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PSEUDO-CONVERGENCES OF SEQUENCES OF MEASURABLE FUNCTIONS ON MONOTONE MULTIMEASURE SPACES

BY

ANCA PRECUPANU and ALINA GAVRILUŢ

Abstract. In this paper, we study different types of pseudo-convergences of sequences of measurable functions with respect to set-valued non-additive monotonic set functions and we establish some pseudo-versions of Egoroff theorem in the set-valued case. We also characterize important structural properties of monotone multimeasures.

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1. Introduction

Convergence theorems for sequences of measurable functions play a central role in classical measure theory.

Relationships among different types of convergences such as almost everywhere convergence, almost uniform convergence and convergence in measure were especially described by the fundamental results contained in the Egoroff, Lebesgue and Riesz theorems (Precupanu [19]).

It is well-known that in non-additive measure theory, developed in the last years by numerous authors as Wang and Klir [29], Pap [18], Denneberg [1], these results do not hold without additional conditions.

In this way, we mention the papers of Li and Yasuda [13], Li [9, 10], Li and Li [11], Li et al. [14], Murofushi et al. [17], Kawabe [6, 7] concerning Egoroff's theorem, or the papers of Li [9], Song and Li [23] for Lebesgue's theorem or Sun [24] for Riesz's theorem and Jiang et al. [5]

or Takahashi et al. [26], Ha et al. [3], Liu [15], Li et al. [12], Li [9], Li et al. [14], Li et al. [8], concerning different convergence theorems of sequences of measurable functions in non-additive measure theory. We also remark the papers of Murofushi [16], Ren et al. [22], Sun [25], Zhang [27], Wang [28] and many others.

More recently, motivated by the applied problems coming from mathematic economics, artificial intelligence, biomathematics and other important fields, some of the above mentioned results were generalized in the set-valued case.

Thus, we remark the paper of Liu [15], in which are given set-valued versions of Egoroff theorem and of Lebesgue theorem for sequences of set-valued measurable functions, our papers [20, 21] concerning Egoroff and Lusin theorems for set-valued fuzzy (i.e., monotone) multimeasures, or the paper of Wu and Liu [30], which contains a set-valued version of Riesz theorem.

The aim of this paper is to investigate, for set-valued non-additive monotonic set functions, some relationships among the main types of convergences of sequences of measurable functions. We especially insist on the different types of pseudo-convergences of sequences of measurable functions, for the set-valued non-additive monotonic set functions, such as, pseudo-almost everywhere (p.a.e.), pseudo-almost uniform (p.a.u) convergences and pseudo-convergence in measure (p.μ) and on the relationships among them, or with almost everywhere, almost uniform convergences and convergence in measure.

Thus, we give a set-valued pseudo-version of Egoroff's theorem and, as a consequence, we obtain a result which emphasizes the non-hereditary character of the pseudo-convergences. We also give characterizations of several important structural properties of monotone multimeasures.

2. Terminology and notations

Let T be an abstract space, $\mathcal{A}$ a σ-algebra of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of $X, \mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of X, $\mathcal{P}_{bfc}(X)$ the family of all bounded, closed, convex nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$ given by:

$$h(M,N) = \max\{e(M,N), e(N,M)\}, \text{ for every } M,N \in \mathcal{P}_f(X),$$

where $e(M,N) = \sup_{x\in M} d(x,N)$. e is called *the excess* of M over N.

It is known that $e(M,N) = 0$ if and only if $M \subset N$. Also, $e(M,N) \leq e(M,P) + e(P,N)$, for every $M,N,P \in \mathcal{P}_f(X)$. On $\mathcal{P}_{bf}(X)$, h becomes a metric [4].

We denote $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X. On $\mathcal{P}_0(X)$ we introduce the Minkowski addition " $\overset{\bullet}{+}$ " defined by:

$$M \overset{\bullet}{+} N = \overline{M+N}, \text{ for every } M,N \in \mathcal{P}_0(X),$$

where $M + N = \{x+y; x \in M, y \in N\}$ and $\overline{M+N}$ is the closure of $M+N$ with respect to the topology induced by the norm of X. We denote $A \cap \mathcal{A} = \{E \subset A, E \in \mathcal{A}\}$, where A is a fixed set in $\mathcal{A}$.

We also recall the following cancelation law [4]:

(C) If $M \overset{\bullet}{+} N = M \overset{\bullet}{+} P$, where $M,N,P \in \mathcal{P}_{bfc}(X)$, then $N = P$.

We shall also use property

$(\tilde{C})$ $h(M \overset{\bullet}{+} N, M \overset{\bullet}{+} P) = h(N,P)$, for every $M,N,P \in \mathcal{P}_{bfc}(X)$.

Obviously, $(\tilde{C})$ implies (C).

By $\mathbb{N}$ we mean the set of all naturals and by $\mathbb{N}^*$ we mean $\mathbb{N}\backslash\{0\}$.

We shall also use the following:

Lemma 2.1. *If $(A_n)_n, (B_n)_n, (C_n)_n$ are sequences of nonvoid closed subsets of T, then:*

i) If $A_n \subset B_n$, for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} h(A_n,A) = 0 = \lim_{n\to\infty} h(B_n,B)$, then $A \subset B$.

ii) If $A_n \subset B_n \subset C_n$, for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} h(A_n,E) = 0 = \lim_{n\to\infty} h(C_n,E)$, then $\lim_{n\to\infty} h(B_n,E) = 0$.

Proof. i) We observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} e(A,B) &\leq e(A,A_n) + e(A_n,B_n) + e(B_n,B) = \\ &= e(A,A_n) + e(B_n,B) \leq h(A,A_n) + h(B_n,B), \end{aligned}$$

whence, by the hypothesis, $e(A,B) = 0$, which means that $A \subset B$.

ii) We observe that, for every $n \in \mathbb{N}$,

$$\begin{array}{rcl} e(B_n, E) & \leq & e(B_n, C_n) + e(C_n, E) = e(C_n, E) \leq h(C_n, E), \\ e(E, B_n) & \leq & e(E, A_n) + e(A_n, B_n) = e(E, A_n) \leq h(E, A_n). \end{array}$$

Consequently, by the hypothesis, $\lim_{n\to\infty} h(B_n, E) = 0$. □

Throughout the paper we shall use the following notions in the set valued case:

Definition 2.2 ([2, 20, 21])**.** A set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is said to be:

i) a *fuzzy multimeasure* if μ is monotone with respect to the inclusion of sets (i.e., $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$) and $\mu(\emptyset) = \{0\}$.

ii) *continuous from below* if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \nearrow A$.

iii) *continuous from above* if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$.

iv) a *fuzzy multimeasure in the sense of Sugeno*, for short (S)-*fuzzy multimeasure,* if μ is a fuzzy multimeasure which is continuous from below and continuous from above.

v) *order continuous* if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$.

vi) *strongly order continuous* if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$ and $\mu(A) = \{0\}$.

vii) *pseudo-order continuous* if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$ and every $B \in \mathcal{A}$, with $A_n \subset B$, for every n, $A_n \searrow A$ and $\mu(B \backslash A) = \mu(B)$.

viii) *null-additive* if $\mu(A \cup B) = \mu(B)$, for every disjoint $A, B \in \mathcal{A}$, with $\mu(A) = \{0\}$.

ix) *pseudo-null-additive* if $\mu(B \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(A \backslash B) = \mu(A)$.

x) a) *autocontinuous from below* (*autocontinuous from above*, respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} h(\mu(A\backslash B_n), \mu(A)) = 0$ ($\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$, respectively).

b) *autocontinuous* if it is autocontinuous from above and autocontinuous from below.

xi) a) *pseudo-autocontinuous from above* (*pseudo-autocontinuous from below,* respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(B_n \cap A), \mu(A)) = 0$, we have $\lim_{n\to\infty} h((\mu(A\backslash B_n) \cup C), \mu(C)) = 0$ (respectively, $\lim_{n\to\infty} h((\mu(B_n \cap C), \mu(C)) = 0$), for every $C \in A \cap \mathcal{A}$.

b) *pseudo-autocontinuous* if it is pseudo-autocontinuous from above and pseudo-autocontinuous from below.

xii) an *additive multimeasure* if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) \stackrel{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

Unless stated otherwise, all over the paper we assume that $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is *a fuzzy* (*i.e., monotone*) *multimeasure.* By $\mathcal{M}$ we denote the class of all $\mathcal{A}$-measurable real-valued functions on $(T, \mathcal{A}, \mu)$, the space with the fuzzy multimeasure μ.

In the following, we point out some relationships among certain types of the above considered continuity, which will be necessary in the other sections.

One can easily verify the following:

Remark 2.3. 1) i) Any additive multimeasure is null-additive.

ii) If μ is pseudo-null-additive, then it is null-additive.

iii) The following statements are equivalent:

a) μ is pseudo-null-additive;

b) $\mu(B \cap C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$;

c) $\mu((A\backslash B) \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$.

2) If μ is strongly order continuous, then it is order continuous.

If, moreover, μ is null-additive, then the converse also holds.

So, if μ is null-additive, then μ is strongly order continuous if and only if it is order continuous.

3) If μ is pseudo-order continuous, then it is also order continuous.

If $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure, by the law of cancelation (C), we also immediately get the converse.

So, if $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure, then μ is pseudo-order continuous if and only if it is order continuous.

4) [30] If $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ is a (S)-fuzzy multimeasure, then the following properties are equivalent:

a) autocontinuous from above;

b) autocontinuous from below;

c) autocontinuous.

By the above considerations we get:

Proposition 2.4. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure. Then the following statements are equivalent:*

i) μ is strongly order continuous;

ii) μ is order continuous;

iii) μ is pseudo-order continuous.

Remark 2.5. If μ is autocontinuous from above, then pseudo-order continuity implies continuity from below. If $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure, the converse also holds.

Consequently, we have:

Proposition 2.6. *If $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure, the following statements are equivalent:*

i) μ is strongly order continuous;

ii) μ is order continuous;

iii) μ is pseudo-order continuous;

iv) μ *is continuous from above;*

v) μ *is continuous from below.*

Definition 2.7. We say that $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ *fulfils:*

i) [20] *property* (S) if for any sequence of sets $(A_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} |\mu(A_n)| = 0$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $\mu(\overline{\lim}_k A_{n_k}) = \{0\}$, where $\overline{\lim}_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

ii) *property* (PS) if for any $A \in \mathcal{A}$ and any sequence of sets $(A_n)_n \subset A \cap \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $h(\mu(\underline{\lim}_k A_{n_k}), \mu(A)) = 0$, where $\underline{\lim}_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$.

3. Relationships among convergences, respectively pseudo–convergences

In this section, we point out, under special conditions, some relationships among almost everywhere, pseudo-almost everywhere, almost uniform and pseudo-almost uniform convergences, generalizing in the set-valued case some results from the real case [29], [18], [3], [9], [14], [26].

Firstly, we shall give the following general result:

Proposition 3.1. *Let be* $A \in \mathcal{A}$ *and* P*, a proposition concerning the points of* A*. Then* P *holds on* A *p.a.e. whenever* P *is true a.e. on* A *if and only if* μ *is null-additive.*

Proof. *Necessity.* To prove that μ is null-additive, let be $A \in \mathcal{A}$ and $E \in \mathcal{A}$, with $\mu(E) = \{0\}$. Let us consider $x \in A\backslash E$ as a proposition $P(x)$. Then P holds a.e. on A. By virtue of the hypothesis, P holds p.a.e. on A and hence, there exists $B \in A \cap \mathcal{A}$, with $\mu(A\backslash B) = \mu(A)$ such that P is true on $A\backslash B$. We observe that $x \in A\backslash B$ implies $x \in A\backslash E$ and so, $A\backslash B \subset A\backslash E \subset A$.

Using the monotonicity of μ, we obtain that $\mu(A\backslash E) = \mu(A)$, which assures that μ is null-additive because $\mu(A) = \mu((A \cup E)\backslash E) = \mu(A \cup E)$, that is, $\mu(A \cup E) = \mu(A)$, for every $E \in \mathcal{A}$, with $\mu(E) = \{0\}$.

Sufficiency. Suppose that μ is null-additive. If P holds a.e. on A, there exists $E \in \mathcal{A}$, with $\mu(E) = \{0\}$ such that P is true on $A\backslash E$. Since μ is null-additive, we have $\mu(A\backslash E) = \mu(A)$ and hence P holds p.a.e. on A. $\square$

Remark 3.2. We observe that a pseudo-almost everywhere property P has not a hereditary character, that is, if P is true p.a.e. on a set A and $B \subset A$, then, generally, P is not true p.a.e. on B.

Definition 3.3. We consider arbitrary $\{f_n\} \subset \mathcal{M}$ and $f \in \mathcal{M}$. We say that:

i) $\{f_n\}$ *converges* μ*-almost everywhere* (respectively, *pseudo-*μ*-almost everywhere*) *to* f *on* A, and denote it by $f_n \xrightarrow[A]{a.e.} f$ (respectively, $f_n \xrightarrow[A]{p.a.e.} f$) if there exists a subset $B \in A \cap \mathcal{A}$ such that $\mu(B) = \{0\}$ (respectively, $\mu(A \backslash B) = \mu(A)$) and $\{f_n\}$ is pointwise convergent to f on $A \backslash B$.

ii) $\{f_n\}$ *converges in* μ*-measure* (respectively, *pseudo in* μ*-measure*) *to* f *on* A, and denote it by $f_n \xrightarrow[A]{\mu} f$ (respectively, $f_n \xrightarrow[A]{p.\mu} f$) if for every $\varepsilon > 0$, $\lim_{n\to\infty} |\mu(A_n(\varepsilon)| = 0$, where $A_n(\varepsilon) = \{t \in A; |f_n(t) - f(t)| \geq \varepsilon\}$ (respectively, $\lim_{n\to\infty} h(\mu(A \backslash A_n(\varepsilon)), \mu(A)) = 0$).

iii) [20] $\{f_n\}$ *converges* μ*-almost uniformly* (respectively, μ*-pseudo-almost uniformly*) *to* f *on* A and denote it by $f_n \xrightarrow[A]{a.u.} f$ (respectively, $f_n \xrightarrow[A]{p.a.u.} f$) if there exists a decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that $\lim_{k\to\infty} |\mu(A_k)| = 0$ (respectively, $\lim_{k\to\infty} h(\mu(A \backslash A_k), \mu(A)) = 0$) and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A \backslash A_k$ ($f_n \xrightarrow[A\backslash A_k]{u} f$).

Using Proposition 3.1 and the law of cancelation (C), we immediately obtain:

Proposition 3.4. 1) *If* $f_n \xrightarrow[A]{a.e.} f$ *and* μ *is null-additive, then* $f_n \xrightarrow[A]{p.a.e.} f$.

2) *If* $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ *is an additive multimeasure,* $f_n \xrightarrow[A]{a.e.} f$ *if and only if* $f_n \xrightarrow[A]{p.a.e.} f$.

Now, we point out the relationships among a.u. and a.e. convergences, on one hand, and p.a.u. and p.a.e. on the other hand:

Proposition 3.5. *i) If* $f_n \xrightarrow[A]{a.u.} f$, *then* $f_n \xrightarrow[A]{a.e.} f$.

ii) If $f_n \xrightarrow[A]{p.a.u.} f$, *then* $f_n \xrightarrow[A]{p.a.e.} f$.

Proof. i) If $f_n \xrightarrow[A]{a.u.} f$, there exists a decreasing sequence $\{C_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that $\lim_{k\to\infty} |\mu(C_k)| = 0$ and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A\backslash C_k$.

Let us denote $C = \bigcap_{k=1}^{\infty} C_k$. Since $C \subset C_k$, for every $k \in \mathbb{N}^*$, by the monotonicity of μ, we have $\mu(C) \subset \mu(C_k)$, for every $k \in \mathbb{N}^*$, so $|\mu(C)| \leq |\mu(C_k)|$. Because $\lim_{k\to\infty} |\mu(C_k)| = 0$, we obtain that $\mu(C) = \{0\}$.

We see that for any $x \in A\backslash C$, there exists $k_0 \in \mathbb{N}^*$ so that $x \in A\backslash C_{k_0}$ and, therefore, $f_n(x)$ converges to $f(x)$, which assures that $f_n \overset{a.e.}{\underset{A}{\to}} f$.

ii) If $f_n \overset{p.a.u.}{\underset{A}{\to}} f$, there exists a sequence $\{C_k\} \subset \mathcal{A}$ so that

$$\text{(1)} \quad \lim_{k\to\infty} h(\mu(A\backslash C_k), \mu(A)) = 0 \text{ and } \{f_n\} \text{ uniformly converges to } f \text{ on } A\backslash C_k, \text{ for any fixed } k \in \mathbb{N}^*.$$

Take $C = \bigcap_{k=1}^{\infty} C_k$. Then $A\backslash C_k \subset A\backslash C \subset A$, for every $k \in \mathbb{N}^*$, which implies by the monotonicity of μ that $\mu(A\backslash C_k) \subset \mu(A\backslash C) \subset \mu(A)$, for every $k \in \mathbb{N}^*$.

From (1), using Lemma 2.1, we obtain that $\mu(A\backslash C) = \mu(A)$. It is easy to see that $\{f_n\}$ converges to f on $A\backslash C$, which assures that $f_n \overset{p.a.e.}{\underset{A}{\to}} f$. □

Theorem 3.6. *i) If μ is autocontinuous from below and if $f_n \xrightarrow[A]{a.u.} f$, then $f_n \xrightarrow[A]{p.a.u.} f$.*

ii) If $\mu : \mathcal{A} \to \mathcal{P}_{bfc}(X)$ is an additive multimeasure, then $f_n \xrightarrow[A]{a.u.} f$ if and only if $f_n \xrightarrow[A]{p.a.u.} f$.

iii) If $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ and $f_n \xrightarrow[A]{p.a.u.} f$ whenever $f_n \xrightarrow[A]{a.u.} f$, then μ is null-additive.

Proof. i) If $f_n \xrightarrow[A]{a.u.} f$, there exists a decreasing sequence $\{C_k\}_{k\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that $\lim_{k\to\infty} |\mu(C_k)| = 0$ and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A\backslash C_k$.

Since μ is autocontinuous from below, we have $\lim_{k\to\infty} h(\mu(A\backslash C_k), \mu(A)) = 0$, which assures that $f_n \xrightarrow[A]{p.a.u.} f$.

ii) The statement is straightforward by i) and property $(\tilde{C})$.

iii) Let be disjoint $A, B \in \mathcal{A}$, with $\mu(B) = \{0\}$. For every $n \in \mathbb{N}$, consider $f_n(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B. \end{cases}$

Obviously, $\{f_n\}_n \subset \mathcal{M}$. We observe that $f_n \xrightarrow[A\cup B]{a.u.} 1$. Then $f_n \xrightarrow[A\cup B]{p.a.u.} 1$, i.e., there exists a decreasing sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $C_k \subset A \cup B$, for every $k \in \mathbb{N}$, $\lim_{k\to\infty} h(\mu((A \cup B)\backslash C_k), \mu(A \cup B)) = 0$ and $f_n \xrightarrow[(A\cup B)\backslash C_k]{u} 1$. Consequently, $(A \cup B)\backslash C_k \subset A$, whence $B\backslash C_k = \emptyset$, for every $k \in \mathbb{N}$, so $(A \cup B)\backslash C_k = A\backslash C_k$.

Therefore, for every $k \in \mathbb{N}$,

$$\begin{aligned} e(\mu(A \cup B), \mu(A)) &\leq e(\mu(A \cup B), \mu((A \cup B)\backslash C_k)) + e(\mu((A \cup B)\backslash C_k), \mu(A)) \\ &\leq h(\mu(A \cup B), \mu((A \cup B)\backslash C_k)) + e(\mu(A\backslash C_k), \mu(A)) \\ &= h(\mu(A \cup B), \mu((A \cup B)\backslash C_k)), \end{aligned}$$

so, for $k \to \infty$, and taking into account that $e(\mu(A), \mu(A \cup B)) = 0$, we get that $h(\mu(A), \mu(A\cup B)) = 0$. Since $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, then μ is null-additive.□

Theorem 3.7. *$f_n \xrightarrow[A]{p.\mu} f$ whenever $f_n \xrightarrow[A]{\mu} f$ if and only if μ is autocontinuous from below.*

Proof. *Necessity.* Suppose $f_n \xrightarrow[A]{p.\mu} f$ whenever $f_n \xrightarrow[A]{\mu} f$ and let be $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} |\mu(B_n)| = 0$.

We define for every $n \in \mathbb{N}$, $f_n(x) = \begin{cases} 1, & \text{if } x \in B_n \\ 0, & \text{if } x \in A\backslash B_n. \end{cases}$

Obviously, $\{f_n\} \subset \mathcal{M}$. Also, it is easy to see that $f_n \xrightarrow[A]{\mu} 0$. By hypothesis, $f_n \xrightarrow[A]{p.\mu} 0$ and then for $\varepsilon = 1$, we have $\lim_{n\to\infty} h(\mu(\{x \in A; |f_n(x)| < 1\}, \mu(A)) = 0$.

We observe that $\{x \in A; |f_n(x)| < 1\} = A\backslash B_n$ and, consequently, we obtain $\lim_{n\to\infty} h(\mu(A\backslash B_n), \mu(A)) = 0$, which says that μ is autocontinuous from below.

Sufficiency. Let μ be autocontinuous from below and suppose that $f_n \xrightarrow[A]{\mu} f$. Then for every $\varepsilon > 0$, we have $\lim_{n\to\infty} |\mu(\{x \in A; |f_n(x) - f(x)| \geq \varepsilon\})| = 0$.

Applying the autocontinuity from below of μ we have $\lim_{n\to\infty} h(\mu(\{x \in A; |f_n(x) - f(x)| < \varepsilon\}), \mu(A)) = \lim_{n\to\infty} h(\mu(A\setminus\{x \in A; |f_n(x) - f(x)| \geq \varepsilon\}), \mu(A)) = 0$ and so $f_n \xrightarrow[A]{p.\mu} f$. $\square$

4. A set-valued pseudo-version of Egoroff's theorem

For the purpose of obtaining a pseudo-version of Egoroff's theorem for fuzzy multimeasures, we shall firstly give a multivalued form of the condition (PE) introduced by Li and Yasuda [13] for real monotonic set functions.

We shall prove that, as in the real case [11], fulfilment of the Egoroff's theorem in the pseudo-version is conditioned by the property (PE).

Definition 4.1. We say that μ *fulfils condition* (PE) if for any $A\in\mathcal{A}$ and every double sequence $\{A_n^{(m)}\}_{m,n\in\mathbb{N}} \subset A\cap\mathcal{A}$ such that for every fixed $m \in \mathbb{N}$, $A_n^{(m)} \nearrow A^{(m)}$ as $n \to \infty$ and $\mu(\bigcap_{m=1}^{\infty} A^{(m)}) = \mu(A)$, there exist increasing sequences $\{n_i\}$ and $\{m_i\}$ of naturals such that $\lim_{k\to\infty} h(\mu(\bigcap_{i=k}^{\infty} A_{n_i}^{(m_i)}), \mu(A)) = 0$.

We remark that continuity from below of μ is a necessary condition for fulfiling (PE), that is:

Proposition 4.2. *If μ fulfils (PE), then μ is continuous from below.*

Proof. Let be $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}$, such that $A_n \nearrow A$. Since μ satisfies property (PE), there exist increasing sequences $\{n_i\}$ and $\{m_i\}$ of naturals such that $\lim_{k\to\infty} h(\mu(\bigcap_{i=k}^{\infty} A_{n_i}), \mu(A))=0$, whence $\lim_{k\to\infty} h(\mu(A_{n_k}), \mu(A)) = 0$. Using the monotonicity of μ and Lemma 2.1-ii), we have that $\lim_{k\to\infty} h(\mu(A_k), \mu(A)) = 0$, which says that μ is continuous from below. $\square$

Now, we can prove a pseudo-form of Egoroff's theorem in the set-valued case:

Theorem 4.3 (Egoroff type)**.** *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. Then $f_n \xrightarrow[A]{p.a.u.} f$ whenever $f_n \xrightarrow[A]{p.a.e.} f$ if and only if μ fulfils condition (PE).*

Proof. *Necessity.* Suppose that for any $A \in \mathcal{A}, f_n \xrightarrow[A]{p.a.e.} f$ implies $f_n \xrightarrow[A]{p.a.u.} f$. Let be a double sequence $\{A_n^{(m)}\}_{m,n\in\mathbb{N}} \subset A \cap \mathcal{A}$ such that for every fixed $m \in \mathbb{N}$, $A_n^{(m)} \nearrow A^{(m)}$ as $n \to \infty$ and

$$\mu(\bigcap_{m=1}^{\infty} A^{(m)}) = \mu(A). \tag{2}$$

We denote by $B_n^{(m)} = \bigcap_{i=1}^{m} A_n^{(i)}$, for every $m, n \in \mathbb{N}$, and by $B^{(m)} = \bigcup_{n=1}^{\infty} B_n^{(m)}$, for every $m \in \mathbb{N}$.

We see that the double sequence $\{B_n^{(m)}\}_{m,n\in\mathbb{N}}$ satisfies the conditions: for every fixed $n \in \mathbb{N}$, $B_n^{(m)} \supset B_n^{(m+1)}$, for any $m \in \mathbb{N}$, and $B_n^{(m)} \nearrow B^{(m)}$ as $n \to \infty$. Since $\bigcap_{m=1}^{\infty} B^{(m)} = \bigcap_{m=1}^{\infty} A^{(m)}$, from (2) we have

$$\mu(\bigcap_{m=1}^{\infty} B^{(m)}) = \mu(A). \tag{3}$$

Now, for every $n \in \mathbb{N}$, we consider

$$f_n(x) = \begin{cases} \frac{1}{m+1}, & \text{if } x \in B_n^{(m)} \setminus B_n^{(m+1)}, m \in \mathbb{N}^* \\ 1, & \text{if } x \in A \setminus B_n^{(1)} \\ 0, & \text{if } x \in A \setminus \bigcap\limits_{m=1}^{\infty} B_n^{(m)}. \end{cases} \tag{4}$$

Then, for every $m \in \mathbb{N}^*$, we have $\{x \in A; |f_n(x)| < \frac{1}{m}\} = B_n^{(m)}$, whence $\{x \in A; |f_i(x)| < \frac{1}{m}, i \geq n\} = \bigcap_{i=n}^{\infty} B_i^{(m)} = B_n^{(m)}$.

If we denote by $C = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \in A; |f_i(x)| \geq \frac{1}{m}\}$, we see that f_n converges to 0 on $A \setminus C$ and $A \setminus C = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} B_i^{(m)} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} B_n^{(m)} = \bigcap_{m=1}^{\infty} B^{(m)}$.

Since by (3) we have $\mu(\bigcap_{m=1}^{\infty} B^{(m)}) = \mu(A)$, we obtain that $\mu(A \setminus C) = \mu(A)$ and hence $f_n \xrightarrow[A]{p.a.e.} f$. Then, by virtue of the hypothesis, $f_n \xrightarrow[A]{p.a.u} 0$.

Consequently, there exists a decreasing sequence $\{C_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} h(\mu(A \setminus C_k), \mu(A)) = 0 \text{ and } f_n \text{ uniformly converges to } f \text{ on } A \setminus C_k, \text{ for every } k \in \mathbb{N}. \tag{5}$$

Now, if $k \in \mathbb{N}^*$, there exists $n_k \in \mathbb{N}$ such that for every $x \in A \setminus C_k$, we have $|f_i(x)| < \frac{1}{k}$, for every $i \geq n_k$ and thus, for every $k \in \mathbb{N}^*$, we have $A \setminus C_k \subset \bigcap_{i=n_k}^{\infty} \{x \in A; |f_i(x)| < \frac{1}{k}\} = \bigcap_{i=n_k}^{\infty} B_i^{(k)} = B_{n_k}^{(k)}$.

Since the sequence $\{A\backslash C_k\}_{k\in\mathbb{N}^*}$ is increasing, we have for every $k \in \mathbb{N}^*$,

$$\mu(A\backslash C_k) = \mu(\bigcap_{i=k}^{\infty}(A\backslash C_i)) \subset \mu(\bigcap_{i=k}^{\infty} B_{n_i}^{(i)}) \subset \mu(A)$$

and using (5) and Lemma 2.1, we obtain

$$\lim_{k\to\infty} h(\mu(\bigcap_{i=k}^{\infty} B_{n_i}^{(i)}), \mu(A)) = 0. \tag{6}$$

But $A_n^{(m)} \supset B_n^{(m)}$, for every $m, n \in \mathbb{N}$, and hence $A_{n_i}^{(i)} \supset B_{n_i}^{(i)}$, for every $i \in \mathbb{N}$, whence, we have $\mu(A) \supset \mu(\bigcap_{i=k}^{\infty} A_{n_i}^{(i)}) \supset \mu(\bigcap_{i=k}^{\infty} B_{n_i}^{(i)})$, for every $i \in \mathbb{N}$.

Taking again into account Lemma 2.1 and (6), we obtain

$$\lim_{k\to\infty} h(\mu(\bigcap_{i=k}^{\infty} A_{n_i}^{(i)}), \mu(A)) = 0, \tag{7}$$

which assures that μ fulfils condition (PE).

Sufficiency. We assume that μ satisfies condition (PE) and $f_n \xrightarrow[A]{p.a.e.} f$. Then there exists a set $E \subset A$ such that $\mu(A\backslash E) = \mu(A)$ and f_n converges to f on $A\backslash E$.

Denoting by B the set of those points $x \in A$ at which $f_n(x)$ converges to $f(x)$, we observe that B can be written as

$$B = \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}\left\{x \in A; |f_i(x) - f(x)| < \frac{1}{m}\right\}. \tag{8}$$

Now, we see that $A\backslash E \subset B \subset A$ and hence, $\mu(B) = \mu(A)$. For every fixed $m \in \mathbb{N}^*$, let us denote by $A_n^{(m)} = \bigcap_{i=n}^{\infty}\{x \in A; |f_i(x) - f(x)| < \frac{1}{m}\}$, for every $n \in \mathbb{N}^*$, and by $A^{(m)} = \bigcup_{n=1}^{\infty} A_n^{(m)}$.

Since the double sequence $\{A_n^{(m)}\}_{n,m\in\mathbb{N}^*}$ is such that $A_n^{(m)} \nearrow A^{(m)}$ for $n \to \infty$ and $\mu(\bigcap_{m=1}^{\infty} A^{(m)}) = \mu(B) = \mu(A)$, we can apply condition (PE).

Consequently, there exist two sequences $\{n_i\}$ and $\{m_i\}$ of naturals such that $\lim_{k\to\infty} h(\mu(\bigcap_{i=k}^{\infty} A_{n_i}^{(m_i)}), \mu(A)) = 0$. Denoting by $C_k = A\backslash\bigcap_{i=k}^{\infty} A_{n_i}^{(k_i)}$, $k \in \mathbb{N}^*$, we observe that $\lim_{k\to\infty} h(\mu(A\backslash C_k), \mu(A)) = 0$.

It remains to prove that f_n converges to f on $A\backslash C_k$, uniformly for any fixed $k \in \mathbb{N}^*$.

Since $A\backslash C_k = \bigcap_{i=k}^{\infty} A_{n_i}^{(m_i)} = \bigcap_{i=k}^{\infty} \bigcap_{j=n_i}^{\infty} \{x \in A; |f_j(x) - f(x)| < \frac{1}{m_i}\}$, then, for every $i \geq k$,

$$A\backslash C_k \subset \bigcap_{j=n_i}^{\infty} \left\{x \in A; |f_j(x) - f(x)| < \frac{1}{m_i}\right\}.$$

If $\varepsilon > 0$ is arbitrary, we can take $i_0 \geq k$ such that $\frac{1}{m_{i_0}} < \varepsilon$.

Now, if $j > m_{i_0}$, for every $x \in A\backslash C_k$, we have $|f_j(x) - f(x)| < \frac{1}{m_{i_0}} < \varepsilon$, which says that $\{f_n\}$ uniformly converges to f on $A\backslash C_k$. Consequently, $f_n \underset{A}{\overset{p.a.u.}{\longrightarrow}} f$. □

By Theorem 4.3 and Proposition 3.5 ii), we get:

Corollary 4.4. *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. Then $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f \Leftrightarrow f_n \underset{A}{\overset{p.a.u.}{\longrightarrow}} f$ if and only if μ fulfils (PE).*

Also, as a corollary of Theorem 4.3, we obtain that continuity from below is a necessary condition for pseudo-form of Egoroff's theorem:

Corollary 4.5. *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. If $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f \Rightarrow f_n \underset{A}{\overset{p.a.u.}{\longrightarrow}} f$, then μ is continuous from below.*

We already remarked at the beginning of Section 3 that a property which holds "p.a.e." is not hereditary and then, it is justified to give the following:

Definition 4.6. Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. We say that $\{f_n\}$ *converges pseudo-almost uniformly* (*pseudo-almost-everywhere,* respectively) *to* f *in* A, denoted by $f_n \overset{p.a.u.}{\longrightarrow} f$ ($f_n \overset{p.a.e.}{\longrightarrow} f$, respectively) in A if $f_n \underset{E}{\overset{p.a.u.}{\longrightarrow}} f$ ($f_n \underset{E}{\overset{p.a.e.}{\longrightarrow}} f$, respectively), for every $E \in A \cap \mathcal{A}$.

Obviously, by Proposition 3.5, if $f_n \overset{p.a.u.}{\longrightarrow} f$ in A, then $f_n \overset{p.a.e.}{\longrightarrow} f$ in A, so $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f$.

We remark that, generally, the converse is not valid. Although, we can give the following:

Theorem 4.7. *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. Then $f_n \overset{p.a.u.}{\longrightarrow} f$ in A whenever $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f$ if and only if μ fulfils condition (PE) and μ is pseudo-null-additive.*

Proof. *Necessity.* Let us assume that $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f$ implies $f_n \overset{p.a.u.}{\longrightarrow} f$ in A.

Obviously, according to Egoroff's Theorem 4.3, μ fulfils condition (PE).

To prove that μ is pseudo-null-additive, let us consider $B \in A \cap \mathcal{A}$, such that $\mu(A \backslash B) = \mu(A)$ and take an arbitrary set $C \in A \cap \mathcal{A}$.

We shall prove that $\mu(B \cup C) = \mu(C)$. Since $A \backslash B \subset A, B \backslash C \subset A$, we have $\mu(A \backslash (B \backslash C)) = \mu(A)$. Let us define for every $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 0, & \text{if } x \in A \backslash (B \backslash C) \\ 1, & \text{if } x \in B \backslash C. \end{cases} \tag{9}$$

We see that $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} 0$ and then, by virtue of the hypothesis, we have $f_n \overset{p.a.u}{\longrightarrow} 0$ in $B \cup C$.

Using Proposition 3.5 ii), we obtain that $f_n \underset{B \cup C}{\overset{p.a.e.}{\longrightarrow}} 0$. Then, there exists $E \subset B \cup C$, with $E \in \mathcal{A}$, such that $\mu((B \cup C) \backslash E) = \mu(B \cup C)$ and $f_n(x)$ converges to 0 at every $x \in (B \cup C) \backslash E$.

By (9) we see that $(B \cup C) \backslash E \subset (B \cup C) \backslash (B \backslash C) = C \subset B \cup C$, which implies that $\mu((B \cup C) \backslash (B \backslash C)) = \mu(B \cup C)$, that is, $\mu(B \cup C) = \mu(C)$, which assures because C is an arbitrary subset of A, that μ is pseudo-null-additive.

Sufficiency. Suppose that μ is pseudo-null-additive and fulfils condition (PE). If $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f$, then, since μ is pseudo-null-additive, we have $f_n \overset{p.a.e.}{\longrightarrow} f$ in A.

Indeed, because $f_n \underset{A}{\overset{p.a.e.}{\longrightarrow}} f$, there exists $E \in A \cap \mathcal{A}$ such that

$$\mu(A \backslash E) = \mu(A) \text{ and } f_n(x) \text{ converges to } f(x) \text{ at every } x \in A \backslash E. \tag{10}$$

Let $C \in A \cap \mathcal{A}$ be an arbitrary set. We shall prove that

$$\mu(C \backslash (C \cap E)) = \mu(C) \tag{11}$$

or, equivalently,

$$\mu(C) = \mu(C \backslash E). \tag{11'}$$

Since $C = (C \cap E) \cup (C \backslash E)$ and $A \backslash E \subset A \backslash (C \cap E) \subset A$, by (10) we have $\mu(A \backslash (C \cap E)) = \mu(A)$.

Now, using the pseudo-null-additivity of μ, we get that $\mu(C) = \mu(C \backslash E)$, that is, ($11'$) holds.

Consequently, there exists the subset $C \cap E$ of C such that $\mu(C) = \mu(C \backslash (C \cap E))$ and f_n converges to f on $C \backslash (C \cap E) = C \backslash E$, which assures that $f_n \xrightarrow{p.a.e.} f$ in A. By Theorem 4.3, we get that $f_n \xrightarrow{p.a.u.} f$ in A. $\square$

By Theorem 4.7 and Proposition 3.5 ii), we get:

Corollary 4.8. *Let be $A \in \mathcal{A}, f \in \mathcal{M}$ and $\{f_n\}_n \subset \mathcal{M}$. Then $f_n \underset{A}{\xrightarrow{p.a.e.}} f$ $\Leftrightarrow f_n \xrightarrow{p.a.u.} f$ in A if and only if μ fulfils condition (PE) and μ is pseudo-null-additive.*

5. Concluding remarks

In this paper, we investigated for set-valued non-additive monotonic set functions, some relationships among the main types of convergences of sequences of measurable functions.

In this way, we insisted on different types of pseudo-convergences of sequences of measurable functions, such as, pseudo-almost everywhere (p.a.e.), pseudo-almost uniform (p.a.u) convergences and pseudo-convergence in measure (p.μ) and on the relationships among them, or with almost everywhere, almost uniform convergences and convergence in measure.

Thus, we gave a set-valued pseudo-version of Egoroff's theorem and, as a consequence, we obtained a result which emphasizes the non-hereditary character of the pseudo-convergence. We also gave characterizations of some important structural properties of monotone multimeasures.

We shall further our study concerning convergences and pseudo-convergences in order to obtain set-valued Lebesgue and Riesz type theorems.

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Faculty of Mathematics,
"Al.I. Cuza" University,
Bd. Carol I, no 11, Iaşi, 700506,
ROMANIA
aprecup@uaic.ro
gavrilut@uaic.ro