

## ON QUASI-EINSTEIN WARPED PRODUCTS

BY

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**Abstract.** We study quasi-Einstein warped product manifolds for arbitrary dimension  $n \geq 3$ .

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### 1. Introduction

A Riemannian manifold  $(M, g)$ ,  $(n \geq 2)$ , is said to be an *Einstein manifold* if its Ricci tensor  $S$  satisfies the condition  $S = \frac{\tau}{n}g$ , where  $\tau$  denotes the *scalar curvature* of  $M$ . A quasi-Einstein manifold was introduced by CHAKI and MAITY in [1]. A non-flat Riemannian manifold  $(M, g)$ ,  $(n \geq 2)$ , is defined to be a *quasi-Einstein manifold* if the condition

$$(1) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y)$$

is fulfilled on  $M$ , where  $\alpha$  and  $\beta$  are scalar functions on  $M$  with  $\beta \neq 0$  and  $A$  is a non-zero 1-form such that

$$(2) \quad g(X, U) = A(X),$$

for every vector field  $X$ ;  $U \in \chi(M)$  being a unit vector field,  $\chi(M)$  is the space of vector fields on  $M$ . If  $\beta = 0$ , then the manifold reduces to an Einstein manifold.

By a contraction from the equation (1), it can be easily seen that  $\tau = \alpha n + \beta$ , where  $\tau$  is the scalar curvature of  $M$ .

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker space-times are quasi-Einstein manifolds. For more information about quasi-Einstein manifolds see [4], [5], [6] and [8].

In [2], CHEN and YANO introduced the notion of a Riemannian manifold  $(M, g)$  of a *quasi-constant sectional curvature* as a Riemannian manifold with the curvature tensor satisfies the condition

$$(3) \quad \begin{aligned} R(X, Y, Z, W) = & a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + b[g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W) \\ & + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)], \end{aligned}$$

where  $a$  and  $b$  are scalar functions with  $b \neq 0$ , where  $\eta$  is a 1-form denoted by  $g(X, E) = \eta(X)$ ,  $E$  is a unit vector field. It can be shown that, if the curvature tensor  $R$  is of the form (3), then the manifold is conformally flat. By a contraction from the equation (3), it can be easily seen that every Riemannian manifold of a quasi-constant sectional curvature is a quasi-Einstein manifold.

Let  $M$  be an  $m$ -dimensional,  $m \geq 3$ , Riemannian manifold and  $p \in M$ . Denote by  $K(\pi)$  or  $K(u, v)$  the sectional curvature of  $M$  associated with a plane section  $\pi \subset T_p M$ , where  $\{u, v\}$  is an orthonormal basis of  $\pi$ . For any  $n$ -dimensional subspace  $L \subseteq T_p M$ ,  $2 \leq n \leq m$ , its scalar curvature  $\tau(L)$  is denoted in [3] by  $\tau(L) = 2 \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$ , where  $\{e_1, \dots, e_n\}$  is any orthonormal basis of  $L$ . When  $L = T_p M$ , then the scalar curvature  $\tau(L)$  is just the scalar curvature  $\tau(p)$  of  $M$  at  $p$ .

## 2. Warped product manifolds

Let  $(B, g_B)$  and  $(F, g_F)$  be two Riemannian manifolds and  $f$  is a positive differentiable function on  $B$ . Consider the product manifold  $B \times F$  with its projections  $\pi : B \times F \rightarrow B$  and  $\sigma : B \times F \rightarrow F$ . The *warped product*  $B \times_f F$  is the manifold  $B \times F$  with the Riemannian structure such that  $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p)) \|\sigma^*(X)\|^2$ , for any vector field  $X$  on  $M$ . Thus we have

$$(4) \quad g = g_B + f^2 g_F$$

holds on  $M$ . The function  $f$  is called the *warping function* of the warped product [10].

Since  $B \times_f F$  is a warped product, then we have  $\nabla_X Z = \nabla_Z X = (X \ln f)Z$  for unit vector fields  $X, Z$  on  $B$  and  $F$ , respectively. Hence, we find  $K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{(\nabla_X X)f - X^2 f\}$ . If we chose a local orthonormal frame  $e_1, \dots, e_n$  such that  $e_1, \dots, e_{n_1}$  are tangent to  $B$  and  $e_{n_1+1}, \dots, e_n$  are tangent to  $F$ , then we have

$$(5) \quad \frac{\Delta f}{f} = \sum_{i=1}^n K(e_j \wedge e_s),$$

for each  $s = n_1 + 1, \dots, n$  [10].

We need the following two lemmas from [10], for later use :

**Lemma 2.1.** *Let  $M = B \times_f F$  be a warped product, with Riemannian curvature tensor  ${}^M R$ . Given fields  $X, Y, Z$  on  $B$  and  $U, V, W$  on  $F$ , then:*

- (1)  ${}^M R(X, Y)Z = {}^B R(X, Y)Z$ ,
- (2)  ${}^M R(V, X)Y = -(H^f(X, Y)/f)V$ , where  $H^f$  is the Hessian of  $f$ ,
- (3)  ${}^M R(X, Y)V = {}^M R(V, W)X = 0$ ,
- (4)  ${}^M R(X, V)W = -(g(V, W)/f)\nabla_X(\text{grad } f)$ ,
- (5)  ${}^M R(V, W)U = {}^F R(V, W)U + (\|\text{grad } f\|^2 / f^2)\{g(V, U)W - g(W, U)V\}$ .

**Lemma 2.2.** *Let  $M = B \times_f F$  be a warped product, with Ricci tensor  ${}^M S$ . Given fields  $X, Y$  on  $B$  and  $V, W$  on  $F$ , then:*

- (1)  ${}^M S(X, Y) = {}^B S(X, Y) - \frac{d}{f}H^f(X, Y)$ , where  $d = \dim F$ ,
- (2)  ${}^M S(X, V) = 0$ ,
- (3)  ${}^M S(V, W) = {}^F S(V, W) - g(V, W) \left[ \frac{\Delta f}{f} + (d-1) \frac{\|\text{grad } f\|^2}{f^2} \right]$ , where  $\Delta f$  is the Laplacian of  $f$  on  $B$ .

Moreover, the scalar curvature  ${}^M \tau$  of the manifold  $M$  satisfies the condition

$$(6) \quad {}^M \tau = {}^B \tau + \frac{1}{f^2} {}^F \tau - \frac{2d}{f} \Delta f - \frac{d(d-1)}{f^2} \|\text{grad } f\|^2,$$

where  ${}^B \tau$  and  ${}^F \tau$  are scalar curvatures of  $B$  and  $F$ , respectively.

In [7], GEBAROWSKI studied Einstein warped product manifolds and proved the following three theorems:

**Theorem 2.3.** *Let  $(M, g)$  be a warped product  $I \times_f F$ ,  $\dim I = 1$ ,  $\dim F = n - 1$  ( $n \geq 3$ ). Then  $(M, g)$  is an Einstein manifold if and only if  $F$  is Einstein with constant scalar curvature  ${}^F\tau$  in the case  $n = 3$  and  $f$  is given by one of the following formulae, for any real number  $b$ ,*

$$f^2(t) = \begin{cases} \frac{4}{a}K \sinh^2 \frac{\sqrt{a}(t+b)}{2} & (a > 0), \\ K(t+b)^2 & (a = 0), \\ -\frac{4}{a}K \sin^2 \frac{\sqrt{-a}(t+b)}{2} & (a < 0), \end{cases}$$

for  $K > 0$ ,  $f^2(t) = b \exp(at)$  ( $a \neq 0$ ), for  $K = 0$ ,  $f^2(t) = -\frac{4}{a}K \cosh^2 \frac{\sqrt{a}(t+b)}{2}$ , ( $a > 0$ ), for  $K < 0$ , where  $a$  is the constant appearing after first integration of the equation  $q''e^q + 2K = 0$  and  $K = \frac{{}^F\tau}{(n-1)(n-2)}$ .

**Theorem 2.4.** *Let  $(M, g)$  be a warped product  $B \times_f F$  of a complete connected  $r$ -dimensional ( $1 < r < n$ ) Riemannian manifold  $B$  and  $(n - r)$ -dimensional Riemannian manifold  $F$ . If  $(M, g)$  is a space of constant sectional curvature  $K > 0$ , then  $B$  is a sphere of radius  $\frac{1}{\sqrt{K}}$ .*

**Theorem 2.5.** *Let  $(M, g)$  be a warped product  $B \times_f I$  of a complete connected  $(n-1)$ -dimensional Riemannian manifold  $B$  and one-dimensional Riemannian manifold  $I$ . If  $(M, g)$  is an Einstein manifold with scalar curvature  ${}^M\tau > 0$  and the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , then*

- (1)  $(B, g_B)$  is an  $(n-1)$ -dimensional sphere of radius  $\rho = \left(\frac{{}^B\tau}{(n-1)(n-2)}\right)^{-\frac{1}{2}}$ .
- (2)  $(M, g)$  is a space of constant sectional curvature  $K = \frac{{}^M\tau}{n(n-1)}$ .

Motivated by the above study by GEBAROWSKI, in the present paper our aim is to generalize Theorem 2.3, Theorem 2.4 and Theorem 2.5 for quasi-Einstein manifolds.

### 3. Quasi-Einstein warped products

In this section, we consider quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

Now, let begin with the following theorem:

**Theorem 3.1.** *Let  $(M, g)$  be a warped product  $I \times_f F$ ,  $\dim I = 1$ ,  $\dim F = n - 1$  ( $n \geq 3$ ), where  $U \in \chi(M)$ . If  $(M, g)$  is a quasi-Einstein manifold with associated scalars  $\alpha$  and  $\beta$ , then  $F$  is a quasi-Einstein manifold.*

**Proof.** Denote by  $(dt)^2$  the metric on  $I$ . Taking  $f = \exp\left\{\frac{q}{2}\right\}$  and making use of the Lemma 2.2, we can write

$$(7) \quad {}^M S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2]$$

and

$$(8) \quad {}^M S(V, W) = {}^F S(V, W) - \frac{1}{4}e^q[2q'' + (n-1)(q')^2]g_F(V, W),$$

for all vector fields  $V, W$  on  $F$ .

Since  $M$  is quasi-Einstein, from (1) we have

$$(9) \quad {}^M S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta A\left(\frac{\partial}{\partial t}\right) A\left(\frac{\partial}{\partial t}\right)$$

and

$$(10) \quad {}^M S(V, W) = \alpha g(V, W) + \beta A(V)A(W).$$

Decomposing the vector field  $U$  uniquely into its components  $U_I$  and  $U_F$  on  $I$  and  $F$ , respectively, we can write  $U = U_I + U_F$ . Since  $\dim I = 1$ , we can take  $U_I = \mu \frac{\partial}{\partial t}$  which gives us  $U = \mu \frac{\partial}{\partial t} + U_F$ , where  $\mu$  is a function on  $M$ . Then we can write

$$(11) \quad A\left(\frac{\partial}{\partial t}\right) = g\left(\frac{\partial}{\partial t}, U\right) = \mu.$$

On the other hand, by the use of (4) and (11), the equations (9) and (10) reduce to

$$(12) \quad {}^M S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha + \mu^2 \beta$$

and

$$(13) \quad {}^M S(V, W) = \alpha e^q g_F(V, W) + \beta A(V)A(W).$$

Comparing the right hand sides of the equations (7) and (12) we get

$$(14) \quad \alpha + \mu^2 \beta = -\frac{(n-1)}{4}[2q'' + (q')^2].$$

Similarly, comparing the right hand sides of (8) and (13) we obtain

$${}^F S(V, W) = \frac{1}{4} e^q [2q'' + (n-1)(q')^2 + 4\alpha] g_F(V, W) + \beta A(V)A(W),$$

which implies that  $F$  is a quasi-Einstein manifold. This completes the proof of the theorem.  $\square$

**Theorem 3.2.** *Let  $(M, g)$  be a warped product  $B \times_f F$  of a complete connected  $r$ -dimensional ( $1 < r < n$ ) Riemannian manifold  $B$  and  $(n-r)$ -dimensional Riemannian manifold  $F$ .*

- (1) *If  $(M, g)$  is a space of quasi-constant sectional curvature, the Hessian of  $f$  is proportional to the metric tensor  $g_B$  and the associated vector field  $E$  is a general vector field on  $M$  or  $E \in \chi(B)$ , then  $B$  is a 2-dimensional Einstein manifold.*
- (2) *If  $(M, g)$  is a space of quasi-constant sectional curvature and the associated vector field  $E \in \chi(F)$ , then  $B$  is an Einstein manifold.*

**Proof.** Assume that  $M$  is a space of quasi-constant sectional curvature. Then from the equation (3) we can write

$$\begin{aligned} {}^M R(X, Y, Z, W) &= a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ (15) \quad &+ b[g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W) \\ &+ g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)], \end{aligned}$$

for all vector fields  $X, Y, Z, W$  on  $B$ .

Decomposing the vector field  $E$  uniquely into its components  $E_B$  and  $E_F$  on  $B$  and  $F$ , respectively, we have

$$(16) \quad E = E_B + E_F.$$

By making use of (4) and (16), we can write

$$(17) \quad \eta(Y) = g(Y, E) = g(Y, E_B) = g_B(Y, E_B).$$

In view of Lemma 2.1 and by the use of (4) and (17), we obtain

$$\begin{aligned} {}^B R(X, Y, Z, W) &= a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)] \\ (18) \quad &+ b[g_B(X, W)g_B(Y, E_B)g_B(Z, E_B) - g_B(X, Z)g_B(Y, E_B)g_B(W, E_B) \\ &+ g_B(Y, Z)g_B(X, E_B)g_B(W, E_B) - g_B(Y, W)g_B(X, E_B)g_B(Z, E_B)]. \end{aligned}$$

By a contraction from the last equation over  $X$  and  $W$  and making use of the equation (17) again, we get

$$(19) \quad {}^B S(Y, Z) = [a(r-1) + b g_B(E_B, E_B)] g_B(Y, Z) + b(r-2) \eta(Y) \eta(Z),$$

which shows us  $B$  is a quasi-Einstein manifold. Contracting from (19) over  $Y$  and  $Z$ , it can be easily seen that

$$(20) \quad {}^B \tau = (r-1)[ar + 2b g_B(E_B, E_B)].$$

Since  $M$  is a space of quasi-constant sectional curvature, in view of (5) and (18) we get

$$(21) \quad \frac{\Delta f}{f} = \frac{ar + b g_B(E_B, E_B)}{2}.$$

On the other hand, since the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , it can be written as follows

$$(22) \quad H^f(X, Y) = \frac{\Delta f}{r} g_B(X, Y).$$

Then, by the use of (20) and (21) in (22) we obtain  $H^f(X, Y) + K f g_B(X, Y) = 0$ , where  $K = \frac{(r-1)b g_B(E_B, E_B) - {}^B \tau}{2r(r-1)}$  holds on  $B$ . So by OBATA's theorem [9],  $B$  is isometric to the sphere of radius  $\frac{1}{\sqrt{K}}$  in the  $(r+1)$ -dimensional Euclidean space. This gives us  $B$  is an Einstein manifold. Since  $b \neq 0$  this implies that  $r = 2$ . Hence  $B$  is a 2-dimensional Einstein manifold.

Assume that the associated vector field  $E \in \chi(B)$ . Then in view of Lemma 2.1 and by making use of (4) and (15) we can write

$$(23) \quad \begin{aligned} {}^B R(X, Y, Z, W) = & a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)] \\ & + b[g_B(X, W)g_B(Y, E)g_B(Z, E) - g_B(X, Z)g_B(Y, E)g_B(W, E) \\ & + g_B(Y, Z)g_B(X, E)g_B(W, E) - g_B(Y, W)g_B(X, E)g_B(Z, E)]. \end{aligned}$$

By a contraction from the last equation over  $X$  and  $W$ , we obtain

$$(24) \quad {}^B S(Y, Z) = [a(r-1) + b] g_B(Y, Z) + b(r-2) g_B(Y, E) g_B(Z, E),$$

which gives us  $B$  is a quasi-Einstein manifold.

By a contraction from (24) over  $Y$  and  $Z$ , we get

$$(25) \quad {}^B \tau = (r-1)[ar + 2b].$$

Since  $M$  is a space of quasi-constant sectional curvature, in view of (5) and (23) we have

$$(26) \quad \frac{\Delta f}{f} = \frac{ar + b}{2}.$$

On the other hand, since the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , it can be written as follows

$$(27) \quad H^f(X, Y) = \frac{\Delta f}{r} g_B(X, Y).$$

Then, by the use of (25) and (26) in (27) we obtain  $H^f(X, Y) + Kfg_B(X, Y) = 0$ , where  $K = \frac{(r-1)b - {}^B\tau}{2r(r-1)}$  holds on  $B$ . So by OBATA's theorem [9],  $B$  is isometric to the sphere of radius  $\frac{1}{\sqrt{K}}$  in the  $(r+1)$ -dimensional Euclidean space. This shows us  $B$  is an Einstein manifold. Since  $b \neq 0$  this implies that  $r = 2$ . Hence  $B$  is a 2-dimensional Einstein manifold.

Assume that the associated vector field  $E \in \chi(F)$ , then the equation (15) reduces to

$${}^M R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

In view of Lemma 2.1 and by the use of (4), the above equation can be written as follows

$${}^B R(X, Y, Z, W) = a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)].$$

By a contraction from the above equation over  $X$  and  $W$ , we get  ${}^B S(Y, Z) = a(r-1)g_B(Y, Z)$ , which implies that  $B$  is an Einstein manifold with the scalar curvature  ${}^B\tau = ar(r-1)$ . Hence, the proof of the theorem is completed.  $\square$

**Theorem 3.3.** *Let  $(M, g)$  be a warped product  $B \times_f I$  of a complete connected  $(n-1)$ -dimensional Riemannian manifold  $B$  and one-dimensional Riemannian manifold  $I$ . If  $(M, g)$  is a quasi-Einstein manifold with constant associated scalars  $\alpha$  and  $\beta$ ,  $U \in \chi(M)$  and the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , then  $(B, g_B)$  is an  $(n-1)$ -dimensional sphere of radius  $\rho = \frac{n-1}{\sqrt{{}^B\tau + \alpha}}$ .*



**Proof.** Assume that  $M$  is a warped product manifold. Then by the use of the Lemma 2.2 we can write

$$(28) \quad {}^B S(X, Y) = {}^M S(X, Y) + \frac{1}{f} H^f(X, Y)$$

for any vector fields  $X, Y$  on  $B$ . On the other hand, since  $M$  is quasi-Einstein we have

$$(29) \quad {}^M S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y).$$

Decomposing the vector field  $U$  uniquely into its components  $U_B$  and  $U_I$  on  $B$  and  $I$ , respectively, we get

$$(30) \quad U = U_B + U_I.$$

In view of (2), (4), (29) and (30) the equation (28) can be written as

$${}^B S(X, Y) = \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) + \frac{1}{f} H^f(X, Y).$$

By a contraction from the above equation over  $X$  and  $Y$ , we find

$$(31) \quad {}^B \tau = \alpha(n-1) + \beta g_B(U_B, U_B) + \frac{\Delta f}{f}.$$

On the other hand, we know from the equation (29) that

$$(32) \quad {}^M \tau = \alpha n + \beta g_B(U_B, U_B).$$

By the use of (32) in (31) we get  ${}^B \tau = {}^M \tau - \alpha + \frac{\Delta f}{f}$ . In view of Lemma 2.2 we also know that

$$(33) \quad -\frac{{}^M \tau}{n} = \frac{\Delta f}{f}.$$

The last two equations give us  ${}^B \tau = \frac{(n-1)}{n} {}^M \tau - \alpha$ . On the other hand, since the Hessian of  $f$  is proportional to the metric tensor  $g_B$ , we can write  $H^f(X, Y) = \frac{\Delta f}{n-1} g_B(X, Y)$ . As a consequence of the equation (33) we have  $\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)} {}^M \tau f$ , which implies that

$$H^f(X, Y) + \frac{{}^B \tau + \alpha}{(n-1)^2} f g_B(X, Y) = 0.$$

So  $B$  is isometric to the  $(n-1)$ -dimensional sphere of radius  $\sqrt{\frac{n-1}{B\tau+\alpha}}$  (see OBATA [9]). Thus our theorem is proved.  $\square$

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