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ON PRIME AND PRIMARY HYPERIDEALS OF A MULTIPLICATIVE HYPERRING

BY

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Abstract. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings. Here we intend to investigate extensively the prime and primary hyperideals of multiplicative hyperrings with absorbing zero. Defining the radical of a hyperideal I of a multiplicative hyperring with absorbing zero, as the intersection of all prime ideals containing I, we obtain a generalized version of Krull's theorem regarding the structure of the radical of a particular class of hyperideals, called $\mathcal{C}$-ideals of a multiplicative hyperring . In the last section of this paper, we describe the prime hyperideals, primary hyperideals and $\mathcal{C}$-ideals of a multiplicative hyperring $\mathbb{Z}_A$ over the ring of integers $\mathbb{Z}$, induced by any $A \in P^*(\mathbb{Z})$.

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1. Introduction

Krasner's hyperring [5] , introduced and studied by Krasner is a hypercompositional structure $(S, +, \cdot)$ where $(S, +)$ is a canonical hypergroup [3], $(S, \cdot)$ is a semigroup in which the zero element is absorbing and the operation $\cdot$ is a two-sided distributive one over the hypercomposition $+$. Chaopraknoi and Kemprasit introduced in 2005, the notion of semihyperring $(S, +, \cdot)$ ([2]), where ($S, +$) is a semihypergroup (i.e., a hypercompositional structure with single associative hyperoperation), $(S, \cdot)$ is a semigroup and the operation $\cdot$ is both left and right distributive across the hyperoperation $+$. Interchanging the modes of the operations involved in the hyperstructure semihyperring, we define in [12] another class of hyperstructure called hypersemiring which is an (additive) commutative semigroup

$(S, +)$ endowed with a hyperoperation $\circ : S \times S \longrightarrow P(S)$ such that for all $x, y, z \in S$, (i) $x \circ (y \circ z) = (x \circ y) \circ z$; (ii) $(x+y) \circ z = x \circ z + y \circ z, x \circ (y+z) = x \circ y + x \circ z$ (where for any $A, B \in P(S), A + B = \{a + b : a \in A, b \in B\}$). In contrary to Krasner's hyperring, another kind of hyper-ring is introduced in [12]. This hyper-ring is a hypersemiring $(S, +, \circ)$, where $(S, +)$ is a commutative group whose identity element 0_S is absorbing in the hypersemiring $(S, +, \circ)$ (in the sense that $0_S \in 0_S \circ x = x \circ 0_S, \forall x \in S$).

A *generalized hyper-ring* or simply a G_H*-ring* [13] is an additive commutative group $(R, +)$ endowed with a hyperoperation $\circ$ such that (i) $(R, \circ)$ is semihypergroup, (ii) $(x + y) \circ z \subseteq x \circ z + y \circ z$, and $x \circ (y + z) \subseteq x \circ y + x \circ z, \forall x, y, z \in R$ and (iii) $0_R \circ x = x \circ 0_R = \{0_R\}$ (absorbing property of 0_R). A G_H-ring $(R, +, \circ)$ is called *commutative* if $x \circ y = y \circ x, \forall x, y \in R$. We proved in [13] that in case of a G_H-ring$(R, +, \circ)$, $(-x) \circ y \cap (-(x \circ y)) \neq \phi$ and $x \circ (-y) \cap (-(x \circ y)) \neq \phi$, for any x, y in R, where for any $A \in P^*(R), -A = \{-a : a \in A\}$. Unlike a ring, the equality of the set-expressions $(-x) \circ y$, $x \circ (-y)$ and $-(x \circ y)$ does not hold in general, on a G_H-ring$(R, +, \circ)$, for any $x, y \in R$. We thus considered and studied in [13] a particular class of G_H-ring, called the G_H-rings with condition $(\mathcal{R})$ (i.e.,$(-x) \circ y = x \circ (-y) = -(x \circ y) \forall x, y \in R$).

In 1982, Rota initiated the study of multiplicative hyperring [11] which was subsequently investigated by Rota [10], Procesi and Rota [8, 9], Olson and Ward [7], Namnak, Triphop and Kemprasit [6] and many others. A multiplicative hyperring is an additive commutative group $(R, +)$ endowed with a hyperoperation $\circ$ which satisfies the first two axioms of G_H-ring along with the condition $(\mathcal{R})$. It is thus clear from the definitions that the class of G_H-rings with condition $(\mathcal{R})$ is precisely that of multiplicative hyperrings with absorbing zero.

If $(R, +, \cdot)$ is a ring, then corresponding to every subset $A \in P^*(R) = P(R) \setminus \{\phi\}$ $(|A| \geq 2)$, there exists a multiplicative hypering with absorbing zero $(R_A, +, \circ)$, where $R_A = R$ and for any $x, y \in R_A, x \circ y = \{x \cdot a \cdot y : a \in A\}$. In the last section we study the multiplicative hypering $\mathbb{Z}_A$ over the ring of integers $\mathbb{Z}$.

For the definitions of subhyperring, (left, right) hyperideal of a multiplicative hyperring and also that of (left, right) hyperideal (resp. $\langle A \rangle)_l, \langle A \rangle_r$) $\langle A \rangle$ generated by a subset A of a multiplicative hyperring, we refer to Chapter 4 of the book [4] by Davvaz and Leoreanu-Fotea. A hyperideal $I(\neq R)$ of a multiplicative hyperring R is maximal in R if for any hyperideal J of R, $I \subsetneq J \subseteq R \Rightarrow J = R$.

2. Prime hyperideals in multiplicative hyperrings

We begin this section with the definition of a typical hyperideal in multiplicative hyperring, called $\mathcal{C}$-ideal which will have a pivotal role in our study of prime and primary hyperideals of multiplicative hyperrings with absorbing zero.

Definition 2.1. Let $\mathcal{C}$ be the class of all finite products of elements of a multiplicative hyperring $(R, +, \circ)$ i.e. $\mathcal{C} = \{r_1 \circ r_2 \circ \ldots \circ r_n : r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a $\mathcal{C}$-ideal of R if, for any $A \in \mathcal{C}, A \cap I \neq \phi \Rightarrow A \subseteq I$.

Example 2.2. (a) Let I be an ideal of a ring $(R, +, \cdot)$. Then $(R, +, \circ)$ [9], where for any $a, b \in R, a \circ b = a \cdot b + I$, is a multiplicative hyperring in which I is itself a $\mathcal{C}$-ideal.

(b) Let $(M, +)$ and $(\Gamma, +)$ be respectively the commutative group of 2×3 matrices and that of 3×2 matrices over the ring of integers $\mathbb{Z}$. Then, M is a Γ-ring [1] with respect to usual matrix multiplication $a\alpha b$ for any $a, b \in M$ and $\alpha \in \Gamma$. Corresponding to any $\Lambda \in P^*(\Gamma)$ with $|\Lambda| \geq 2$, $(M, +, \circ)$ is a multiplicative hyperring with absorbing zero, where $a \circ b = \{a\alpha b : \alpha \in \Lambda\}$, for any $a, b \in M$. Let this multiplicative hyperring be denoted by M_Λ. Then the set $I = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} : a \in \mathbb{Z} \right\}$ is a $\mathcal{C}$-ideal of the multiplicative hyperring M_Λ, where

$$\Lambda = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{Z} \setminus \{0_M\} \right\}.$$

(c) Let A be the set of all positive odd integers. Then the set E of all even integers is a $\mathcal{C}$-ideal of the multiplicative hyperring $(\mathbb{Z}_A, +, \circ)$ over the ring $(\mathbb{Z}, +, \cdot)$ of all integers, induced by A.

(d) The set $12\mathbb{Z} = \{12n : n \in \mathbb{Z}\}$ is a hyperideal, but not a $\mathcal{C}$-ideal of the multiplicative hyperring of integers $\mathbb{Z}_A$ when $A = \{2, 4\}$.

Remark 2.3. Since the intersection of $\mathcal{C}$-ideals of a multiplicative hyperring R is also a $\mathcal{C}$-ideal of R and R is itself a $\mathcal{C}$-ideal, so the smallest $\mathcal{C}$-ideal containing a hyperideal I of R, being called as the $\mathcal{C}$-closure of I and denoted by $\mathcal{C}(I)$, exists and is the intersection of all $\mathcal{C}$-ideals containing I. Clearly, $\mathcal{C}(\mathcal{C}(I)) = \mathcal{C}(I)$.

Example 2.4. For the hyperideal $12\mathbb{Z}$ of the multiplicative hyperring $\mathbb{Z}_A$ $(A = \{2, 4\})$, $\mathcal{C}(12\ \mathbb{Z}) = 3\mathbb{Z}$.

Definition 2.5. A non-empty finite subset $\mathcal{E}_l$ (resp. $\mathcal{E}_r$)$=\{e_1, e_2, \ldots, e_n\}$ of a multiplicative hyperring $(R, +, \circ)$ is called a *left* (resp. *right*) *identity set* (or *i- set*, in short) [13] of R if (i) $e_i \neq 0_R$ for at least one $i = 1, 2, \ldots, n$, and (ii) for any $a \in R$, $a \in \sum_{i=1}^{n} e_i \circ a$ (resp. $a \in \sum_{i=1}^{n} a \circ e_i$). A non-empty finite subset $\mathcal{E}$ of a multiplicative hyperring $(R, +, \circ)$ is called an *i-set* of R if it is both a left i-set and a right i-set of R.

Example 2.6. The multiplicative hyperring Z_A over the ring of integers Z has an i-set $\mathcal{E} = \{5, -3\}$, when $A = \{2, 3\}$. Z_A does not have any i-set when $A = \{6, 9\}$.

Definition 2.7. A non-empty subset A of a multiplicative hyperring $(R, +, \circ)$ is said to be a multiplicative set if $a, b \in A \Rightarrow a \circ b \cap A \neq \phi$.

Example 2.8. Let $\Im(\neq \phi)$ be the collection of all i-sets of a commutative multiplicative hyperring $(R, +, \circ)$. Then, $\mathbf{E} = \bigcup_{\mathcal{E} \in \Im} \mathcal{E}$ is a multiplicative set in R. In fact, let $a, b \in \mathbf{E}$. Then, there exist $\mathcal{E}$ and $\mathcal{E}'$ in $\Im$ such that $a \in \mathcal{E}$ and $b \in \mathcal{E}'$. Let $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ and $\mathcal{E}' = \{e'_1, e'_2, \ldots, e'_m\}$. Then for some $k, l(1 \leq k \leq n, 1 \leq l \leq m), a = e_k$ and $b = e'_l$. Now, for any $i(= 1, 2, \ldots, n), e_i \in \sum_{j=1}^{m} e_i \circ e'_j$. So, for each $i(= 1, 2, \ldots, n)$ there is $e_{ij} \in e_i \circ e'_j$ such that $e_i = \sum_{j=1}^{m} e_{ij}$. Thus, for any $x \in R$ and for any i, $x \circ e_i = x \circ (\sum_{j=1}^{m} e_{ij}) \subseteq \sum_{j=1}^{m} x \circ e_{ij}$. Hence, $x \in \sum_{i=1}^{n} x \circ e_i \subseteq \sum_{i=1}^{n} \sum_{j=1}^{m} x \circ e_{ij}$. So, $\mathcal{F} = \{e_{ij} : i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \in \Im$. Thus, $e_{kl} \in \mathbf{E}$ and $e_{kl} \in e_k \circ e'_l = a \circ b$. Hence, $a \circ b \cap \mathbf{E} \neq \phi$.

Procesi and Rota conceptualized in [9] the notion of primeness of hyperideal in a multiplicative hyperring, which is formally defined as follows:

Definition 2.9. A hyperideal $I(\neq R)$ of a multiplicative hyperring $(R, +, \circ)$ is called a prime hyperideal of R if, for any $a, b \in R, a \circ b \subseteq I \Rightarrow a \in I$ or $b \in R$.

Proposition 2.10. *A hyperideal $I(\neq R)$ of a multiplicative hyperring $(R, +, \circ)$ is prime if and only if $R \setminus I$ is a multiplicative set.*

Proof. Let I be a hyperideal of the multiplicative hyperring $(R, +, \circ)$ such that $R \setminus I$ be a multiplicative set. Suppose that $a \circ b \subseteq I$. Then

$(a \circ b) \cap (R \setminus I) = \phi$. Hence $a \notin R \setminus I$ or $b \notin R \setminus I$ (since $R \setminus I$ is a multiplicative set), i.e., $a \in I$ or $b \in I$. Thus I is a prime hyperideal of R.

Conversely, let I be a prime hyperideal and $a, b \in R \setminus I$. Then, $a \circ b \nsubseteq I$, whence $(a \circ b) \cap (R \setminus I) \neq \phi$, i.e., $R \setminus I$ is a multiplicative set. □

Lemma 2.11. *For two hyperideals A and B of a multiplicative hyperring $(R, +, \circ)$, $AB = \cup\{\sum_{i=1}^{n} a_i \circ b_i : a_i \in A, b_i \in B$ and $n \in \mathbb{N}\}$ is also a hyperideal of R.*

Proof. Straightforward. □

Proposition 2.12. *If I is a prime hyperideal of a multiplicative hyperring $(R, +, \circ)$, then for any hyperideals A, B of R, $AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.*

Proof. Straightforward. □

Proposition 2.13. *Let $(R, +, \circ)$ be a multiplicative hyperring. For any $A \in P^*(R)$, suppose that $\mathcal{LC}_A = \{\sum_{i=1}^{n} x_i \circ a_i : x_i \in R, a_i \in A$ and $n \in \mathbb{N}\}$, $\mathcal{RC}_A = \{\sum_{i=1}^{n} a_i \circ x_i : x_i \in R, a_i \in A$ and $n \in \mathbb{N}\}$ and $\mathcal{C}_A = \{\sum_{i=1}^{n} x_i \circ a_i + \sum_{j=1}^{m} b_j \circ y_j + \sum_{k=1}^{l} r_k \circ c_k \circ s_k : x_i, y_j, r_k, s_k \in R, a_i, b_j, c_k \in A$ and $n, m, l \in \mathbb{N}\}$. Then,*

$$\langle A\rangle_l = (A) + \cup\mathcal{LC}_A, \langle A\rangle_r = (A) + \cup\mathcal{RC}_A \text{ and } \langle A\rangle = (A) + \cup\mathcal{C}_A,$$

where (A) is the subgroup of the group $(R, +)$, generated by the set A.

Proof. For any $a, b \in \cup\mathcal{LC}_A$, we have $a \in \sum_{i=1}^{n} x_i \circ a_i$ and $b \in \sum_{j=1}^{m} y_j \circ b_j$ for some $x_i, y_j \in R$ and $a_i, b_j \in A$. Then, $a - b \in \sum_{i=1}^{n} x_i \circ a_i - \sum_{j=1}^{m} y_j \circ b_j = \sum_{i=1}^{n} x_i \circ a_i + \sum_{j=1}^{m}(-y_j) \circ b_j \Rightarrow a - b \in \cup\mathcal{LC}_A$. Again, for any $r \in R, r \circ a \subseteq r \circ (\sum_{i=1}^{n} x_i \circ a_i) \subseteq \sum_{i=1}^{n} r \circ (x_i \circ a_i) \subseteq \sum_{i=1}^{n}(r \circ x_i) \circ a_i$. Hence, for any $x \in r \circ a$ there exist $r_i \in r \circ x_i (i = 1, 2, \ldots, n)$ such that $x \in \sum_{i=1}^{n} r_i \circ a_i \subseteq \cup\mathcal{LC}_A$. Thus, $r \circ a \subseteq \cup\mathcal{LC}_A$, for any $r \in R$ and $a \in \cup\mathcal{LC}_A$, i.e., $\cup\mathcal{LC}_A$ is a left hyperideal of R. Moreover, for any $r \in R$ and $a = \sum_{i=1}^{n}(m_i a_i) \in (A)$ (for some $m_i \in \mathbb{Z}$ and $a_i \in A$), $r \circ \sum_{i=1}^{n}(m_i a_i) \subseteq \sum_{i=1}^{n} m_i(r \circ a_i)) \subseteq \cup\mathcal{LC}_A$, i.e., $r \circ (A) \subseteq \cup\mathcal{LC}_A$.

Hence, $(A) + \cup\mathcal{LC}_A$ is a left hyperideal of R. Since $A \subseteq (A)$ and $0 \in \mathcal{LC}_A$, so $A \subseteq (A) + \cup\mathcal{LC}_A$. So clearly $(A) + \cup\mathcal{LC}_A$ is the smallest left hyperideal of R containing A i.e., $\langle A\rangle_l = (A) + \cup\mathcal{LC}_A$. The cases for $\langle A\rangle_r$ and $\langle A\rangle$ can be proved similarly. □

Remark 2.14. (i) Let $(R, +, \circ)$ be a multiplicative hyperring. For any $a \in R$, if we write $\mathcal{C}_{\{a\}}$ simply as $\mathcal{C}_a$, then the principal hyperideal of R generated by a is given by

$$\langle a \rangle = (a) + \cup\mathcal{C}_a = \{pa : p \in \mathbb{Z}\} + \Big\{ \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{l} z_k : \forall i, j, k,$$
$$\exists r_i, s_j, t_k, u_k \in R \text{ such that } x_i \in r_i \circ a, y_j \in a \circ s_j, z_k \in t_k \circ a \circ u_k \Big\}$$

(ii) If the multiplicative hyperring $(R, +, \circ)$ has an i-set $\mathcal{E} = \{e_1, e_2, \ldots, e_q\}$ (for some $q \in \mathbb{N}$), then for any $a \in R$

$$\langle a \rangle = \cup\mathcal{C}_a = \Big\{ \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{l} z_k : \forall i, j, k,$$
$$\exists r_i, s_j, t_k, u_k \in R \text{ such that } x_i \in r_i \circ a, y_j \in a \circ s_j, z_k \in t_k \circ a \circ u_k \Big\}.$$

Indeed, $a \in \sum_{\imath=1}^{q} e_\imath \circ a \in \cup\mathcal{C}_a \Rightarrow (a) \subseteq \cup\mathcal{C}_a$.

(iii) This is stated in [4] that, in a multiplicative hyperring $(R, +, \circ)$,

$$\langle 0 \rangle = \Big\{ \sum_i x_i + \sum_j y_j + \sum_k z_k : \text{ each sum is finite and for each } i, j, k,$$
$$\text{there exist } r_i, s_j, t_k, u_k \in R \text{ such that}$$
$$x_i \in r_i \circ 0, y_j \in 0 \circ s_j, z_k \in t_k \circ 0 \circ u_k \Big\}.$$

This is here only to point out that the above form of $\langle 0 \rangle$ follows straight from (i), since $(0) = \{0\} \subseteq \cup\mathcal{C}_0$.

(iv) Let $(R, +, \circ)$ be a multiplicative hyperring. Then, for any $A \in P^*(R)$ and for any positive integer n, we write $nA = \underbrace{A + A + \ldots + A}_{n \text{ times}}$ and $(-n)A = n(-A)$. With this notational convenience, we have for any a, b, r in R, and for any $k_1, k_2 \in \mathbb{Z}$, $(k_1 a) \circ (k_2 b) \subseteq k_1 k_2 (a \circ b)$, $(k_1 a) \circ r \subseteq k_1 (a \circ r)$ and $r \circ (k_2 b) \subseteq k_2 (r \circ b)$.

Proposition 2.15. *Let $(R, +, \circ)$ be a commutative multiplicative hyperring. Then, for any $a, b \in R$, $\langle a \rangle \langle b \rangle \subseteq \langle a \circ b \rangle$.*

Proof. By proposition 2.13, for any $a, b \in R$ we have that $\langle a \rangle = (a) + \cup\mathcal{C}_a$, $\langle b \rangle = (b) + \cup\mathcal{C}_b$ and $\langle a \circ b \rangle = (a \circ b) + \cup\mathcal{C}_{a \circ b}$. Since R is commutative,

$\mathcal{C}_A = \mathcal{L}\mathcal{C}_A = \mathcal{R}\mathcal{C}_A$ for any $A \in P^*(R)$. Then, $d \in \langle a\rangle \circ \langle b\rangle \Rightarrow d \in (k_1a+r)\circ(k_2b+s) \subseteq k_1k_2(a\circ b)+k_1(a\circ s)+k_2(b\circ r)+r\circ s$ (for some $r \in \cup\mathcal{C}_a$, $s \in \cup\mathcal{C}_b$ and $k_1, k_2 \in \mathbb{Z}$) $\Rightarrow d \in k_1k_2(a\circ b) + k_1(a\circ(\sum_{j=1}^m b\circ y_j)) + k_2(b\circ(\sum_{i=1}^n a\circ x_i)) + (\sum_{i=1}^n a\circ x_i)\circ(\sum_{j=1}^m b\circ y_j) \subseteq k_1k_2(a\circ b) + k_1(\sum_{j=1}^m (a\circ b)\circ y_j)) + k_2(\sum_{i=1}^n (a\circ b)\circ x_i)) + \sum_{i=1}^n \sum_{j=1}^m (x_i\circ y_j)\circ(a\circ b)$ (for some $x_i, y_j \in R$) $\Rightarrow d \in k_1k_2(a\circ b) + k_1\sum_{j=1}^m a_j\circ y_j + k_2\sum_{i=1}^n b_i\circ x_i + \sum_{i=1}^n\sum_{j=1}^m z_{ij}\circ c_{ij}$ (for some $a_j, b_i, c_{ij} \in a\circ b$ and $z_{ij} \in x_i\circ y_j$) $\Rightarrow d \in (a\circ b) + \mathcal{C}_{a\circ b} = \langle a\circ b\rangle$. Thus, $\langle a\rangle\circ\langle b\rangle \subseteq \langle a\circ b\rangle$. Consequently, $\langle a\rangle\langle b\rangle \subseteq \langle a\circ b\rangle$. □

Proposition 2.16. *Let $(R, +, \circ)$ be a commutative multiplicative hyperring. If $I(\neq R)$ is a hyperideal of R such that for any hyperideals A, B of R, $AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$, then I is a prime hyperideal of R.*

Proof. Let $a, b \in R$ be such that $a\circ b \subseteq I$. Then, $\langle a\circ b\rangle \subseteq I$. Hence, by proposition 2.15, $\langle a\rangle\langle b\rangle \subseteq I$ (since R is commutative), whence $\langle a\rangle \subseteq I$ or $\langle b\rangle \subseteq I$ i.e., $a \in I$ or $b \in I$. Hence, I is a prime hyperideal of R. □

Proposition 2.17. *Let S be a multiplicative subset of a commutative multiplicative hyperring $(R, +, \circ)$ and I be a hyperideal of R disjoint from S. Then there exists a hyperideal P which is maximal in the set of all hyperideals of R disjoint from S, containing I. Any such hyperideal P is a prime hyperideal of R.*

Proof. Let $\mathcal{S}$ be the set of all hyperideals of R disjoint from S, containing I. Then $\mathcal{S} \neq \phi$, since $I \in \mathcal{S}$. So $\mathcal{S}$ is a partially ordered set with respect to set inclusion relation. By Zorn's lemma, there is a hyperideal P which is maximal in $\mathcal{S}$. Let A and B be two hyperideals of R such that $AB \subseteq P$. If $A \nsubseteq P$ and $B \nsubseteq P$, then each of the two hyperideals $P + A$ and $P + B$ properly contain P. So, by maximality of P in $\mathcal{S}$, we have that $(P + A) \cap S \neq \phi$ and $(P + B) \cap S \neq \phi$. Then, there exist $p_1, p_2 \in P, a \in A$ and $b \in B$ such that $p_1 + a \in S$ and $p_2 + b \in S$. Then $(p_1 + a)\circ(p_2 + b) \cap S \neq \phi$ (since S is a multiplicative set). Now $(p_1 + a)\circ(p_2 + b) \subseteq p_1\circ p_2 + a\circ p_2 + p_1\circ b + a\circ b \subseteq P + AB \subseteq P$ (since $AB \subseteq P$). Thus, $P \cap S \neq \phi$ which is contradictory to the fact that $P \in \mathcal{S}$. Hence, by proposition 2.16, P is a prime hyperideal (since R is commutative). □

Proposition 2.18. *Every maximal hyperideal of a commutative multiplicative hyperring with an i-set, is a prime hyperideal.*

Proof. Let P be a maximal hyperideal of a commutative multiplicative hyperring $(R, +, \circ)$ with an i-set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$. Suppose, A and B are two hyperideals of R such that $AB \subseteq P$, but $A \nsubseteq P$. Then $P + A$ is a hyperideal of R properly containing P. Hence by maximality of P, $P + A = R$. So, $\mathcal{E} \subseteq P + A$. Thus for each $e_i \in \mathcal{E}$, there exist $p_i \in P$ and $a_i \in A$ such that $e_i = p_i + a_i$. Then, for each $i(= 1, 2, \ldots, n)$, and for any $b \in B, e_i \circ b \subseteq (p_i + a_i) \circ b \subseteq p_i \circ b + a_i \circ b \subseteq P$. Hence, $b \in \sum_{i=1}^{n} e_i \circ b \subseteq P$, i.e., $B \subseteq P$. So, by proposition 2.16, P is a prime hyperideal of R (since R is a commutative multiplicative hyperring). □

Proposition 2.19. *Let K be a subhyperring of a multiplicative hyperring $(R, +, \circ)$. If $P_1, P_2, \ldots, P_n$ are prime $\mathcal{C}$-ideals of R such that $K \subseteq \bigcup_{i=1}^{n} P_i$, then $K \subseteq P_i$ for some i.*

Proof. If $n = 1$, then there is nothing left to be proved. So, suppose that $n > 1$. If possible let $K \nsubseteq P_i$ for any i. If $K \cap P_i = \phi$ for some i, then, $K \subseteq \bigcup_{j=1}^{m} Q_j \subseteq \bigcup_{i=1}^{n} P_i$, where $Q_j = P_{k_j}$ for some $k_j(1 \leq k_j \leq n, j = 1, 2, \ldots m)$ such that $K \cap P_{k_j} \neq \phi$ for any j. Then, for each j, $K \nsubseteq \bigcup_{l \neq j} Q_l$; but $K \cap (\bigcup_{l \neq j} Q_l) \neq \phi$, as otherwise, since $K \subseteq \bigcup_{j=1}^{m} Q_j$ we arrive at a contradiction that $K \subseteq Q_j$ for some j. So for each j, we choose $a_j \in K \setminus (\bigcup_{k \neq j} Q_k)$. Since $K \subseteq \bigcup_{j=1}^{m} Q_j$, so each $a_j \in Q_j$. We consider the set $\{a_1\} + a_2 \circ a_3 \circ \ldots \circ a_m$. Clearly, $\{a_1\} + a_2 \circ a_3 \circ \ldots \circ a_m \subseteq K$ (since K is a subhyperring of the multiplicative hyperring R) and hence $\{a_1\} + a_2 \circ a_3 \circ \ldots \circ a_m \subseteq \bigcup_{j=1}^{m} Q_j$. Thus for each $a \in a_2 \circ a_3 \circ \ldots \circ a_m$, there exists one Q_j such that $a_1 + a \in Q_j$. If $j > 1$, then $a \in a_2 \circ a_3 \circ \ldots \circ a_j \circ \ldots \circ a_m \subseteq Q_j$ (since Q_j is a hyperideal), whence $a_1 \in Q_j$, which is a contradiction. If $j = 1$, then $a_1 + a \in Q_1$ and thus $a \in Q_1$ (since $a_1 \in Q_1$). So, $(a_2 \circ a_3 \circ \ldots \circ a_m) \cap Q_1 \neq \phi$ and thus $a_2 \circ a_3 \circ \ldots \circ a_m \subseteq Q_1$ (since Q_1 is a $\mathcal{C}$-ideal of R). Hence, $a_k \in Q_1$ for some $k = 2, 3, \ldots, m$ (since Q_1 is a prime hyperideal), which is also a contradiction. Thus $K \subseteq P_i$ for some i. □

3. Prime radicals of hyperideals

At the on-set of the study of the prime radical of a hyperideal of a multiplicative hyperring, we fix up for any element r of a multiplicative hyperring R the notation that (for any positive integer $n > 1$) $r^n = \underbrace{r \circ r \circ \ldots \circ r}_{n \text{ times}}$ and $r^1 = \{r\}$.

Definition 3.1. Let I be a hyperideal of a multiplicative hyperring $(R, +, \circ)$. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by $Rad(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $Rad(I) = R$.

Proposition 3.2. *Let I be a hyperideal of a commutative multiplicative hyperring $(R, +, \circ)$. Then, $D \subseteq Rad(I)$ where $D = \{r \in R\colon r^n \subseteq I$ for some $n \in \mathbb{N}\}$. The equality holds when I is a $\mathcal{C}$-ideal of R.*

Proof. If $Rad(I) = R$, then $D \subseteq Rad(I)$. Assume that $Rad(I) \neq R$. Let $r \in D$. Then $r^n \subseteq I$ for some $n \in \mathbb{N}$. Hence, for any prime hyperideal P of R, containing I, $r^n \subseteq P$ and hence $r \in P$. So, $r \in Rad(I)$, i.e., $D \subseteq Rad(I)$.

Now suppose that I is a $\mathcal{C}$-ideal. Let $r \notin D$. Then, for any $n \in \mathbb{N}$, $r^n \nsubseteq I$. Thus $r^n \cap I = \phi$ for all $n \in \mathbb{N}$ (since, I is a $\mathcal{C}$-ideal). Let $S = \cup\{r^n + I : n \in \mathbb{N}\}$. Then, for any $a, b \in S$, $a \circ b \subseteq S$. Hence, S is a multiplicative set. Here, $S \cap I = \phi$, as otherwise, if $p \in S \cap I$, then there exist $x \in I$ and $y \in r^n$ for some $n \in \mathbb{N}$, such that $p = x + y$, implying that $y \in I$ which is contradictory to the fact that $r^n \cap I = \phi$ for all $n \in \mathbb{N}$. Hence, by the proposition 2.17, there is a prime hyperideal P containing I, disjoint from S. Hence $r^n \cap P = \phi$ for any $n \in \mathbb{N}$. So, $r \notin P$ i.e. $r \notin Rad(I)$. Hence, $Rad(I) \subseteq D$. □

Proposition 3.3. *Let $I, I_1, I_2, \ldots, I_n$ are some $\mathcal{C}$-ideals of a commutative multiplicative hyperring $(R, +, \circ)$. Then,*

(*i*) $Rad(Rad\, I) = Rad(I)$,

(*ii*) $Rad(I_1 I_2 \ldots I_n) = Rad(\bigcap_{j=1}^{n} I_j) = \bigcap_{j=1}^{n} Rad(I_j)$.

Proof. (i) Let $r \in Rad(Rad\, I)$. Then there exists $n \in \mathbb{N}$ such that $r^n \subseteq Rad\, I$. So, for any $x \in r^n$ there exists $n_x \in \mathbb{N}$ such that $x^{n_x} \subseteq I$. Again, $x \in r^n \Rightarrow x^{n_x} \subseteq (r^n)^{n_x} = r^{nn_x}$. Hence, $r^{nn_x} \cap I \neq \phi$. Thus, $r^{nn_x} \subseteq I$ (since, I is a $\mathcal{C}$-ideal), whence $Rad(Rad\, I) \subseteq Rad(I)$. Clearly, $Rad(I) \subseteq Rad(Rad\, I)$. Thus, $Rad(Rad\, I) = Rad(I)$.

(ii) Here $I_1 I_2 \ldots I_n \subseteq \bigcap_{j=1}^{n} I_j$. So, $Rad(I_1 I_2 \ldots I_n) \subseteq Rad(\bigcap_{j=1}^{n} I_j)$. Now since each I_j is a $\mathcal{C}$-ideal, so is $\bigcap_{j=1}^{n} I_j$. Thus, by proposition 3.2, for any $x \in Rad(\bigcap_{j=1}^{n} I_j)$, we have $x^m \subseteq \bigcap_{j=1}^{n} I_j$ (for some $m \in \mathbb{N}$) $\Rightarrow$

$x^m \subseteq I_j$ for all $j(= 1, 2, \ldots, n) \Rightarrow x \in Rad(I_j) \Rightarrow x \in \bigcap_{j=1}^n Rad(I_j)$. So, $Rad(\bigcap_{j=1}^n I_j) \subseteq \bigcap_{j=1}^n Rad(I_j)$. Finally, let $x \in \bigcap_{j=1}^n Rad(I_j)$. Then, for each $j(= 1, 2, \ldots, n)$ there exists $m_j \in \mathbb{N}$ such that $x^{m_j} \subseteq I_j \Rightarrow x^{\sum_{j=1}^n (m_j)} \subseteq I_1 I_2 \ldots I_n \Rightarrow x \in Rad(I_1 I_2 \ldots I_n)$. Hence, $\bigcap_{j=1}^n Rad(I_j) \subseteq Rad(I_1 I_2 \ldots I_n)$. □

Definition 3.4. A hyperideal $Q(\neq R)$ in a commutative multiplicative hyperring $(R, +, \circ)$ is called a primary hyperideal of R if for any $a, b \in R$; $a \circ b \subseteq Q$ and $a \notin Q \Rightarrow b^n \subseteq Q$, for some $n \in \mathbb{N}$.

Example 3.5. Every prime hyperideal of a commutative multiplicative hyperring is a primary hyperideal. The set E of all even integers, is not a prime hyperideal, but is a primary hyperideal of the multiplicative hyperring of integers $\mathbb{Z}_A$ over the ring of integers $\mathbb{Z}$, induced by the set A of all positive even integers.

Proposition 3.6. *If Q is a primary $\mathcal{C}$-ideal of a commutative multiplicative hyperring $(R, +, \circ)$, then $Rad(Q)$ is a prime hyperideal.*

Proof. Let, $a \circ b \subseteq Rad(Q)$ and $a \notin Rad(Q)$. Then, by proposition 3.2, for any $x \in a \circ b$ there exists $n_x \in \mathbb{N}$ such that $x^{n_x} \subseteq Q$. Again, $x^{n_x} \subseteq (a \circ b)^{n_x} = a^{n_x} \circ b^{n_x}$ (since R is commutative). So, $(a^{n_x} \circ b^{n_x}) \cap Q \neq \phi$ and thus, $a^{n_x} \circ b^{n_x} \subseteq Q$ (since, Q is a $\mathcal{C}$-ideal). Now $a \notin Rad(Q) \Rightarrow a^{n_x} \nsubseteq Q$ (by proposition 3.2)$\Rightarrow a^{n_x} \cap Q = \phi$. Thus, for any $p \in a^{n_x}$ and $q \in b^{n_x}$, we have that $p \notin Q$ and $p \circ q \subseteq a^{n_x} \circ b^{n_x} \subseteq Q$. Thus, $q^{n_q} \subseteq Q$ for some $n_q \in \mathbb{N}$ (since Q is a primary hyperideal). Again $q \in b^{n_x} \Rightarrow q^{n_q} \subseteq (b^{n_x})^{n_q} = b^{n_x n_q}$. Hence, $b^{n_x n_q} \cap Q \neq \phi$ and thus, $b^{n_x n_q} \subseteq Q$, whence $b \in Rad(Q)$. So, $Rad(Q)$ is a prime hyperideal. □

For a $\mathcal{C}$-ideal Q of a commutative multiplicative hyperring R , we refer to the prime hyperideal $P = Rad(Q)$ as the associated prime hyperideal of Q and on the other hand Q is referred to as a P-primary $\mathcal{C}$-ideal of R.

Proposition 3.7. *Let Q be a $\mathcal{C}$-ideal and P be a hyperideal of a commutative multiplicative hyperring $(R, +, \circ)$. Then, Q is a P-primary $\mathcal{C}$-ideal of R if and only if*

(*i*) $Q \subseteq P \subseteq Rad(Q)$;

(*ii*) $a \circ b \subseteq Q$ *and* $a \notin Q \Rightarrow b \in P$.

Proof. Suppose (i) and (ii) hold. Let $a \circ b \subseteq Q$ and $a \notin Q$. Then, $b \in P \subseteq Rad(Q)$, whence, $b^n \subseteq Q$ for some $n \in \mathbb{N}$. So, Q is a primary $\mathcal{C}$-ideal of R. Now let $c \in Rad(Q)$. Suppose n be the least positive integer such that $c^n \subseteq Q$. If $n = 1$, then $c \in \{c\} = c^1 \subseteq Q \subseteq P$. If $n > 1$, $c^{n-1} \nsubseteq Q$ by the minimality of n and thus, $c^{n-1} \cap Q = \phi$ (since, Q is a $\mathcal{C}$-ideal). Then, for any $x \in c^{n-1}, x \circ c \subseteq c^{n-1} \circ c = c^n \subseteq Q$. Thus by (ii), $c \in P$ (since $x \notin Q$). So, $Rad(Q) \subseteq P$ whence $P = Rad(Q)$ (by (i)). Hence, Q is a P-primary $\mathcal{C}$-ideal of R. The converse part is immediate. □

Proposition 3.8. *If $Q_1, Q_2, \ldots, Q_n$ are primary $\mathcal{C}$-ideals of a commutative multiplicative hyperring R, all of which are P-primary for a prime hyperideal P, then $\bigcap_{i=1}^{n} Q_i$ is also a P-primary $\mathcal{C}$-ideal of the multiplicative hyperring R.*

Proof. Straightforward. □

Definition 3.9. A $\mathcal{C}$-ideal I of a commutative multiplicative hyperring R, is said to have a $\mathcal{C}$-primary decomposition if $I = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ for some primary $\mathcal{C}$-ideals Q_i of R. If no Q_i contains $Q_1 \cap Q_2 \cap \ldots \cap Q_{i-1} \cap Q_{i+1} \cap \ldots \cap Q_n$ and the radicals of the Q_i are all distinct, then the $\mathcal{C}$-primary decomposition is said to be reduced. A hyperideal I of R is said to have a $\mathcal{C}$-primary decomposition (resp. reduced $\mathcal{C}$-primary decomposition) if the $\mathcal{C}$-closure $\mathcal{C}(I)$ of I has a $\mathcal{C}$-primary decomposition (resp. reduced $\mathcal{C}$-primary decomposition).

Proposition 3.10. *If a hyperideal I of a commutative multiplicative hyperring R has a $\mathcal{C}$-primary decomposition, then I has a reduced $\mathcal{C}$-primary decomposition.*

Proof. Straightforward. □

4. Prime, primary hyperideals and $\mathcal{C}$-ideals in $\mathbb{Z}_A$

Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. Corresponding to every subset $A \in P^*(\mathbb{Z}) = P(R) \setminus \{\phi\}$ $(|A| \geq 2)$, there exists a multiplicative hyperring $(\mathbb{Z}_A, +, \circ)$, where $\mathbb{Z}_A = \mathbb{Z}$ and for any $x, y \in \mathbb{Z}_A, x \circ y = \{x \cdot a \cdot y : a \in A\}$. For the sake of brevity, the product $a \cdot b$ of any two elements a, b in the ring of integers $(\mathbb{Z}, +, \cdot)$, will be written simply as ab.

This is immediate to observe that the principal hyperideal $\langle a \rangle$ of the multiplicative hyperring $\mathbb{Z}_A$ over the ring of integers $\mathbb{Z}$, induced by any $A \in P^*(\mathbb{Z})$ is identical with the principal ideal generated by a in the ring of integers $\mathbb{Z}$. Moreover, every hyperideal of a $\mathbb{Z}_A$ is a principal hyperideal. But, unlike in the ring of integers $\mathbb{Z}$, a principal hyperideal in a multiplicative hyperring of integers $\mathbb{Z}_A$, generated by a prime integer may not be a prime hyperideal of $\mathbb{Z}_A$, as is shown in the following example.

Example 4.1. In the multiplicative hyperring of integers $\mathbb{Z}_A$ with $A = \{14, 21\}$, the principal hyperideal $\langle 7 \rangle = \{7n : n \in \mathbb{Z}\}$ is not a prime hyperideal. In fact, $1 \circ 1 = \{14, 21\} \subseteq \langle 7 \rangle$, but $1 \notin \langle 7 \rangle$.

Example 4.2. In the multiplicative hyperring of integers $\mathbb{Z}_A$ with $A = \{2, 3\}$, every principal hyperideal generated by prime integer is a prime hyperideal.

Proposition 4.3. *In a multiplicative hyperring of integers $\mathbb{Z}_A$, a principal hyperideal $\langle p \rangle$ generated by a positive integer p, is a prime hyperideal of $\mathbb{Z}_A$ if and only if p is a prime integer and $A \not\subseteq \langle p \rangle$.*

Proof. Let p be a prime integer and $A \not\subseteq \langle p \rangle$. Then, there exists $\alpha \in A \setminus \langle p \rangle$. Now suppose that $a \circ b \subseteq \langle p \rangle$ and $a \notin \langle p \rangle$. Then, $a\alpha b \in \langle p \rangle$ and hence $b \in \langle p \rangle$ (since $\langle p \rangle$ is a prime ideal of Z, $\alpha \notin \langle p \rangle$ and $a \notin \langle p \rangle$). Thus $\langle p \rangle$ is a prime hyperideal of the multiplicative hyperring $\mathbb{Z}_A$.

Conversely let p be a positive integer and the hyperideal $\langle p \rangle$ be a prime hyperideal of $\mathbb{Z}_A$. Suppose that $a, b \in \mathbb{Z}$ such that $p \mid ab$ and $p \nmid a$. Then, $a \notin \langle p \rangle$. But $ab \in \langle p \rangle$ and so, for any $\alpha \in A, a\alpha b = ab\alpha \in \langle p \rangle$. Hence $a \circ b \subseteq \langle p \rangle$ and thus $b \in \langle p \rangle$ (since $\langle p \rangle$ is a prime hyperideal of multiplicative hyperring $\mathbb{Z}_A$). Hence $p \mid b$ and thus p is a prime integer. If $A \subseteq \langle p \rangle$ then, for any $a \notin \langle p \rangle, b \notin \langle p \rangle$ and for any $\alpha \in A$, $a\alpha b \in \langle p \rangle$, i.e., $a \circ b \subseteq \langle p \rangle$ which is a contradiction. □

Proposition 4.4. *In a multiplicative hyperring of integers $\mathbb{Z}_A$, a principal hyperideal $\langle p \rangle$ generated by a positive integer p, is a prime $\mathcal{C}$-ideal of $\mathbb{Z}_A$ if and only if p is a prime integer and $A \cap \langle p \rangle = \phi$.*

Proof. Let p be a prime integer and $A \cap \langle p \rangle = \phi$. Then, by proposition 4.3, $\langle p \rangle$ is a prime hyperideal of $\mathbb{Z}_A$. Now let $r_i \in \mathbb{Z}(i = 1, 2, \ldots, n)$ be such that $r_1 \circ r_2 \circ \ldots \circ r_n \cap \langle p \rangle \neq \phi$. Then, there are some $\alpha_i \in A$ such that $r_1\alpha_1 r_2\alpha_2\ldots\alpha_{n-1}r_n \in \langle p\rangle$, i.e., $(r_1r_2\ldots r_n)(\alpha_1\alpha_2\ldots\alpha_{n-1}) \in \langle p\rangle$. So, $p \mid (r_1r_2\ldots r_n)(\alpha_1\alpha_2\ldots\alpha_{n-1})$ and $p \nmid (\alpha_1\alpha_2\ldots\alpha_{n-1})$ (since $A\cap\langle p\rangle = \phi$). Hence $p \mid (r_1r_2\ldots r_n)$ and thus $p \mid r_i$ for some i. So, $r_1\circ r_2\circ\ldots\circ r_n \subseteq \langle p\rangle$. Consequently, $\langle p\rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$.

Conversely let, for a positive integer p, $\langle p\rangle$ be a prime $\mathcal{C}$-ideal of $\mathbb{Z}_A$. Then, by proposition 4.3, p is a prime integer (and $A \nsubseteq \langle p\rangle$). If $A\cap\langle p\rangle \neq \phi$, there is an $\alpha\in A$ such that $\alpha\in\langle p\rangle$. Moreover, $\alpha = 1\alpha 1 \in 1\circ 1$. Thus, $(1\circ 1)\cap\langle p\rangle \neq \phi$, but $1\circ 1 \nsubseteq \langle p\rangle$ (since, $\langle p\rangle$ is a prime hyperideal of multiplicative hyperring $\mathbb{Z}_A$ and $1\notin\langle p\rangle$). This is a contradiction (since $\langle p\rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$). So, $A\cap\langle p\rangle = \phi$. □

Proposition 4.5. *For a positive integer a, if the principal hyperideal $\langle a\rangle$ of a multiplicative hyperring of integers $\mathbb{Z}_A$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$, then either $A\subseteq\langle a\rangle$ or else $A\cap\langle a\rangle = \phi$.*

Proof. For a positive integer a, let the principal hyperideal $\langle a\rangle$ of a multiplicative hyperring of integers $\mathbb{Z}_A$ is such that $A\nsubseteq\langle a\rangle$ and $A\cap\langle a\rangle \neq \phi$. Then there are $\alpha,\beta\in A$ such that $\alpha\in\langle a\rangle$ and $\beta\in\mathbb{Z}\setminus\langle a\rangle$. So, $\alpha = 1\alpha 1 \in 1\circ 1\cap\langle a\rangle$, i.e., $1\circ 1\cap\langle a\rangle \neq\phi$, whereas $1\circ 1\nsubseteq\langle a\rangle$, since $\beta = 1\beta 1\in 1\circ 1$ and $\beta\notin\langle a\rangle$. Hence, $\langle a\rangle$ is not a $\mathcal{C}$-ideal of $\mathbb{Z}_A$. □

Proposition 4.6. *In a multiplicative hyperring of integers $\mathbb{Z}_A$ if $A\subseteq\langle a\rangle(\neq\mathbb{Z})$ for some positive integer a, then the principal hyperideal $\langle a\rangle$ is a non-prime primary $\mathcal{C}$-ideal of the multiplicative hyperring $\mathbb{Z}_A$.*

Proof. since $A\subseteq\langle a\rangle$, so for any $r_i\in\mathbb{Z}(i=1,2,\ldots,n;n\in\mathbb{N})$, $r_1\circ r_2\circ\ldots\circ r_n = \{r_1\alpha_1r_2\alpha_2\ldots\alpha_{n-1}r_n : \alpha_i\in A, i=1,2,\ldots,n; n\in\mathbb{N}\}\subseteq\langle a\rangle$. Hence, vacuously, $\langle a\rangle$ is a $\mathcal{C}$-ideal of the multiplicative hyperring $\mathbb{Z}_A$. By the same argument, since for any $r\in\mathbb{Z}, r^2 = r\circ r\subseteq\langle a\rangle$, we can say that $\langle a\rangle$ is a primary hyperideal of $\mathbb{Z}_A$. Again, for $1\notin\langle a\rangle$ (since $\langle a\rangle\neq\mathbb{Z}$), $1\circ 1\subseteq\langle a\rangle$. Thus $\langle a\rangle$ is not a prime hyperideal of the multiplicative hyperring $\mathbb{Z}_A$. □

For any $A\in P^*(\mathbb{Z})$ and any positive integer a, the condition that $A\cap\langle a\rangle = \phi$ is not a sufficient one for the principal hyperideal $\langle a\rangle$ to be a $\mathcal{C}$-ideal of the multiplicative hyperring $\mathbb{Z}_A$.

Example 4.7. Let $a = 12$ and $A = \{2,3\}$. Then, $12 = 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 3\cdot 1\in 1\circ 1\circ 1\circ 1$, i.e., $1\circ 1\circ 1\circ 1\cap\langle a\rangle\neq\phi$, whereas $1\circ 1\circ 1\circ 1\nsubseteq\langle a\rangle$, since $16 = 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\in 1\circ 1\circ 1\circ 1$ and $16\notin\langle a\rangle$. Hence, for

$A = \{2, 3\}$, though $A \cap \langle 12 \rangle = \phi$, the hyperideal $\langle 12 \rangle$ is not a $\mathcal{C}$-ideal of the multiplicative hyperring of integers $\mathbb{Z}_A$.

Corollary 4.8. *A principal hyperideal $\langle p \rangle$ of a multiplicative hyperring of integers $\mathbb{Z}_A$, generated by a prime integer p, is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$ if and only if either $A \subseteq \langle p \rangle$ or else $A \cap \langle p \rangle = \phi$.*

Proof. If $A \subseteq \langle p \rangle$, then by proposition 4.6, $\langle p \rangle$ is a $\mathcal{C}$-ideal of the multiplicative hyperring $\mathbb{Z}_A$. If $A \cap \langle p \rangle = \phi$, then by proposition 4.4, $\langle p \rangle$ is a $\mathcal{C}$-ideal of a multiplicative hyperring $\mathbb{Z}_A$. The converse part follows from proposition 4.5. □

Corollary 4.9. *In a multiplicative hyperring of integers $\mathbb{Z}_A$, if a principal hyperideal $\langle p \rangle$, generated by a prime integer p, is a $\mathcal{C}$-ideal, then $\langle p \rangle$ is a primary hyperideal of $\mathbb{Z}_A$.*

Proof. It follows from corollary 4.8, proposition 4.4 and proposition 4.6. □

Following is an example of a primary hyperideal, generated by a prime integer, which is not a $\mathcal{C}$-ideal of a multiplicative hyperring $\mathbb{Z}_A$.

Example 4.10. In the multiplicative hyperring of integers $\mathbb{Z}_A$ with $A = \{2, 3\}$, the hyperideal $\langle 2 \rangle$ is a prime hyperideal (by proposition 4.3, since $A \not\subseteq \langle 2 \rangle$) and thus a primary hyperideal; but it is not a $\mathcal{C}$-ideal (by proposition 4.4, since $A \cap \langle 2 \rangle \neq \phi$).

Proposition 4.11. *Let a be a positive integer and the principal hyperideal $\langle a \rangle (\neq \mathbb{Z})$ be a $\mathcal{C}$-ideal of a multiplicative hyperring of integers $\mathbb{Z}_A (|A| > 1)$. Then for each prime factor p of a, the principal hyperideal $\langle p \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$.*

Proof. Since $\langle a \rangle \neq \mathbb{Z}$, $a > 1$. If a is a prime integer then there is nothing to prove. Thus we suppose that p is a prime factor of a and $a = p^m p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, where p_i's are distinct prime integers, different from p ; m and m_i are some positive integers. If possible let $A \not\subseteq \langle p \rangle$ and $A \cap \langle p \rangle \neq \phi$. Then there exist $\alpha, \beta \in A$ such that $\alpha \in \langle p \rangle$ and $\beta \in \mathbb{Z} \setminus \langle p \rangle$.

We claim that $\alpha \in \langle a \rangle$. In fact, if $\alpha \notin \langle a \rangle$ then $\alpha = p^l p_1^{l_1} p_2^{l_2} \dots p_k^{l_k} b$, where b is a positive integer such that $p \nmid b$, $p_i \nmid b$ (for any i) and l, l_i ($l \geq 1$ and $l_i \geq 0$) are integers such that either $l < m$ or $l_i < m_i$ for some i. Now we choose some integers n, n_i in such a way that $n = 0$ or $m - l$ according as $l \geq m$ or $l < m$ and also $n_i = 0$ or $m_i - l_i$ according as $l_i \geq m_i$ or $l_i < m_i$. We consider now the integer $c = p^n p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$. Then, $\alpha c = bp^{l+n} p_1^{l_1+n_1} p_2^{l_2+n_2} \ldots p_k^{l_k+n_k} = bp^{m'} p_1^{m'_1} p_2^{m'_2} \ldots p_k^{m'_k}$, where $m' = l + n \geq m$ and $m'_i = l_i + n_i \geq m_i$ for all i (due to the choice of n and n_i). So, $p^{m'} p_1^{m'_1} p_2^{m'_2} \ldots p_k^{m'_k} = ad$ for some $d \in \mathbb{Z}$ whence $\alpha c = abd \in \langle a \rangle$. Also, $\alpha c = 1\alpha c \in 1 \circ c$ (since, $\alpha \in A$). Thus $(1 \circ c) \cap \langle a \rangle \neq \phi$ and hence $1 \circ c \subseteq \langle a \rangle$ (since $\langle a \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$). So, for $\beta \in A$, $1\beta c \in 1 \circ c \subseteq \langle a \rangle$. Hence, $\beta p^n p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} = p^m p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k} s$ (for some $s \in \mathbb{Z}$) $\Rightarrow \beta = p^{m-n} p_1^{m_1-n_1} p_2^{m_2-n_2} \ldots p_k^{m_k-n_k} s$. Now we see that $m - n$ is either $m(\geq 1)$ or $l(\geq 1)$ (and that $m_i - n_i$ is either $m_i(\geq 1)$ or $l_i(\geq 0)$). Thus $p \mid \beta$ which is a contradiction (since $\beta \notin \langle p \rangle$). Thus, $\alpha \in \langle a \rangle$.

Now since $\beta \notin \langle p \rangle$, so $\beta \notin \langle a \rangle$. But $\beta = 1\beta 1 \in 1 \circ 1$. So, $1 \circ 1 \nsubseteq \langle a \rangle$, whereas $\alpha = 1\alpha 1 \in (1 \circ 1) \cap \langle a \rangle$ which is again a contradiction (since $\langle a \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$). Thus, our initial assumptions that $A \nsubseteq \langle p \rangle$ and $A \cap \langle p \rangle \neq \phi$ are not simultaneously ture. So, either $A \subseteq \langle p \rangle$ or else $A \cap \langle p \rangle = \phi$. Hence, by corollary 4.8, $\langle p \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$. So, the hyperideals of $\mathbb{Z}_A$, generated by the prime factors of a, are $\mathcal{C}$-ideals of $\mathbb{Z}_A$. □

Example 4.12. Let $a = 5400$ and $A = \{6, 216\}$. The only prime factors of a are $2, 3, 5$ and we see that $A \subseteq \langle 2 \rangle$, $A \subseteq \langle 3 \rangle$ and $A \cap \langle 5 \rangle = \phi$. Thus by corollary 4.8, the hyperideals of the multiplicative hyperring of integers $\mathbb{Z}_A$, generated by the prime factors of a, are $\mathcal{C}$-ideals of the multiplicative hyperring $\mathbb{Z}_A$. But here $\langle a \rangle$ is itself not a $\mathcal{C}$-ideal of the multiplicative hyperring of integers $\mathbb{Z}_A$. In fact, $5400 = 5 \cdot 216 \cdot 5 \in 5 \circ 5 \cap \langle a \rangle$, i.e., $5 \circ 5 \cap \langle a \rangle \neq \phi$, whereas $5 \circ 5 \nsubseteq \langle a \rangle$, since $150 = 5 \cdot 6 \cdot 5 \in 5 \circ 5$ and $150 \notin \langle a \rangle$.

Proposition 4.13. *For a positive integer $a(> 1)$, the principal hyperideal $\langle a \rangle$ of a multiplicative hyperring of integers $\mathbb{Z}_A$, is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$ if and only if exactly one of the following two conditions hold true.*

I. All the principal hyperideals of $\mathbb{Z}_A$, generated by the prime factors of a are prime $\mathcal{C}$-ideals of $\mathbb{Z}_A$.

II. $a = p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n}$ is a representation of a as a product of distinct prime integers $p_i (i = 1, 2, \ldots n; n \in \mathbb{N})$($m_i$'s being some positive integers) corresponding to which there exists a positive integer $k (1 \leq k \leq n)$ such that

(i) $\alpha \in A \Rightarrow \alpha = p_1^{t_1} p_2^{t_2} \ldots p_k^{t_k} r_\alpha$ for some $t_j \in \mathbb{N}(j = 1, 2, \ldots, k)$ and $r_\alpha \in \mathbb{Z}$ with $p_i \nmid r_\alpha$ for any i, and also

(ii) $\alpha = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k} r_\alpha \in A$ and $\beta = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k} r_\beta \in A \Rightarrow$ for any j, $t_j = s_j$ whenever $t_j < m_j$.

Proof. For a positive integer $a(> 1)$, let the principal hyperideal $\langle a \rangle$ of a multiplicative hyperring of integers $\mathbb{Z}_A$ be a $\mathcal{C}$-ideal of $\mathbb{Z}_A$. Then, by proposition 4.11, for each prime factor p of a, the hyperideal $\langle p \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$. So, by corollary 4.8, for each prime factor p of a, either $A \subseteq \langle p \rangle \dots (1)$ or else $A \cap \langle p \rangle = \phi \dots (2)$. If (2) is true for any prime factor of a, then all the hyperideals of $\mathbb{Z}_A$, generated by the prime factors of a are prime $\mathcal{C}$-ideals of $\mathbb{Z}_A$ (by proposition 4.4). Suppose, (2) is not true for some prime factors of a. Then, we can write $a = p_1^{m_1} p_2^{m_2} \dots p_n^{m_n}$ (where p_i's are some distinct prime integers and m_i's are some positive integers) such that $A \subseteq (p_j)$ for some $j = 1, 2, \dots, k; (1 \le k \le n)$ and in case when $k < n, A \cap (p_j) = \phi$ for $j = k+1, \dots, n$. Thus, for each $\alpha \in A$, there are some $t_j \in \mathbb{N}(j = 1, 2, \dots, k)$ and $r_\alpha \in \mathbb{Z}$ with $p_i \nmid r_\alpha$ (for any $i = 1, 2, \dots, n$) such that $\alpha = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k} r_\alpha$. Let $\alpha = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k} r_\alpha$ and $\beta = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k} r_\beta$ be two elements of A and suppose that $t_l < m_l$ for some $l \in \{1, 2, \dots k\}$. Then we have the following cases.

1. If $s_l < t_l$, then for $r = p_1^{m_1} \dots p_{l-1}^{m_{l-1}} p_l^{m_l - t_l} p_{l+1}^{m_{l+1}} \dots p_n^{m_n}$, we see that $\alpha r \in 1 \circ r \cap \langle a \rangle$, i.e., $1 \circ r \cap \langle a \rangle \neq \phi$; whereas $1 \circ r \not\subseteq \langle a \rangle$ since $\beta r \in 1 \circ r$ and $\beta r \notin \langle a \rangle$. Thus we arrive at a contradiction (since, $\langle a \rangle$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$).
2. If $s_l > t_l$ and $s_l < m_l$, then for $r = p_1^{m_1} \dots p_{l-1}^{m_{l-1}} p_l^{m_l - s_l} p_{l+1}^{m_{l+1}} \dots p_n^{m_n}$, we see that $\beta r \in 1 \circ r \cap \langle a \rangle$, i.e., $1 \circ r \cap \langle a \rangle \neq \phi$; whereas $1 \circ r \not\subseteq \langle a \rangle$ (since $\alpha r \in 1 \circ r$ and $\alpha r \notin \langle a \rangle$) - a contradiction.
3. If $s_l > t_l$ and $s_l > m_l$, then for $r = p_1^{m_1} \dots p_{l-1}^{m_{l-1}} p_{l+1}^{m_{l+1}} \dots p_n^{m_n}$, we see that $\beta r \in 1 \circ r \cap \langle a \rangle$; whereas $1 \circ r \not\subseteq \langle a \rangle$ (since, $t_l < m_l$) - a contradiction.

Thus, for any j, $t_j = s_j$ whenever $t_j < m_j$.

Conversely, let the condition (I) hold true for the positive integer a. Then, by proposition 4.4, $A \cap \langle p \rangle = \phi$ for any prime factors of a. Suppose that $r_1 \circ r_2 \circ \dots \circ r_m \cap \langle a \rangle \neq \phi$ for some $r_\imath \in \mathbb{Z}(\imath = 1, 2, \dots, m; m \in \mathbb{N}, m > 1)$. Then, there are $\alpha_\imath \in A(\imath = 1, 2, \dots, m-1)$ such that $(r_1 r_2 \dots r_m)(\alpha_1 \alpha_2 \dots \alpha_{m-1}) \in \langle a \rangle$. Now, since none of $\alpha_\imath$'s is divisible by any prime factor of a, so $r_1 r_2 \dots r_m \in \langle a \rangle$ and hence, $r_1 \circ r_2 \circ \dots \circ r_m \subseteq \langle a \rangle$. Thus, the principal hyperideal $\langle a \rangle$ of the multiplicative hyperring $\mathbb{Z}_A$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$.

Now let the condition (II) hold true for the positive integer $a = p_1^{m_1} p_2^{m_2} \dots p_n^{m_n}$ (where p_i's are distinct prime integers and m_i's are some positive integers). Then, by condition (II)(i) there is a positive integer $k(1 \le k \le n)$ such that, for any $\alpha \in A$, $\alpha = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k} r_\alpha$ (for some $t_j \in$

$\mathbb{N}(j = 1, 2, \ldots, k)$ and $r_\alpha \in \mathbb{Z}$ with $p_i \nmid r_\alpha$ for any i). If there is $\beta = p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k} r_\beta \in A$, with $s_j \geq m_j$ for all $j = 1, 2, \ldots k$, then, by virtue of the condition (II)(ii), we obtain that, for any $r_\imath \in \mathbb{Z}(\imath = 1, 2, \ldots, m; m \in \mathbb{N}, m > 1)$, $r_1 \circ r_2 \circ \ldots \circ r_m \subseteq \langle a \rangle$ or $r_1 \circ r_2 \circ \ldots \circ r_m \cap \langle a \rangle = \phi$ according as $r_1 r_2 \ldots r_m \in \langle b \rangle$ or not (where $b = p_{k+1}^{m_{k+1}} \ldots p_n^{m_n}$). Thus, in this case, the principal hyperideal $\langle a \rangle$ of the multiplicative hyperring $\mathbb{Z}_A$ is a $\mathcal{C}$-ideal of $\mathbb{Z}_A$. Suppose on the contrary that there is an $l(1 \leq l \leq k)$ such that $s_j < m_j$ for $j = 1, 2, \ldots l$ and (in case when $l < k$) $s_j \geq m_j$, for $j = l+1, \ldots k$. Then, by the condition (II)(ii),

$$A \subseteq \{p_1^{s_1} p_2^{s_2} \ldots p_l^{s_l} p_{l+1}^{m'_{l+1}} \ldots p_k^{m'_k} r : m'_j > m_j, r \in \mathbb{Z} \text{ with } p_i \nmid r \text{ for any } i\}.$$

So, for any $r_\imath \in \mathbb{Z}(\imath = 1, 2, \ldots, m; m \in \mathbb{N}, m > 1)$, we have that $r_1 \circ r_2 \circ \ldots \circ r_m \subseteq S$ where

$$S = \{(r_1 r_2 \ldots r_m)(p_1^{(m-1)s_1} p_2^{(m-1)s_2} \ldots p_l^{(m-1)s_l} p_{l+1}^{\mu_{l+1}} \ldots p_k^{\mu_k}) u : \mu_j > m_j, \\ u \in \mathbb{Z} \text{ with } p_i \nmid u, \forall i\}.$$

Now suppose that $r_1 \circ r_2 \circ \ldots \circ r_m \cap \langle a \rangle \neq \phi$. Then, for some $\mu_j > m_j$, $\lambda_i \geq m_i$ and $u, v \in \mathbb{Z}$ with $p_i \nmid u$ and $p_i \nmid v$ for any i, we have

$$(r_1 r_2 \ldots r_m)(p_1^{(m-1)s_1} p_2^{(m-1)s_2} \ldots p_l^{(m-1)s_l} p_{l+1}^{\mu_{l+1}} \ldots p_k^{\mu_k}) u = p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_n^{\lambda_n} v.$$

Since, $p_i \nmid v$ for any i, so $(m-1)s_j \leq \lambda_j$ and $\mu_j \leq \lambda_j$ and thus

$$r_1 r_2 \ldots r_m = p_1^{\lambda_1 - (m-1)s_1} p_2^{\lambda_2 - (m-1)s_2} \ldots p_l^{\lambda_l - (m-1)s_l} \cdot \\ \cdot p_{l+1}^{\lambda_{l+1} - \mu_{l+1}} \ldots p_k^{\lambda_k - \mu_k} p_{k+1}^{\lambda_{k+1}} \ldots p_n^{\lambda_n} w \ldots (*)$$

where $w = \frac{v}{u} \in \mathbb{Z}$, since $p_i \nmid u$ for any i. So, for any

$$x = (r_1 r_2 \ldots r_m)(p_1^{(m-1)s_1} p_2^{(m-1)s_2} \ldots p_l^{(m-1)s_l} p_{l+1}^{\nu_{l+1}} \ldots p_k^{\nu_k}) z \in r_1 \circ r_2 \circ \ldots \circ r_m,$$

we have from the relation $(*)$ that

$$x = p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_l^{\lambda_l} p_{l+1}^{\lambda_{l+1} - \mu_{l+1} + \nu_{l+1}} \ldots p_k^{\lambda_k - \mu_k + \nu_k} p_{k+1}^{\lambda_{k+1}} \ldots p_n^{\lambda_n} wz \in \langle a \rangle.$$

Hence, the principal hyperideal $\langle a \rangle$ is a $\mathcal{C}$-ideal of the multiplicative hyperring of integers $\mathbb{Z}_A$. □

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