{\Omega} |d\Psi|^2 v_g,$$

for all compact domains $\Omega \subseteq M$. Here $|d\Psi|$ denotes the Hilbert-Schmidt norm of $d\Psi$ and v_g is the volume form of g (see [14]).

For any compact domain $\Omega \subseteq M$, the bienergy is defined by

$$E_2(\Psi) = \frac{1}{2} \int_{\Omega} |\tau(\Psi)|^2 v_g.$$

Then a smooth map Ψ is called biharmonic if it is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. The first variation formula for the bienergy is ([20], [21])

$$\frac{d}{dt} E_2(\Psi_t; \Omega)|_{t=0} = \int_{\Omega} \overline{g}(\tau_2(\Psi), w) v_g,$$

where v_g is the volume element, w is the variational vector field associated to the variation $\{\Psi_t\}$ of Ψ and

$$\tau_2(\Psi) = -J(\tau_2(\Psi)) = -\Delta^{\Psi}\tau(\Psi) - \operatorname{trace} \overline{R}(d\Psi, \tau(\Psi))d\Psi. \tag{2.1.2}$$

$\tau_2(\Psi)$ is called bitension field of Ψ. Here Δ^Ψ is the rough Laplacian on the sections of the pull-back bundle $\Psi^{-1}TN$ which is defined by

$$\Delta^\Psi V = -\operatorname{trace}\{\nabla^\Psi \nabla^\Psi V - \nabla^\Psi_\nabla V\}, \quad V \in \Gamma(\Psi^{-1}TN), \tag{2.1.3}$$

where ∇^Ψ is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\overline{R}$ is the Riemannian curvature operator of $\overline{M}$. It is obvious from (2.1.2) that any harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps. When the target manifold is semi-Riemannian manifold, the bienergy and bitension field can be defined in the same way.

A submanifold M of $(\overline{M}, \overline{g})$ is called a biharmonic submanifold if the inclusion map $\Psi : (M, g) \to (\overline{M}, \overline{g})$ is a biharmonic isometric immersion where $g = \Psi^* \overline{g}$.

Let $\Psi : M \to \overline{M}$ be an isometric immersion. Then the pull-back bundle can be written by $\Psi^{-1}T\overline{M} = \sigma M \oplus \nu M$, as an orthogonal decomposition of vector bundles. Here σM and νM denotes the tangent and normal bundles, respectively. $d\Psi$ can be used to identify TM with its image σM in the pull back bundle. Then we have $\nabla^\Psi_X(d\Psi(Y)) = \overline{\nabla}_X Y$, for all vector fields X, $Y \in \Gamma(TM)$. By using (2.1.1), we get $d\Psi\,(\nabla_X Y) + \nabla d\Psi(X, Y) = \overline{\nabla}_X Y$. Hence $\nabla d\Psi(X, Y)$ equals to the normal component of $\overline{\nabla}_X Y$. This is the second fundamental form $B(X, Y)$ of the immersed submanifold $\Psi(M)$ in $\overline{M}$. Therefore the second fundamental form of an isometric immersion $\Psi : M \to \overline{M}$ is equal to the second fundamental form of the immersed submanifold $\Psi(M)$ in $\overline{M}$ (see [3]). An isometric immersion is minimal if and only if it is harmonic. Hence, minimal submanifolds are automatically biharmonic. Then non-minimal biharmonic submanifolds are called proper biharmonic submanifolds.

2.2. Lorentzian Para-Sasakian manifolds

Let $\overline{M}$ be an $(m+1)$-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\overline{M}$ such that [24]

$$\eta(\xi) = -1, \tag{2.2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2.2}$$

where I denotes the identity map of $T_p\overline{M}$ and $\otimes$ is the tensor product. The equations (2.2.1) and (2.2.2) imply that

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \quad \text{rank}(\phi) = m. \tag{2.2.3}$$

Then $\overline{M}$ admits a Lorentzian metric $\overline{g}$, i.e., $\overline{g}$ is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p \in \overline{M}$, the tensor $\overline{g}_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate inner product of index 1, where $T_p\overline{M}$ is the tangent space of $\overline{M}$ at the point p, such that

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) + \eta(X)\eta(Y), \tag{2.2.4}$$

and $\overline{M}$ is said to admit a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$. Then we get

$$\begin{aligned} &\overline{g}(X, \xi) = \eta(X), \\ &\Phi(X, Y) = \overline{g}(X, \phi Y) = \overline{g}(\phi X, Y) = \Phi(Y, X), \\ &(\overline{\nabla}_X \Phi)(Y, Z) = g(Y, (\overline{\nabla}_X \phi)Z) = (\overline{\nabla}_X \Phi)(Z, Y), \end{aligned} \tag{2.2.5}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to $\overline{g}$. A non-zero vector $X_p \in T_p\overline{M}$ is called spacelike, null or timelike, if it satisfies $\overline{g}_p(X_p, X_p) > 0$, $\overline{g}_p(X_p, X_p) = 0$ $(X_p \neq 0)$ or $\overline{g}_p(X_p, X_p) < 0$, respectively. It is clear that the Lorentzian metric $\overline{g}$ makes ξ a timelike unit vector field, i.e, $\overline{g}(\xi, \xi) = -1$. The manifold $\overline{M}$ equipped with a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian almost paracontact manifold (for short LAP-manifold) [23], [24].

In equations (2.2.1) and (2.2.2) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by Satō [35].

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian paracontact manifold (for short LP-manifold) [24] if

$$\Phi(X, Y) = \frac{1}{2}((\overline{\nabla}_X \eta)Y + (\overline{\nabla}_Y \eta)X). \tag{2.2.6}$$

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian para-Sasakian manifold (for short LP-Sasakian) [24] if

$$(\overline{\nabla}_X \phi)Y = \eta(Y)X + \overline{g}(X, Y)\xi + 2\eta(X)\eta(Y)\xi. \tag{2.2.7}$$

In an LP-Sasakian manifold the 1-form η is closed and

$$\overline{\nabla}_X \xi = \phi X. \tag{2.2.8}$$

Also, an LP-Sasakian manifold $\overline{M}$ is said to be η-Einstein if its Ricci tensor $\overline{S}$ satisfies

$$\overline{S}(X,Y) = a\overline{g}(X,Y) + b\eta(X)\eta(Y), \tag{2.2.9}$$

for any vector fields X, Y where a, b are functions on $\overline{M}$. The Ricci tensor of an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold is given by [27]

$$\overline{S}(X,Y) = (\frac{\overline{r}}{m} - 1)\overline{g}(X,Y) + (\frac{\overline{r}}{m} - (m+1))\eta(X)\eta(Y), \tag{2.2.10}$$

where $\overline{r}$ is the scalar curvature of the manifold.

In an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$ with the structure $(\phi, \xi, \eta, \overline{g})$, the following relations hold [11], [24]:

$$\overline{g}(\overline{R}(X,Y)Z, \xi) = \eta(\overline{R}(X,Y)Z) = \overline{g}(Y,Z)\eta(X) - \overline{g}(X,Z)\eta(Y), \tag{2.2.11}$$

$$\overline{R}(\xi, X)Y = \overline{g}(X,Y)\xi - \eta(Y)X \tag{2.2.12}$$

$$\overline{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.2.13}$$

$$\overline{R}(\xi, X)\xi = X + \eta(X)\xi, \tag{2.2.14}$$

$$\overline{S}(X, \xi) = m\eta(X), \tag{2.2.15}$$

$$\overline{S}(\phi X, \phi Y) = \overline{S}(X,Y) + m\eta(X)\eta(Y) \tag{2.2.16}$$

for any vector fields X, Y, Z in $\overline{M}$ where $\overline{R}$ and $\overline{S}$ are the Riemannian curvature and the Ricci tensors of $\overline{M}$, respectively.

A semi-Riemannian hypersurface of a semi-Riemannian manifold is just a semi-Riemannian submanifold of codimension 1. It is well known that a Lorentzian manifold is a semi-Riemannian manifold with a symmetric nondegenerate $(0,2)$ tensor field, namely metric tensor, of index 1. Let M be a hypersurface of a Lorentzian manifold $\overline{M}$. If the normal vector field of M is timelike (respectively, spacelike) then M is called a spacelike (respectively, timelike) hypersurface of $\overline{M}$ (see [29]).

Let M be a hypersurface of an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$. The Gauss and Weingarten formula formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.2.17}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2.18}$$

for each X, $Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, B is the second fundamental form of M and A_N is the shape operator with respect to the normal section N. We can write $B(X,Y) = b(X,Y)N$, where b is the function-valued second fundamental form of M. Then the second fundamental form B and the shape operator of the hypersurface with respect to the unit normal vector field N are related by

$$B(X,Y) = \varepsilon\overline{g}(\overline{\nabla}_X Y, N)N = -\varepsilon\overline{g}(Y, \overline{\nabla}_X N)N = \varepsilon\overline{g}(A_N X, Y)N \tag{2.2.19}$$

and

$$\overline{g}(A_N X, Y) = \overline{g}(B(X,Y), N) = \overline{g}(b(X,Y)N, N) = \varepsilon b(X,Y), \tag{2.2.20}$$

where X, $Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$ and $\varepsilon = \overline{g}(N,N)$.

3. Biharmonic spacelike hypersurfaces in LP-Sasakian manifolds

In this section we consider that the characteristic vector field of the LP-Sasakian manifold is the unit normal vector field of the hypersurface. Hence, we characterize the spacelike biharmonic hypersurfaces in a Lorentzian para-Sasakian (LP-Sasakian) manifold.

Theorem 3.1. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and $\Psi : M \to \overline{M}$ be an isometric immersion with $\dim M = m$. Assume that the characteristic vector field ξ is the unit normal vector field of the hypersurface M. Then the spacelike hypersurface M is biharmonic if and only if*

$$\Delta H - 2mH = 0, \quad 2A(\mathrm{grad}H) - \frac{m}{2}(\mathrm{grad}H^2) = 0, \tag{3.1}$$

where A is the shape operator of the hypersurface with respect to the unit normal vector field ξ and $\mu = H\xi$ is the mean curvature vector.

Proof. Let M be a hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field ξ and $\Psi : M \to \overline{M}$ be an isometric immersion. Assume that $\{e_i\}_{i=1}^m$ is a local orthonormal frame of M such that $\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$ is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$. We identify $d\Psi(X)$ by X and $\nabla^{\Psi}_X W$ by $\overline{\nabla}_X W$ for all $X \in \Gamma(TM)$, $W \in \Gamma(\Psi^{-1}T\overline{M})$. Note that the tension field of Ψ is $\tau(\Psi) = mH\xi$. Then the bitension field of $\Psi : M \to \overline{M}$ is as follows:

$$\begin{aligned}
\tau_2(\Psi) &= \sum_{i=1}^{m}\{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}\tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}\tau(\Psi) - \overline{R}(d\Psi(e_i), \tau(\Psi))d\Psi(e_i)\}\\
&= \sum_{i=1}^{m}\{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}(mH\xi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\}\\
&= \sum_{i=1}^{m}\{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(mH\xi) - \overline{\nabla}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\}\\
&= m\sum_{i=1}^{m}\{\overline{\nabla}_{e_i}\left(e_i(H)\xi + H\overline{\nabla}_{e_i}\xi\right) - (\nabla_{e_i}e_i)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi\\
&\quad - H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\} \qquad (3.2)\\
&= m\sum_{i=1}^{m}\{e_ie_i(H)\xi + 2e_i(H)\overline{\nabla}_{e_i}\xi + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi\\
&\quad - (\nabla_{e_i}e_i)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi - H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\}\\
&= -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(\mathrm{grad}H)\\
&\quad + mH\sum_{i=1}^{m}\overline{R}(\xi, d\Psi(e_i))d\Psi(e_i).
\end{aligned}$$

Since $\overline{M}$ is a LP-Sasakian manifold then from (2.2.12), we have

$$\sum_{i=1}^{m}\overline{R}(\xi, d\Psi(e_i))d\Psi(e_i) = m\xi. \tag{3.3}$$

By writing (3.3) in (3.2), we get

$$\tau_2(\Psi) = -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(\mathrm{grad}H) + m^2H\xi. \tag{3.4}$$

Now, to compute the tangential and normal parts of the bitenson field, it suffices to find only the normal and tangential parts of $\Delta^{\Psi}\xi$:

From (2.2.8) we have

$$\begin{aligned}
\overline{g}(\Delta^{\Psi}\xi, \xi) &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi, \xi) = -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi, \xi)\\
&= \sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi, \overline{\nabla}_{e_i}\xi) = \sum_{i=1}^{m}\overline{g}(\phi e_i, \phi e_i). \qquad (3.5)
\end{aligned}$$

By using (2.2.4) in (3.5), then the normal part of $\Delta^{\Psi}\xi$ is

$$\left(\Delta^{\Psi}\xi\right)^{\perp} = -\overline{g}(\Delta^{\Psi}\xi,\xi)\xi = -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi,\overline{\nabla}_{e_i}\xi)\xi = -m\xi. \tag{3.6}$$

The tangential part of $\Delta^{\Psi}\xi$ can be calculated by

$$\begin{aligned}\left(\Delta^{\Psi}\xi\right)^{\top} &= -\sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi, e_k)e_k\\ &= \sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}Ae_i - A(\nabla_{e_i}e_i), e_k)e_k\\ &= \sum_{i,k=1}^{m}\{e_i\overline{g}(Ae_i,e_k) - \overline{g}(Ae_i,\nabla_{e_i}e_k) - \overline{g}(A(\nabla_{e_i}e_i),e_k)\}e_k \qquad (3.7)\\ &= \sum_{i,k=1}^{m}\{-e_ib(e_i,e_k) + b(e_i,\nabla_{e_i}e_k) + b(\nabla_{e_i}e_i,e_k)\}e_k\\ &= -\sum_{i,k=1}^{m}\{\nabla_{e_i}b(e_k,e_i)\}e_k.\end{aligned}$$

By Codazzi-Mainardi equation, we have

$$\begin{aligned}\sum_{i=1}^{m}(\nabla_{e_i}b(e_k,e_i) - \nabla_{e_k}b(e_i,e_i)) &= -\sum_{i=1}^{m}\overline{g}\left(\overline{R}(e_i,e_k)e_i,\xi\right)\\ &= \overline{S}(\xi,e_k). \qquad (3.8)\end{aligned}$$

Since $\overline{S}(\xi,e_k) = 0$, (3.8) implies that

$$\sum_{i=1}^{m}(\nabla_{e_i}b(e_k,e_i) - \nabla_{e_k}b(e_i,e_i)) = 0. \tag{3.9}$$

If we write (3.9) in (3.7), we get $\left(\Delta^{\Psi}\xi\right)^{\top} = -m\,\mathrm{grad}H$. Finally by considering all these parts, we have the tangential and normal components of the bitension field as follows:

$$\begin{aligned}(\tau_2(\Psi))^{\top} &= -2mA(\mathrm{grad}H) + \frac{m^2}{2}(\mathrm{grad}H^2),\\ (\tau_2(\Psi))^{\perp} &= (-m(\Delta H) + 2m^2H)\xi.\end{aligned}$$

This completes the proof. □

Example 3.1. Let $\overline{M} = \mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x, y, z). We define

$$\eta = dz\ , \quad \xi = -\frac{\partial}{\partial z}, \quad \phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x},$$
$$\phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0, \quad \overline{g} = e^{-2z}\,(dx)^2 + e^{2z}\,(dy)^2 - (dz)^2\,.$$

Then $\overline{M}$ is an LP-Sasakian manifold with an LP-Sasakian structure $(\phi, \xi, \eta, \overline{g})$.

Let M be a surface of $\overline{M}$ defined by $z = c$ where $c > 0$ is a constant and $\Psi : M \to \overline{M}$ be the isometric immersion with $\Psi(x, y) = (x, y, c)$. We can easily check that the induced metric is given by

$$g_{11} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \overline{g}\left(d\Psi\left(\frac{\partial}{\partial x}\right), d\Psi\left(\frac{\partial}{\partial x}\right)\right) \circ \Psi = e^{-2z},$$
$$g_{12} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \overline{g}\left(d\Psi\left(\frac{\partial}{\partial x}\right), d\Psi\left(\frac{\partial}{\partial y}\right)\right) \circ \Psi = 0,$$
$$g_{22} = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \overline{g}\left(d\Psi\left(\frac{\partial}{\partial y}\right), d\Psi\left(\frac{\partial}{\partial y}\right)\right) \circ \Psi = e^{2z}.$$

One can also see that

$$f_1 = e^z\frac{\partial}{\partial x}, \quad f_2 = e^{-z}\frac{\partial}{\partial y}, \quad f_3 = \frac{\partial}{\partial z}$$

constitute an orthonormal frame on $\overline{M}$ adapted to the surface M with $\xi = -f_3$ being unit normal vector field. Thus M becomes a spacelike surface of the LP-Sasakian manifold $\overline{M}$. By a further computation we have the following Lie brackets

$$[f_1, f_2] = 0, \qquad [f_1, f_3] = -e^z f_1, \qquad [f_2, f_3] = -e^{-z} f_2,$$

and the coefficients of Levi-Civita connection

$$\begin{array}{lll} \overline{\nabla}_{f_1} f_1 = -f_3, & \overline{\nabla}_{f_1} f_2 = 0, & \overline{\nabla}_{f_1} f_3 = -f_1 \\ \overline{\nabla}_{f_2} f_1 = 0, & \overline{\nabla}_{f_2} f_2 = f_3, & \overline{\nabla}_{f_2} f_3 = -f_2 \\ \overline{\nabla}_{f_3} f_1 = 0, & \overline{\nabla}_{f_3} f_2 = 0, & \overline{\nabla}_{f_3} f_3 = 0. \end{array}$$

Since $\xi = -f_3$ is the unit normal vector field we can compute the components of the second fundamental form as

$$b(f_1, f_1) = -\overline{g}(\overline{\nabla}_{f_1} f_1, \xi) = -1,$$
$$b(f_1, f_2) = -\overline{g}(\overline{\nabla}_{f_1} f_2, \xi) = 0, \tag{3.10}$$
$$b(f_2, f_2) = -\overline{g}(\overline{\nabla}_{f_2} f_2, \xi) = 1.$$

From (3.10) we obtain $H = \frac{1}{2}\left[b(f_1, f_1) + b(f_2, f_2)\right] = 0$, where H is the mean curvature of the isometric immersion Ψ. Hence, the upper half space with the LP-Sasakian structure $(\phi, \xi, \eta, \overline{g})$ is foliated by minimal so biharmonic planes $z = c$.

Corollary 3.1. *A spacelike hypersurface of an LP-Sasakian manifold with a harmonic mean curvature is biharmonic if and only if it is minimal.*

From Corollary 3.1. it is obvious that biharmonic spacelike hypersurfaces of LP-Sasakian manifolds with a constant mean curvature are minimal.

Corollary 3.2. *Let M be a spacelike hypersurface of an $(m+1)$ dimensional LP-Sasakian manifold with satisfying $\Delta H = 2mH$. Then M is a biharmonic spacelike hypersurface if and only if*

$$A(\text{grad}H) = \frac{m}{4}(\text{grad}H^2). \tag{3.11}$$

Theorem 3.2. *Let M be a totally umbilic biharmonic spacelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with dimension $(m+1)$. Then M is minimal.*

Proof. Assume that $\{e_i\}_{i=1}^m$ is a local orthonormal frame of M such that $\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$ is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$ where $\Psi : M \to \overline{M}$ is an isometric immersion. By identifying $d\Psi(X)$ by X, for all X in TM, we have an orthonormal basis$\{e_1, e_2, ..., e_m, \xi\}$ for the ambient manifold $\overline{M}$ such that $Ae_i = \lambda_i e_i$, where A is the shape operator of M and λ_i, $(1 \leq i \leq m)$, is the principal curvatures in the direction of e_i. Since M is totally umbilical then all the principal curvatures at any point p of M are equal to the same number $\lambda(p)$. Then by taking ξ instead of N in (2.2.19) we have

$$H = -\frac{1}{m}\sum_{i=1}^{m} g(B(e_i, e_i), \xi) = -\frac{1}{m}\sum_{i=1}^{m} g(Ae_i, e_i) \tag{3.12}$$

$$= -\frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i, e_i) = -\lambda.$$

On the other hand by using (3.12), we get

$$A(\text{grad}H) = -\frac{1}{2}\text{grad}\lambda^2. \tag{3.13}$$

Since M is a biharmonic spacelike hypersurface of $\overline{M}$, from (3.1), (3.12) and (3.13), we obtain $\Delta\lambda - 2m\lambda = 0$, $(2+m)\text{grad}\lambda^2 = 0$, which completes the proof. □

4. Biharmonic timelike hypersurfaces in LP-Sasakian manifolds

Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be a hypersurface of $\overline{M}$. Assume that the characteristic vector field of $\overline{M}$ belongs to the tangent hyperplane of the hypersurface M and N is the unit normal vector field of the manifold. Since N is spacelike then M becomes timelike hypersurface of $\overline{M}$.

We note that the tension field of the isometric immersion $\Psi : M \to \overline{M}$ is $\tau(\Psi) = m\mu$, where $\mu = HN$ is the mean curvature vector field with the mean curvature function H.

Theorem 4.1. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be its timelike hypersurface. Then M is a biharmonic hypersurface of $\overline{M}$ if and only if*

$$\begin{gathered} \tfrac{m}{2}(\text{grad}H^2) + 2A(\text{grad}H) - 2H(\overline{Q}(N)) = 0, \\ \Delta H + H\,|A|^2 - H(\overline{S}(N,N)) = 0, \end{gathered} \tag{4.1}$$

where $\overline{S}$ is the Ricci curvature of the LP-Sasakian manifold $\overline{M}$, $\overline{Q}$ is the Ricci operator of $\overline{M}$ defined by $\overline{g}(\overline{Q}X, Y) = \overline{S}(X, Y)$ and A is the shape operator of the hypersurface with respect to the unit normal vector field N.

Proof. Assume that M is a timelike hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field N and $\Psi : M \to \overline{M}$ be an isometric immersion. Consider $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an local orthonormal basis for the hypersurface. Since the tension field of Ψ is $\tau(\Psi) = mHN$, we have

$$
\begin{aligned}
\tau_2(\Psi) &= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}\tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}\tau(\Psi) - \overline{R}(d\Psi(e_i),\tau(\Psi))d\Psi(e_i)\} \\
&= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}(mHN) - \nabla^{\Psi}_{\nabla_{e_i}e_i}(mHN) \\
&- \overline{R}(d\Psi(e_i), mHN)d\Psi(e_i)\} \\
&= \sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(mHN) - \overline{\nabla}_{\nabla_{e_i}e_i}(mHN) \\
&- \overline{R}(d\Psi(e_i), mHN)d\Psi(e_i)\} \\
&= m\sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i}\left(e_i(H)N + H\overline{\nabla}_{e_i}N\right) - (\nabla_{e_i}e_i)(H)N \\
&- H\overline{\nabla}_{\nabla_{e_i}e_i}N - H\overline{R}(d\Psi(e_i), N)d\Psi(e_i)\} \\
&= m\sum_{i=1}^{m} \varepsilon_i \{e_ie_i(H)N + 2e_i(H)\overline{\nabla}_{e_i}N + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N \\
&- (\nabla_{e_i}e_i)(H)N - H\overline{\nabla}_{\nabla_{e_i}e_i}N - H\overline{R}(d\Psi(e_i), N)d\Psi(e_i)\} \\
&= -m(\Delta H)N - mH\Delta^{\Psi}N - 2mA(\mathrm{grad}H) \\
&- mH\left\{\sum_{i=1}^{m-1}\overline{R}(d\Psi(e_i), N)d\Psi(e_i) - \overline{R}(d\Psi(\xi), N)d\Psi(\xi)\right\},
\end{aligned} \tag{4.2}
$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$, $\overline{R}$ is the Riemannian curvature tensor of $\overline{M}$ and ∇^{Ψ} is the pull-back connection.

Now we shall compute the tangential and normal components of the $\Delta^{\Psi}N$ and the curvature term, respectively:

The tangential part of $\Delta^{\Psi}N$ can be calculated by

$$
\begin{aligned}
\left(\Delta^{\Psi}N\right)^{\top} &= -\sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N - \overline{\nabla}_{\nabla_{e_i}e_i}N, e_k)e_k \\
&= \sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}Ae_i - A(\nabla_{e_i}e_i), e_k)e_k \\
&= \sum_{i,k=1}^{m} \{e_i\overline{g}(Ae_i, e_k) - \overline{g}(Ae_i, \nabla_{e_i}e_k) - \overline{g}(A(\nabla_{e_i}e_i), e_k)\}e_k
\end{aligned} \tag{4.3}
$$

$$= \sum_{i,k=1}^{m} \{e_i b(e_i, e_k) - b(e_{i,e_i} e_k) - b(\nabla_{e_i} e_i, e_k)\} e_k$$
$$= \sum_{i,k=1}^{m} \{\nabla_{e_i} b(e_k, e_i)\} e_k,$$

where ∇ is the induced connection of the hypersurface and $b : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M, R)$ is the function valued second fundamental form such that $B(X, Y) = b(X, Y)N$, for all vector fields X, Y on M. By Codazzi-Mainardi equation, we have

$$\sum_{i=1}^{m} (\nabla_{e_i} b(e_k, e_i) - \nabla_{e_k} b(e_i, e_i)) = \sum_{i=1}^{m} \overline{g}\,(\overline{R}(e_i, e_k) e_i, N) = -\overline{S}(N, e_k). \tag{4.4}$$

which implies that

$$\sum_{i=1}^{m} \nabla_{e_i} b(e_k, e_i) = \sum_{i=1}^{m} \nabla_{e_k} b(e_i, e_i) - \overline{S}(N, e_k). \tag{4.5}$$

By using (4.5) in (4.3), we obtain

$$\left(\Delta^\Psi N\right)^\top = \sum_{i,k=1}^{m} \{\nabla_{e_k} b(e_i, e_i) - \overline{S}(N, e_k)\} e_k = m(\mathrm{grad} H) - (\overline{Q}(N)). \tag{4.6}$$

By straightforward computations, the normal part of the $\Delta^\Psi N$ is

$$\begin{aligned} (\Delta^\Psi N)^\perp &= \overline{g}(\Delta^\Psi N, N) N \\ &= -\sum_{i=1}^{m} \{\varepsilon_i \overline{g}(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N - \overline{\nabla}_{\nabla_{e_i} e_i} N, N)\} N \\ &= \sum_{i=1}^{m} \{\varepsilon_i \overline{g}(\overline{\nabla}_{e_i} N, \overline{\nabla}_{e_i} N)\} N = |A|^2 N, \end{aligned} \tag{4.7}$$

where $\varepsilon_i = \overline{g}(e_i, e_i)$, $1 \le i \le m$.

On the other hand, since

$$\begin{aligned} -\sum_{k=1}^{m-1} \overline{S}(N, e_k) e_k &= \sum_{i,k=1}^{m-1} \overline{g}(\overline{R}(d\Psi(e_i), N) d\Psi(e_i), e_k) e_k \\ &\quad - \sum_{i=1}^{m-1} \overline{g}(\overline{R}(d\Psi(e_i), N) d\Psi(e_i), \xi)\xi = (\overline{Q}(N))\xi \end{aligned} \tag{4.8}$$

and

$$\sum_{i=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),N)=-\overline{S}(N,N)+1 \tag{4.9}$$

then the tangential and normal components of the first curvature term in (4.2) are equal to $(\overline{Q}(N))$ and $-\overline{S}(N,N)+1$, respectively. Also, from (2.2.14) we have

$$\overline{R}(d\Psi(\xi),N)d\Psi(\xi)=N. \tag{4.10}$$

Finally, by reorganizing all the tangent and normal parts of the bitension field, we get

$$\begin{aligned}(\tau_2(\Psi))^{\top} &= -m\left[\frac{m}{2}(\mathrm{grad}H^2)+2A(\mathrm{grad}H)-2H(\overline{Q}(N))\right],\\ (\tau_2(\Psi))^{\perp} &= -m\left[(\Delta H)+H\left|A\right|^2-H\overline{S}(N,N)\right]N.\end{aligned}$$

This completes the proof. □

Corollary 4.1. *Let M be a timelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with constant mean curvature. Then M is a biharmonic timelike hypersurface if and only if either it is minimal or*

$$(\overline{Q}(N))=0 \ \ \textit{and}\ \overline{S}(N,N)=\left|A\right|^2. \tag{4.11}$$

In particular, if $\overline{M}$ has a nonpositive Ricci curvature, a timelike hypersurface of with constant mean curvature is biharmonic if and only if it is minimal.

Example 4.1. Let $\overline{M}=\mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x,y,z) and a LP-Sasakian structure $(\phi,\xi,\eta,\overline{g})$ given in the Example 3.1. Assume that M is a surface of $\overline{M}$ defined by $x=f(y)$. We can easily see that

$$\left\{f_1=\left(\frac{f'}{\sqrt{e^{-2z}(f')^2+e^{2z}}},\frac{1}{\sqrt{e^{-2z}(f')^2+e^{2z}}},0\right),\ f_2=(0,0,1)\right\}$$

constitute an orthonormal frame on M and

$$N=\left(\frac{e^{2z}}{\sqrt{(f')^2e^{-2z}+e^{2z}}},\frac{-f'e^{-2z}}{\sqrt{(f')^2e^{-2z}+e^{2z}}},0\right)$$

is the unit normal vector field of M. Since $f_2 = -\xi$ then M becomes a timelike surface of the LP-Sasakian manifold $\overline{M}$. By further computations we have

$$\begin{aligned}
&\overline{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -e^{-2z}\frac{\partial}{\partial z}, \quad \overline{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \quad \overline{\nabla}_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = -\frac{\partial}{\partial x},\\
&\overline{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 0, \quad \overline{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = e^{2z}\frac{\partial}{\partial z}, \quad \overline{\nabla}_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = \frac{\partial}{\partial y},\\
&\overline{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x} = -\frac{\partial}{\partial x}, \quad \overline{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}, \quad \overline{\nabla}_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = 0.
\end{aligned} \tag{4.12}$$

From (4.12) one can easily see that

$$\begin{aligned}
\overline{\nabla}_{f_1}f_1 = &\frac{-\left(f'\right)^2 e^{-2z}}{(e^{-2z}(f')^2 + e^{2z})}\frac{\partial}{\partial z}\\
&+\frac{1}{\sqrt{e^{-2z}(f')^2 + e^{2z}}}\frac{\partial}{\partial y}\left(\frac{f'}{\sqrt{e^{-2z}(f')^2 + e^{2z}}}\right)\frac{\partial}{\partial x}\\
&+\frac{1}{\sqrt{e^{-2z}(f')^2 + e^{2z}}}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{e^{-2z}(f')^2 + e^{2z}}}\right)\frac{\partial}{\partial y}\\
&+\frac{e^{2z}}{(e^{-2z}(f')^2 + e^{2z})}\frac{\partial}{\partial z}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\overline{\nabla}_{f_1}f_1 = &\frac{f''e^{2z}}{(e^{-2z}(f')^2 + e^{2z})^2}\frac{\partial}{\partial x}\\
&-\frac{f''f'e^{-2z}}{(e^{-2z}(f')^2 + e^{2z})^2}\frac{\partial}{\partial y} + \frac{e^{2z} - \left(f'\right)^2 e^{-2z}}{(e^{-2z}(f')^2 + e^{2z})}\frac{\partial}{\partial z}.
\end{aligned}$$

Also one can easily see that $\overline{\nabla}_{f_2}f_2 = 0$. Since N is the unit normal vector field of M we can compute the components of the second fundamental form as

$$b(f_1, f_1) = g(\overline{\nabla}_{f_1}f_1, N) = \frac{f''}{(e^{-2z}(f')^2 + e^{2z})^{\frac{3}{2}}}, \tag{4.13}$$

$$b(f_2, f_2) = g(\overline{\nabla}_{f_2}f_2, N) = 0. \tag{4.14}$$

From (4.13) and (4.14) we obtain

$$H = \frac{f''}{2\,(e^{-2z}(f')^2 + e^{2z})^{\frac{3}{2}}} \tag{4.15}$$

where H is the mean curvature of the isometric immersion Ψ. Hence M is minimal and so biharmonic if and only if $f(y) = cy + d$, where c, d are nonzero constants.

Corollary 4.2. *A timelike hypersurface of a Ricci flat LP-Sasakian manifold with a constant mean curvature is biharmonic if and only if it is minimal.*

Theorem 4.2. *Let $\overline{M}$ be an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold and M be a timelike hypersurface of $\overline{M}$. Then M is biharmonic if and only if*

$$(4.16)\quad \frac{m}{2}(\mathrm{grad}H^2)+2A(\mathrm{grad}H) = 0, \qquad (\Delta H)+H\,|A|^2-H(\frac{\overline{r}}{m}-1) = 0,$$

where $\overline{r}$ is the scalar curvature of $\overline{M}$. Particularly, if M is a timelike hypersurface with $0 \neq H =$constant, then M is a non-minimal biharmonic timelike hypersurface if and only if

$$(4.17)\qquad |A|^2 = \frac{\overline{r}}{m} - 1.$$

Proof. Assume that $\overline{M}$ be an $(m + 1)$-dimensional η-Einstein LP-Sasakian manifold. Then by using 2.2.10, we have

$$(4.18)\qquad \overline{S}(N,N) = \frac{\overline{r}}{m} - 1.$$

On the other hand

$$(4.19)\qquad (\overline{Q}(N)) = 0.$$

By using (4.18) and (4.19) in (4.1), we obtain the assertion of the theorem.□

Theorem 4.3. *Let $\overline{M}$ be an $(m + 1)$-dimensional LP-Sasakian space form and M be a timelike hypersurface of $\overline{M}$. Then M is biharmonic if and only if*

$$(4.20)\quad \frac{m}{2}(\mathrm{grad}H^2) + 2A(\mathrm{grad}H) = 0, \qquad (\Delta H) + H\,|A|^2 - mH = 0.$$

In particular, M is a hypersurface of $\overline{M}$ with a constant mean curvature, then M is a non-minimal biharmonic timelike hypersurface if and only if $|A|^2 = m$.

Proof. In an $(m+1)$-dimensional LP-Sasakian space form $\overline{M}$, since $\overline{S}(X,Y) = m\overline{g}(X,Y)$, for all vector fields X, Y , then $\overline{M}$ is an η-Einstein manifold with

$$\overline{r} = m(m+1). \tag{4.21}$$

Therefore, the biharmonic equation (4.16) reduces to (4.20). □

Theorem 4.4. *Let $\overline{M}$ be an $(m+1)$-dimensional* ($\dim \overline{M} > 2$) *η-Einstein LP-Sasakian manifold. A totally umbilical biharmonic timelike hypersurface of $\overline{M}$ has constant mean curvature.*

Proof. Let $\{e_1, e_2, ..., e_{m-1}, e_m = \xi, N\}$ be a local orthonarmal basis of η-Einstein LP-Sasakian manifold $\overline{M}$ such that $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an orthonormal frame for the hypersurface M. Since $\overline{M}$ is totally umbilical, we have $A = \lambda I$, where λ is a smooth function. Then

$$\begin{aligned} H &= \frac{1}{m}\sum_{i=1}^{m} \varepsilon_i \overline{g}(B(e_i,e_i),N) = \frac{1}{m}\sum_{i=1}^{m-1} \overline{g}(Ae_i,e_i) = \frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i,e_i) \\ &= \frac{m-1}{m}\lambda. \end{aligned} \tag{4.22}$$

From (4.22), we can write

$$A(\mathrm{grad} H) = \frac{m-1}{2m}\mathrm{grad}\lambda^2. \tag{4.23}$$

On the other hand, by straightforward calculations one can easily see that

$$|A|^2 = m\lambda^2. \tag{4.24}$$

By using (4.22), (4.23) and (4.24) in (4.16), we obtain

$$\left(\frac{m^2-1}{2m}\right)\mathrm{grad}\lambda^2 = 0, \qquad \Delta\lambda + m\lambda^3 - \left(\frac{\overline{r}}{m} - 1\right)\lambda = 0. \tag{4.25}$$

From (4.25), we have either $\lambda = 0$ and therefore $H = 0$, or $\lambda = \pm\frac{1}{m}\sqrt{\overline{r} - m} =$ constant. □

Corollary 4.3. *A totally umbilical timelike hypersurface of an η-Einstein LP-Sasakian manifold $\overline{M}$* ($\dim \overline{M} = m+1 > 2$) *with $\overline{r} < m$ is biharmonic if and only if it is minimal.*

Theorem 4.5. *A totally umbilical biharmonic timelike hypersurface of a Ricci flat LP-Sasakian manifold is minimal.*

Proof. Let $\overline{M}$ be an $(m+1)$-dimensional ($\dim \overline{M} > 2$) Ricci flat LP-Sasakian manifold and M be a totally umbilical biharmonic timelike hypersurface of $\overline{M}$. By using (4.22), (4.23) and (4.24) in (4.1), we have

$$\left(\frac{m^2-1}{2m}\right) \operatorname{grad} \lambda^2 = 0, \qquad (m-1)\left(\frac{1}{m}\Delta\lambda + \lambda^3\right) = 0. \tag{4.26}$$

Since M is biharmonic and $m > 1$ then by solving (4.26) we obtain $\lambda = 0$, hence $H = 0$. □

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REFERENCES

1. Arslan, K.; Ezentas, R.; Murathan, C.; Sasahara, T. – *Biharmonic submanifolds in 3-dimensional (κ,μ)-manifolds,* Int. J. Math. Math. Sci., 22 (2005), 3575–3586.

2. Arslan, K.; Ezentas, R.; Murathan, C.; Sasahara, T. – *Biharmonic anti-invariant submanifolds in Sasakian space forms,* Beiträge Algebra Geom., 48 (2007), 191–207.

3. Baird, P.; Wood, J.C. – *Harmonic Morphisms Between Riemannian Manifolds,* London Mathematical Society Monographs. New Series, 29, The Clarendon Press, Oxford University Press, Oxford, 2003.

4. Balmuş, A.; Montaldo, S.; Oniciuc, C. – *Classification results for biharmonic submanifolds in spheres,* Israel J. Math., 168 (2008), 201–220.

5. Balmuş, A.; Oniciuc, C. – *Biharmonic surfaces of $\mathbb{S}^4$,* Kyushu J. Math., 63 (2009), 339–345.

6. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds of S^3*, Internat. J. Math., 12 (2001), 867–876.

7. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds in spheres,* Israel J. Math., 130 (2002), 109–123.

8. Balmuş, A.; Montaldo, S.; Oniciuc, C. – *Biharmonic hypersurfaces in 4-dimensional space forms,* Math. Nachr., 283 (2010), 1696–1705.

9. Chen, B.-Y. – *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math., 17 (1991), 169–188.

10. Chen, B.-Y.; Ishikawa, S. – *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces,* Kyushu J. Math., 52 (1998), 167–185.

11. De, U.C.; Matsumoto, K.; Shaikh, A.A. – *On Lorentzian para-Sasakian Manifolds,* Rendiconti del Seminario Mat. di Messina, 3 (1999), 149–156.

12. Defever, F.; Kaimakamis, G.; Papantoniou, V. – *Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space* $\mathbb{E}^4_s$, J. Math. Anal. Appl., 315 (2006), 276–286.

13. Dimitrić, I. – *Submanifolds of* E^m *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica, 20 (1992), 53–65.

14. Eells, J. Jr.; Sampson, J.H. – *Harmonic mappings of Riemannian manifolds,* Amer. J. Math., 86 (1964), 109–160.

15. Fetcu, D. – *Biharmonic curves in the generalized Heisenberg group,* Beiträge Algebra Geom., 46 (2005), 513–521.

16. Fetcu, D.; Oniciuc, C. – *Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres,* Abh. Math. Sem. Univ. Hamburg, 77 (2007), 179–190.

17. Hasanis, Th.; Vlachos, Th. – *Hypersurfaces in* E^4 *with harmonic mean curvature vector field*, Math. Nachr., 172 (1995), 145–169.

18. Inoguchi, J.-I. – *Biharmonic curves in Minkowski 3-space*, Int. J. Math. Sci., 21 (2003), 1365–1368.

19. Ishikawa, S. – *Biharmonic* W*-surfaces in* 4*-dimensional pseudo-Euclidean space,* Mem. Fac. Sci. Kyushu Univ. Ser. A, 46 (1992), 269–286.

20. Jiang, G.Y. – *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A, 7 (1986), 130–144.

21. Jiang, G.Y. – *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A, 7 (1986), 389–402.

22. Keleş, S.; Perktaş, S.Y.; Kiliç, E. – *Biharmonic curves in Lorentzian para-Sasakian manifolds,* Bull. Malays. Math. Sci. Soc., 33 (2010), 325–344.

23. Matsumoto, K.; Mihai, I.; Roşca, R. – ξ*-null geodesic gradient vector fields on a Lorentzian para-Sasakian manifold*, J. Korean Math. Soc., 32 (1995), 17–31.

24. Matsumoto, K. – *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci., 12 (1989), 151–156.

25. Matsumoto, K.; Mihai, I. – *On a certain transformation in a Lorentzian para-Sasakian manifold*, Tensor (N.S.), 47 (1988), 189–197.

26. Mihai, I.; Roşca, R. – *On Lorentzian* P*-Sasakian manifolds,* Classical analysis (Kazimierz Dolny, 1991), 155–169, World Sci. Publ., River Edge, NJ, 1992.

27. Mihai, I.; Shaikh, A.A.; De, U.C. – *On Lorentzian para-Sasakian Manifolds,* Korean J. Math. Sciences, 6 (1999), 1–13.

28. Montaldo, S.; Oniciuc, C. – *A short survey on biharmonic maps between Riemannian manifolds,* Rev. Un. Mat. Argentina, 47 (2006), 1–22.

29. O'Neill, B. – *Semi-Riemannian Geometry. With Applications to Relativity,* Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

30. Ou, Y.-L. – *p-harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps*, J. Geom. Phys., 56 (2006), 358–374.

31. Ou, Y.-L. – *Biharmonic hypersurfaces in Riemannian manifolds,* Pacific J. Math., 248 (2010), 217–232.

32. Ou, Y.-L. – *On conformal biharmonic immersions,* Ann. Global Anal. Geom., 36 (2009), 133–142.

33. Ouyang, C.Z. – 2*-harmonic space-like submanifolds of a pseudo-Riemannian space form,* (Chinese) Chinese Ann. Math. Ser. A, 21 (2000), 649–654.

34. Sasahara, T. – *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen, 67 (2005), 285–303.

35. Satō, I. – *On a structure similar to the almost contact structure,* Tensor (N.S.), 30 (1976), 219–224.

36. Sun, H. – 2*-harmonic spacelike submanifolds in de Sitter space*, Journal of Southern Institute of Metallurgy, 21 (2000), 138–142.

37. Tripathi, M.M.; De, U.C. – *Lorentzian almost paracontact manifolds and their submanifolds,* J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 8 (2001), 101–125.

38. Tripathi, M.M.; Shukla, S.S. – *On submanifolds of Lorentzian almost paracontact manifolds,* Publ. Math. Debrecen, 59 (2001), 327–338.

39. *The bibliography of biharmonic maps*, `http://beltrami.sc.unica.it/biharmonic/`.

40. Zhang, W. – *Biharmonic space-like hypersurfaces in pseudo-Riemannian space*, arxiv:math/0808.1346v1.

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