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CONSIDERATIONS ON (PSEUDO-) CONVERGENCES OF SEQUENCES OF MEASURABLE FUNCTIONS ON MONOTONE SET MULTIFUNCTION SPACE

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Abstract. In this paper, we further a previous study concerning convergences and pseudo-convergences of sequences of measurable functions on monotone set multifunction spaces. Various considerations concerning operations and the uniqueness of the limit with respect to such convergences are given and several asymptotic structural properties of certain set multifunctions are, in this way, characterized.

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1. Introduction

As it is well known, convergences and pseudo-convergences of sequences of measurable real-valued functions with respect to monotone set (multi)functions is a very important area of measure theory (see, for instance, [19]), due to its various theoretical and practical applications.

In non-additive measure theory, we mention in this sense the contributions of DENNEBERG [1], HA, WANG and WU [2], JIANG ET AL. [5], KAWABE [6, 7], LI ET AL. [8], LI [9, 10], LI and LI [11], LI ET AL. [12], LI and YASUDA [13], LI ET AL. [14], LIU [15], MUROFUSHI [16], MUROFUSHI ET AL. [17], PAP [18], REN ET AL. [24], SONG and LI [25], SUN [26, 27], TAKAHASHI ET AL. [28], WANG [29], WANG and KLIR [30], WU and LIU [31], ZHANG [32] and many others.

Recently, due to various necessities coming from mathematical economics, artificial intelligence, biomathematics etc., some results from the

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above mentioned papers were generalized to the set-valued case in Hausdorff topology (see, for instance, PRECUPANU and GAVRILUŢ [20-23], WU and LIU [31]) for $\mathcal{P}_f(X)$ -valued monotone set multifunctions, X being a real normed space and $\mathcal{P}_f(X)$ the family of all nonvoid closed sets of X.

The aim of this paper is to further the study [22] concerning convergences and pseudo-convergences of sequences of real-valued measurable functions with respect to $\mathcal{P}_{bf}(X)$ -valued monotone set multifunctions, X being a Banach space and $\mathcal{P}_{bf}(X)$ the family of all nonvoid closed, bounded sets of X. Considerations concerning operations and uniqueness of the limit with respect to such convergences are given and asymptotic structural properties of certain monotone set multifunctions are characterized.

2. Terminology, notations and basic results

Let T be an abstract nonvoid space, \mathcal{A} a σ -algebra of subsets of T, X a Banach space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of $X, \mathcal{P}_f(X)$ the family of closed, nonvoid sets of X, $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$ given by:

$$h(M, N) = \max\{e(M, N), e(N, M)\}, \text{ for every } M, N \in \mathcal{P}_f(X),$$

where $e(M, N) = \sup_{x \in M} d(x, N)$ is the excess of M over N.

It is known that if $M, N \in \mathcal{P}_f(X)$, then e(M, N) = 0 if and only if $M \subset N$. Consequently, e(M, N) = h(M, N), for every $M, N \in \mathcal{P}_f(X)$, with $N \subset M$. Also, $e(M, N) \leq e(M, P) + e(P, N)$, for every $M, N, P \in \mathcal{P}_f(X)$. On $\mathcal{P}_{bf}(X)$, h becomes a metric [3].

We denote $A \cap \mathcal{A} = \{E \subset A, E \in \mathcal{A}\}$, where A is a fixed set in \mathcal{A} . By cA we mean $T \setminus A$. By \mathbb{N} we denote the set of all naturals and by \mathbb{N}^* we mean $\mathbb{N} \setminus \{0\}$.

All over the paper, a limit of the type $\lim_{n\to\infty} \mu(M_n) = \{0\}$ or $\lim_{n\to\infty} \mu(M_n) = \mu(M)$ (where $(M_n)_n, M \subset \mathcal{A}$) will be understood with respect to h.

Throughout the paper we shall use the following notions in the setvalued case:

Definition 2.1 ([20, 21, 22, 23]). A set multifunction $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$, with $\mu(\emptyset) = \{0\}$ is said to be:

i) monotone if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$.

- ii) continuous from below if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \nearrow A$.
- iii) continuous from above if $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$.
- iv) Sugeno-continuous, for short (S)-continuous if μ is monotone, continuous from below and continuous from above.
- v) order continuous if $\lim_{n\to\infty} \mu(A_n) = \{0\}$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow \emptyset$.
- vi) strongly order continuous if $\lim_{n\to\infty} \mu(A_n) = \{0\}$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$, with $A_n \searrow A$ and $\mu(A) = \{0\}$.
- vii) pseudo-order continuous if $\lim_{n\to\infty} \mu(A_n) = \{0\}$, for every sequence of sets $(A_n)_n \subset \mathcal{A}$ and every $B \in \mathcal{A}$, with $A_n \subset B$, for every n, $A_n \searrow A$ and $\mu(B \setminus A) = \mu(B)$.
- viii) null-additive if $\mu(A \cup B) = \mu(B)$, for every disjoint $A, B \in \mathcal{A}$, with $\mu(A) = \{0\}$. double null-additive (or, null-null-additive) if $\mu(A \cup B) = \{0\}$, for every (disjoint) $A, B \in \mathcal{A}$, with $\mu(A) = \mu(B) = \{0\}$.
 - ix) single asymptotic null-additive if for every $A \in \mathcal{A}$ with $\mu(A) = \{0\}$ and every sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, with $\lim_{n \to \infty} \mu(B_n) = \{0\}$, we have $\lim_{n \to \infty} \mu(A \cup B_n) = \{0\}.$
 - x) double asymptotic null-additive if $\lim_{m,n\to\infty} \mu(A_n \cup B_m) = \{0\}$, whenever $(A_n)_n, (B_m)_m \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \lim_{m\to\infty} \mu(B_m) = \{0\}$.
 - xi) pseudo-null-additive if $\mu(B \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}$, $C \in A \cap \mathcal{A}$ and $\mu(A \setminus B) = \mu(A)$.
- xii) a) autocontinuous from below (autocontinuous from above, respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(B_n) = \{0\}$, we have $\lim_{n\to\infty} h(\mu(A \setminus B_n), \mu(A)) = 0$ ($\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$, respectively).

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- b) *autocontinuous* if it is autocontinuous from above and autocontinuous from below.
- xiii) a) pseudo-autocontinuous from above (pseudo-autocontinuous from below, respectively) if for every $A \in \mathcal{A}$ and every $(B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(B_n \cap A), \mu(A)) = 0$, we have $\lim_{n\to\infty} h((\mu(A \setminus B_n) \cup C), \mu(C)) = 0$ (respectively, $\lim_{n\to\infty} h((\mu(B_n \cap C), \mu(C)) = 0)$, for every $C \in A \cap \mathcal{A}$.
 - b) *pseudo-autocontinuous* if it is pseudo-autocontinuous from above and pseudo-autocontinuous from below.
- xiv) uniformly pseudo-autocontinuous from below if for every $\varepsilon > 0$, there exists $\delta_{\varepsilon} > 0$ so that for every $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$, with $h(\mu(A), \mu(B)) < \delta$, we have $h(\mu(C), \mu(B \cap C)) < \varepsilon$.

Unless stated otherwise, all over the paper we assume that $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ is monotone.

The following example emphasizes the importance of the set-valued framework:

Example 2.2. Suppose X is an AL-space [4] (i.e., a real Banach space equipped with a lattice order relation, which is compatible with the linear structure, such that the norm $\|\cdot\|$ on X is monotone, that is, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, for every $x, y \in X$, and also satisfying the supplementary condition $\|x + y\| = \|x\| + \|y\|$, for every $x, y \in X$, with $x, y \geq 0$).

For instance, \mathbb{R} , $L_1(\mu)$, l_1 are usual examples of AL-spaces.

Let Λ be the positive cone of X. As usual, by [x, y] we mean the interval consisting of all $z \in X$ so that $x \leq z \leq y$.

Suppose $m : \mathcal{A} \to \Lambda$ is an arbitrary set function, with $m(\emptyset) = 0$. We consider the induced set multifunction $\mu : \mathcal{A} \to \mathcal{P}_{bf}(X)$ defined for every $A \in \mathcal{A}$ by $\mu(A) = [0, m(A)]$.

We observe that $h(\mu(A), \{0\}) = \sup_{0 \le x \le m(A)} ||x|| = ||m(A)||$, for every $A, B \in \mathcal{A}$.

We remark that the set-valued framework is a very good direction study, because when we use a proper set multifunction (for example, the induced set multifunction), it allows us to get back our considerations to important particular spaces, as, for instance, *AL*-spaces. **Definition 2.3** ([22, 23]). We say that μ fulfils:

- i) property (S) if for any sequence of sets $(A_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \{0\}$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $\mu(\overline{\lim_{k\to\infty}}A_{n_k}) = \{0\}$, where $\overline{\lim_{n\to\infty}}E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.
- ii) property (PS) if for any $A \in \mathcal{A}$ and any sequence of sets $(A_n)_n \subset A \cap \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, there exists a subsequence $(A_{n_k})_k$ of $(A_n)_n$ such that $\mu(\underline{\lim}_{k\to\infty}A_{n_k}) = \mu(A)$, where $\underline{\lim}_{n\to\infty}E_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}E_k$.

Remark 2.4. I) i) [31] If μ is (S)-continuous, then it is autocontinuous from above if and only if it is null-additive and has property (S), if and only if it is autocontinuous from below.

ii) If μ is null-additive, then it is single asymptotic null-additive.

If μ is single asymptotic null-additive, then it is null-null-additive.

If μ is double asymptotic null-additive, then it is single asymptotic null-additive.

iii) [22] If μ is (S)-continuous, then μ is pseudo-autocontinuous from below if and only if it is pseudo-null-additive and has property (PS), if and only if it is pseudo-autocontinuous from above.

II) μ is double asymptotic null-additive if and only if $\lim_{n\to\infty} \mu(A_n \cup B_n) = \{0\}$, whenever $(A_n)_n, (B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \{0\}$.

III) [22] The following statements are equivalent:

- a) μ is pseudo-null-additive;
- b) $\mu(B \cap C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$;
- c) $\mu((A \setminus B) \cup C) = \mu(C)$, whenever $A \in \mathcal{A}, B \in A \cap \mathcal{A}, C \in A \cap \mathcal{A}$ and $\mu(B) = \mu(A)$.

Proposition 2.5. If μ is (S)-continuous and autocontinuous from above, then it is double asymptotic null-additive.

Proof. First, suppose that, on the contrary, μ is autocontinuous from above, but it is not double asymptotic null-additive. Consequently, there exist $\varepsilon_0 > 0$ and two sequences $(A_n)_n, (B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \{0\}$ and $e(\mu(A_n \cup B_n), \{0\}) \geq \varepsilon_0$, for every $n \in \mathbb{N}$.

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By Remark 2.4 I-i), μ has property (S), so, there exist two subsequences $(A_{n_k})_k$ of $(A_n)_n, (B_{n_l})_l$ of $(B_n)_n$ such that $\mu(\limsup_{k\to\infty} A_{n_k}) = \{0\}$ and $\mu(\limsup_{l\to\infty} B_{n_l}) = \{0\}.$

Again, by Remark 2.4 I-i), μ is null-additive. In consequence, because μ is continuous from above, then $0 = e(\mu((\limsup_{k\to\infty} A_{n_k})\cup(\limsup_{l\to\infty} B_{n_l})), \{0\}) \ge \limsup_{k,l\to\infty} e(\mu(A_{n_k}\cup B_{n_l}), \{0\}) \ge \varepsilon_0$, which is a contradiction. \Box

In what follows, by \mathcal{M} we denote the class of all \mathcal{A} -measurable realvalued functions on (T, \mathcal{A}, μ) , which is the measurable space (T, \mathcal{A}) endowed with the monotone set multifunction μ .

Definition 2.6. I) Let be arbitrary $f, g \in \mathcal{M}$. We say that:

- i) f = g almost everywhere (for short, a.e.) if there exists $A \in \mathcal{A}$ such that $\mu(A) = \{0\}$ and f = g on $T \setminus A$.
- ii) f = g pseudo-almost everywhere (for short, p.a.e.) if there exists $A \in \mathcal{A}$ such that $\mu(T) = \mu(T \setminus A)$ and f = g on $T \setminus A$.
- II) We consider arbitrary $\{f_n\} \subset \mathcal{M}$ and $f \in \mathcal{M}$. We say that:
 - i) [22] {f_n} converges μ-almost everywhere (respectively, pseudo-μ-almost everywhere) to f on A, and denote it by f_n ^{a.e.}/_A f (respectively, f_n ^{p.a.e.}/_A f) if there exists a subset B ∈ A ∩ A such that μ(B) = {0} (respectively, μ(A\B) = μ(A)) and {f_n} is pointwise convergent to f on A\B.
 - ii) [22] $\{f_n\}$ converges in μ -measure (respectively, pseudo in μ -measure) to f on A, and denote it by $f_n \xrightarrow{\mu}{A} f$ (respectively, $f_n \xrightarrow{p.\mu}{A} f$) if for every $\varepsilon > 0$, $\lim_{n \to \infty} \mu(A_n(\varepsilon) = \{0\}$, where $A_n(\varepsilon) = \{t \in A; |f_n(t) - f(t)| \ge \varepsilon\}$ (respectively, $\lim_{n \to \infty} h(\mu(A \setminus A_n(\varepsilon)), \mu(A)) = 0$).
 - iii) [22] $\{f_n\}$ converges μ -almost uniformly (respectively, μ -pseudo-almost uniformly) to f on A and denote it by $f_n \xrightarrow[A]{A} f$ (respectively, $f_n \xrightarrow[A]{A} f$) if there exists a decreasing sequence $\{A_k\}_{k\in\mathbb{N}} \subset A \cap A$ such that $\lim_{k\to\infty} \mu(A_k) = \{0\}$ (respectively, $\lim_{k\to\infty} h(\mu(A \setminus A_k), \mu(A)) = 0$) and for every fixed $k \in \mathbb{N}$, $\{f_n\}$ uniformly converges to f on $A \setminus A_k$ ($f_n \xrightarrow[A \setminus A_k] f$).

iv) $\{f_n\}$ is a.e. (respectively, a.u., μ , p.a.e., p. μ , p.a.u.)-fundamental if $\lim_{m,n\to\infty} |f_n - f_m| = 0$ a.e. (respectively, a.u., μ , p.a.e., p. μ , p.a.u.).

In the sequel, we discuss the uniqueness of the limit of convergence and pseudo-convergence with respect to a monotone set multifunction. First, we shall need the following lemma:

Lemma 2.7. If $\lim_{n\to\infty} \mu(M_n) = \{0\}$ and for every $n \in \mathbb{N}$, $N_n \subset M_n$, then $\lim_{n\to\infty} \mu(N_n) = \{0\}$.

Proof. The statement easily follows since

$$h(\mu(N_n), \{0\}) = e(\mu(N_n), \{0\}) \le e(\mu(N_n), \mu(M_n)) + e(\mu(M_n), \{0\})$$

= $h(\mu(M_n), \{0\}) \to 0.$

Theorem 2.8. Let us consider arbitrary $\{f_n\} \subset \mathcal{M}$ and $f, g \in \mathcal{M}$ so that f = g a.e. Then μ is single asymptotic null-additive if and only if $f_n \xrightarrow{\mu} f$ implies $f_n \xrightarrow{\mu} g$.

Proof. Necessity. Since f = g a.e., there exists $A \in \mathcal{A}$ so that $\mu(A) = \{0\}$ and f(t) = g(t), for every $t \in T \setminus A$. Then $\mu(\{t; f(t) \neq g(t)\}) = \{0\}$.

Since μ is single asymptotic null-additive and $f_n \xrightarrow{\mu} f$, then for every $\varepsilon > 0$, $\lim_{n \to \infty} \mu(\{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\} \cup \{t; f(t) \ne g(t)\}) = \{0\}.$

By Lemma 2.7, because

$$\{t; |f_n(t) - g(t)| \ge \varepsilon\} \subset \{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\} \cup \{t; f(t) \ne g(t)\},\$$

we get that $f_n \xrightarrow{\mu} g$.

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Sufficiency. Let $(A_n)_n \subset \mathcal{A}, A \in \mathcal{A}$ be so that $\lim_{n \to \infty} \mu(A_n) = \{0\}$ and $\mu(A) = \{0\}$.

For every $n \in \mathbb{N}$ and every $t \in T$, we consider $f_n(t) = \chi_{A_n \setminus A}(t)$, $g(t) = \chi_A(t)$ and f(t) = 0.

We observe that f = g a.e. and for every $n \in \mathbb{N}$ and every $\varepsilon > 0$,

$$\mu(\{t; |f_n(t) - f(t)| \ge \varepsilon\}) = \mu(A_n \setminus A) \subset \mu(A_n),$$

whence, by Lemma 2.7, $f_n \xrightarrow{\mu} f$. Then, by the hypothesis, $f_n \xrightarrow{\mu} g$.

Consequently, $\lim_{n\to\infty} \mu(A \cup A_n) = \lim_{n\to\infty} \mu(\{t; |f_n(t) - g(t)| \ge 1\}) = \{0\}$, i.e., μ is single asymptotic null-additive.

 $\begin{aligned} \text{Theorem 2.9. Let us consider arbitrary } \{f_n\} \subset \mathcal{M} \text{ and } f, g \in \mathcal{M} \text{ so} \\ \text{that } f = g \text{ p.a.e. If } \mu \text{ is pseudo-null-additive and } f_n \xrightarrow{p,\mu} f, \text{ then } f_n \xrightarrow{p,\mu} g. \\ \text{Proof. Since } f = g \text{ p.a.e, there exists } A \in \mathcal{A} \text{ so that } \mu(T) = \mu(T \setminus A) \\ \text{and } f(t) = g(t), \text{ for every } t \in T \setminus A. \\ \text{Because } f_n \xrightarrow{p,\mu} f, \text{ then for every } \varepsilon > 0, \lim_{n \to \infty} h(\mu(T \setminus A_n(\varepsilon)), \mu(T)) = \\ 0, \text{ where } A_n(\varepsilon) = \{t \in T; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\}. \\ \text{Since } \widetilde{A}_n(\varepsilon) = \{t \in T; |f_n(t) - g(t)| \ge \varepsilon\} \subset A_n(\varepsilon) \cup A, \text{ we have} \\ \lim_{n \to \infty} e(\mu(T), \mu(T \setminus \widetilde{A}_n(\varepsilon))) \\ & \leq \lim_{n \to \infty} [e(\mu(T), \mu(T \setminus A_n(\varepsilon))) + e(\mu(T \setminus A_n(\varepsilon)), \mu(T \setminus \widetilde{A}_n(\varepsilon)))] \\ & = \lim_{n \to \infty} e(\mu(T \setminus A_n(\varepsilon)), \mu(T \setminus \widetilde{A}_n(\varepsilon))) \\ & \leq \lim_{n \to \infty} [e(\mu(T \setminus A_n(\varepsilon)), \mu(T \setminus \widetilde{A}_n(\varepsilon)))] \\ & = \lim_{n \to \infty} e(\mu(C \setminus A_n(\varepsilon)), \mu(T \setminus \widetilde{A}_n(\varepsilon)))] \\ & = \lim_{n \to \infty} e(\mu(C \setminus A_n(\varepsilon)), \mu(C \setminus \widetilde{A}_n(\varepsilon)))] \end{aligned}$

because μ is pseudo-null-additive and $\mu(T) = \mu(cA)$, so, by Remark 2.3 III), we have $\mu(D) = \mu(D \cap cA)$, for every $D \in \mathcal{A}$.

Consequently, $\lim_{n\to\infty} h(\mu(T), \mu(T \setminus \widetilde{A}_n(\varepsilon))) = 0$ and this means $f_n \xrightarrow{p.\mu} g.\square$

Theorem 2.10. Let be arbitrary $\{\{f_n\}, f, g\} \subset \mathcal{M}$. If μ is double asymptotic null-additive and continuous from below, $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then f = g a.e.

Proof. Because $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then for every $m \in \mathbb{N}^*$, $\lim_{n\to\infty} \mu((\{t; |f_n(t) - f(t)| \ge \frac{1}{2m}\}) = \{0\}$ and $\lim_{n\to\infty} \mu(\{t; |f_n(t) - g(t)| \ge \frac{1}{2m}\}) = \{0\}$.

Since μ is double asymptotic null-additive and for every $m \in \mathbb{N}^*$,

$$\{t; |f(t) - g(t)| \ge \frac{1}{m}\} \subseteq \{t; |f_n(t) - f(t)| \ge \frac{1}{2m}\} \\ \cup \{t; |f_n(t) - g(t)| \ge \frac{1}{2m}\},$$

by Lemma 2.7 we get that $\lim_{m\to\infty} \mu(\{t; |f(t) - g(t)| \ge \frac{1}{m}\}) = \{0\}$. Taking into account that μ is continuous from below and

$$\{t; |f(t) - g(t)| > 0\} = \bigcup_{m=1}^{\infty} \{t; |f(t) - g(t)| \ge \frac{1}{m}\},\$$

we obtain that $\lim_{m\to\infty} h(\mu(\{t; |f(t) - g(t)| > 0\}), \mu(\{t; |f(t) - g(t)| \ge \frac{1}{m}\})) = 0$, which yields $\mu(\{t; |f(t) - g(t)| > 0\}) = \{0\}$, so, f = g a.e. \Box

Lemma 2.11. Suppose μ is (S)-continuous. Then μ is pseudo-autocontinuous from below if and only if for every $A \in \mathcal{A}$ and every $(B_n)_n, (C_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} e(\mu(A), \mu(A\cap B_n)) = 0$ and $\lim_{n\to\infty} e(\mu(A), \mu(A\cap C_n)) = 0$, we have $\lim_{n\to\infty} e(\mu(A), \mu(A\cap B_n\cap C_n)) = 0$.

Proof. Necessity. By the definition of lim sup of the excess, there exists a subsequence $(n_k)_k$ so that $\limsup_n e(\mu(A), \mu(A \cap B_n \cap C_n)) = \lim_{k\to\infty} e(\mu(A), \mu(A \cap B_{n_k} \cap C_{n_k})).$

On the other hand, by Remark 2.4 I)-iii), μ has property (PS), so there exists a subsequence $(n_{k_s})_s$ of $(n_k)_k$ so that

$$\mu(A \cap \liminf B_{n_{k_s}}) = \mu(A \cap \liminf C_{n_{k_s}}) = \mu(A).$$

By Remark 2.4 I)-iii), μ is also pseudo-null-additive, hence, by Remark 2.4 III), $\mu(A \cap \liminf_{s} B_{n_{k_s}} \cap \liminf_{s} C_{n_{k_s}}) = \mu(A)$. Now, we observe that

$$\begin{split} \limsup_{n} e(\mu(A), \mu(A \cap B_{n} \cap C_{n})) \\ &= \lim_{k \to \infty} e(\mu(A), \mu(A \cap B_{n_{k}} \cap C_{n_{k}})) \\ &= \lim_{s \to \infty} e(\mu(A), \mu(A \cap B_{n_{k_{s}}} \cap C_{n_{k_{s}}})) \\ &= \lim_{s \to \infty} e(\mu(A \cap \liminf_{s} B_{n_{k_{s}}} \cap \liminf_{s} C_{n_{k_{s}}}), \mu(A \cap B_{n_{k_{s}}} \cap C_{n_{k_{s}}})) = 0, \end{split}$$

because μ is continuous from below. Consequently, $\limsup_n e(\mu(A), \mu(A \cap B_n \cap C_n)) = 0$, whence $\lim_{n \to \infty} e(\mu(A), \mu(A \cap B_n \cap C_n)) = 0$, as claimed.

Sufficiency. Let us consider arbitrary $A \in \mathcal{A}, C \in A \cap \mathcal{A}, (B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} h(\mu(B_n \cap A), \mu(A)) = 0$. We prove that $\lim_{n\to\infty} h((\mu(B_n \cap C), \mu(C))) = 0$.

We shall apply the hypothesis for $C_n = [(A \setminus B_n) \cap C] \Delta C$, with $n \in \mathbb{N}$. One can easily check that $C_n \supset A \cap B_n$, so $A \cap C_n \supset A \cap B_n$. Also, $A \cap B_n \cap C_n = C \cap B_n$. This implies

$$\lim_{n \to \infty} e(\mu(A), \mu(A \cap C_n))$$

$$\leq \lim_{n \to \infty} [e(\mu(A), \mu(A \cap B_n)) + e(\mu(A \cap B_n), \mu(A \cap C_n))]$$

$$= \lim_{n \to \infty} e(\mu(A), \mu(A \cap B_n)) = 0,$$

so $\lim_{n\to\infty} e(\mu(A), \mu(A \cap C_n)) = 0.$

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Consequently, by the hypothesis, we have $\lim_{n\to\infty} e(\mu(A), \mu(A \cap B_n \cap C_n)) = 0$, whence

$$\lim_{n \to \infty} h((\mu(B_n \cap C), \mu(C))) = \lim_{n \to \infty} e(\mu(C), \mu(B_n \cap C))$$

$$\leq \lim_{n \to \infty} [e(\mu(C), \mu(A)) + e(\mu(A), \mu(B_n \cap C))]$$

$$= \lim_{n \to \infty} e(\mu(A), \mu(B_n \cap C)) = \lim_{n \to \infty} e(\mu(A), \mu(A \cap B_n \cap C_n)) = 0.$$

Therefore, we get that $\lim_{n\to\infty} h((\mu(B_n\cap C), \mu(C)) = 0$, that is, μ is pseudo-autocontinuous from below.

Theorem 2.12. Let be arbitrary $\{\{f_n\}, f, g\} \subset \mathcal{M}$. If μ is pseudoautocontinuous from below and (S)-continuous, $f_n \xrightarrow{p.\mu} f$ and $f_n \xrightarrow{p.\mu} g$, then f = g p.a.e.

Proof. Because $f_n \xrightarrow{p,\mu} f$ and $f_n \xrightarrow{p,\mu} g$, then for every $m \in \mathbb{N}^*$, $\lim_{n\to\infty} h(\mu(T), \mu((\{t; |f_n(t) - f(t)| < \frac{1}{2m}\})) = 0$ and $\lim_{n\to\infty} h(\mu(T), \mu((\{t; |f_n(t) - g(t)| < \frac{1}{2m}\}) = 0$, whence, by Lemma 2.11, we get $\lim_{n\to\infty} h(\mu(T), \mu(\{t; |f_n(t) - f(t)| < \frac{1}{2m}\} \cap \{t; |f_n(t) - g(t)| < \frac{1}{2m}\})) = 0$. Because for every $m, n \in \mathbb{N}^*$,

$$\{t; |f_n(t) - f(t)| < \frac{1}{2m}\} \cap \{t; |f_n(t) - g(t)| < \frac{1}{2m}\} \subset \{t; |f(t) - g(t)| < \frac{1}{m}\},\$$

we get that

$$\begin{split} &\lim_{n \to \infty} h(\mu(T), \mu(\{t; | f(t) - g(t)| < \frac{1}{m}\}) \\ &= \lim_{n \to \infty} e(\mu(T), \mu(\{t; | f(t) - g(t)| < \frac{1}{m}\}) \\ &\leq \lim_{n \to \infty} [e(\mu(T), \mu(\{t; | f_n(t) - f(t)| < \frac{1}{2m}\}) \\ &\cap \{t; | f_n(t) - g(t)| < \frac{1}{2m}\})) + e(\mu(\{t; | f_n(t) - f(t)| < \frac{1}{2m}\}) \\ &\cap \{t; | f_n(t) - g(t)| < \frac{1}{2m}\}), \mu(\{t; | f(t) - g(t)| < \frac{1}{m}\}))] \\ &= \lim_{n \to \infty} e(\mu(T), \mu(\{t; | f_n(t) - f(t)| < \frac{1}{2m}\} \cap \{t; | f_n(t) - g(t)| < \frac{1}{2m}\})) \\ &= \lim_{n \to \infty} h(\mu(T), \mu(\{t; | f_n(t) - f(t)| < \frac{1}{2m}\} \cap \{t; | f_n(t) - g(t)| < \frac{1}{2m}\})) = 0. \end{split}$$

$$\{t; |f(t) - g(t)| = 0\} = \bigcap^{\infty} \{t; |f(t) - g(t)| < \frac{1}{m}\},\$$

$$m=1$$

we finally get that $h(\mu(T), \mu(\{t; |f(t) - g(t)| = 0\}) = 0$, i.e., f = g p.a.e. \Box

3. Algebraic operations with (pseudo-) convergent sequences of measurable functions

In this section, we present several results concerning the inheriting of convergence and pseudo-convergence with respect to monotone set multifunctions, under addition and multiplication. Several asymptotic structural properties of the monotone set multifunction are in this way characterized.

Theorem 3.1. Consider arbitrary $\{\{f_n\}, \{g_n\}, f, g\} \subset \mathcal{M}$.

 μ is double asymptotic null-additive if and only if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ imply $f_n + g_n \xrightarrow{\mu} f + g$.

Proof. Necessity. Since $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, then for every $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\}) = \{0\}$ and $\lim_{n\to\infty} \mu(\{t; |g_n(t) - g(t)| \ge \frac{\varepsilon}{2}\}) = \{0\}$.

Because μ is double asymptotic null-additive and

$$\begin{aligned} \{t; |(f_n + g_n)(t) - (f + g)(t)| \ge \varepsilon\} \subset \{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\} \\ \cup \{t; |g_n(t) - g(t)| \ge \frac{\varepsilon}{2}\}, \end{aligned}$$

then, by Lemma 2.7, $\lim_{n\to\infty} \mu(\{t; |(f_n+g_n)(t)-(f+g)(t)| \ge \varepsilon\}) = \{0\}$, that is, $f_n + g_n \xrightarrow{\mu} f + g$.

Sufficiency. Suppose that, on the contrary, there exist $(A_n)_n, (B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \{0\}$ and $\lim_{n\to\infty} \mu(A_n \cup B_n) \supseteq \{0\}$. For every $n \in \mathbb{N}$, consider $f_n(t) = \chi_{A_n(t)}, g_n(t) = \chi_{B_n(t)}, f(t) = 0$ and g(t) = 0. Evidently, $(f_n + g_n)(t) = \chi_{A_n \cup B_n}(t)$ and (f + g)(t) = 0. We observe that for every $\varepsilon > 0$ and every $n \in \mathbb{N}, \mu(\{t; |f_n(t) - f(t)| \ge \varepsilon\}) \subset \mu(A_n)$ and $\mu(\{t; |g_n(t) - g(t)| \ge \varepsilon\}) \subset \mu(B_n)$.

Since $\lim_{n\to\infty} \mu(A_n) = \{0\} = \lim_{n\to\infty} \mu(B_n)$, by Lemma 2.7, we get that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. By the hypothesis, $f_n + g_n \xrightarrow{\mu} f + g$, so,

$$\lim_{n \to \infty} \mu(A_n \cup B_n) = \lim_{n \to \infty} \mu(\{t; |(f_n + g_n)(t) - (f + g)(t)| \ge 1\}) = \{0\},\$$

which is a contradiction.

Theorem 3.1'. Consider arbitrary $\{\{f_n\}, \{g_n\}, f, g\} \subset \mathcal{M}$.

If μ is (S)-continuous and pseudo-autocontinuous from below, $f_n \xrightarrow{p.\mu} f$ and $g_n \xrightarrow{p.\mu} g$, then $f_n + g_n \xrightarrow{p.\mu} f + g$.

Proof. Since $f_n \xrightarrow{p.\mu} f$ and $g_n \xrightarrow{p.\mu} g$, then for every $\varepsilon > 0$, we have $\lim_{n\to\infty} e(\mu(T), \mu(\{t; |f_n(t) - f(t)| < \frac{\varepsilon}{2}\})) = 0$ and $\lim_{n\to\infty} e(\mu(T), \mu(\{t; |g_n(t) - g(t)| < \frac{\varepsilon}{2}\})) = 0$. For every $n \in \mathbb{N}$, we denote $A_n = \{t; |f_n(t) - f(t)| < \frac{\varepsilon}{2}\}$, $B_n = \{t; |g_n(t) - g(t)| < \frac{\varepsilon}{2}\}$ and $C_n = \{t; |(f_n + g_n)(t) - (f + g)(t)| < \varepsilon\}$. By Lemma 2.11, we have $\lim_{n\to\infty} e(\mu(T), \mu(A_n \cap B_n)) = 0$.

Since for every $n \in \mathbb{N}$, $A_n \cap B_n \subset C_n$, then $\lim_{n \to \infty} e(\mu(T), \mu(C_n)) = 0$, that is, $f_n + g_n \xrightarrow{\mu} f + g$.

Applying the notions from Definition 2.6 II)-ii), one can easily verify the following results:

Proposition 3.2. i) Consider arbitrary $c \in \mathbb{R}^*$ and $\{\{f_n\}, f\} \subset \mathcal{M}$. If $f_n \xrightarrow{\mu} f(f_n \xrightarrow{p.\mu} f, respectively)$, then $cf_n \xrightarrow{\mu} cf(cf_n \xrightarrow{p.\mu} cf, respectively)$. ii) If $\{\{f_n\}, f\} \subset \mathcal{M}$ and $f_n \xrightarrow{p.\mu} f$, then $|f_n| \xrightarrow{p.\mu} |f|$.

Lemma 3.3. Consider arbitrary $\{\{f_n\}, f, h\} \subset \mathcal{M}$. If $f_n \xrightarrow{\mu} f$, $\inf_{t \in T} |h(t)| > 0$ and $\sup_{t \in T} |h(t)| < \infty$, then $f_n h \xrightarrow{\mu} f h$.

Proof. Since $f_n \xrightarrow{\mu} f$, then for every $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{\sup_{t\in T} |h(t)|}\}) = \{0\}$. Because for every $\varepsilon > 0$,

$$\{t; |f_n(t)h(t) - f(t)h(t)| \ge \varepsilon \} = \{t; |f_n(t) - f(t)| \cdot |h(t)| \ge \varepsilon \} \subseteq$$

$$\le \{t; |f_n(t) - f(t)| \cdot \sup_{t \in T} |h(t)| \ge \varepsilon \} = \left\{t; |f_n(t) - f(t)| \ge \frac{\varepsilon}{\sup_{t \in T} |h(t)|} \right\},$$

the conclusion follows according to Lemma 2.7.

Lemma 3.3'. Consider arbitrary $\{\{f_n\}, f, h\} \subset \mathcal{M}$. If $\sup_{t \in T} |h(t)| < \infty$ and $f_n \xrightarrow{p.\mu} f$, then $f_n h \xrightarrow{p.\mu} fh$.

Proof. If $h \equiv 0$, the statement is evident. Suppose now that $\sup_{t \in T} |h(t)| > 0$. Since $f_n \xrightarrow{p \cdot \mu} f$, then for every $\varepsilon > 0$, $\lim_{n \to \infty} e(\mu(T), \mu(\{t;$

$$\begin{aligned} f_n(t) - f(t)| &< \frac{\varepsilon}{\sup_{t \in T} |h(t)|} \}) = 0. \text{ Because} \\ \lim_{n \to \infty} e(\mu(T), \mu(\{t; |f_n(t)h(t) - f(t)h(t)| < \varepsilon\})) \\ &\leq \lim_{n \to \infty} e(\mu(T), \mu(\{t; |f_n(t) - f(t)| < \frac{\varepsilon}{\sup_{t \in T} |h(t)|}\})) = 0, \end{aligned}$$

the conclusion follows immediately.

Lemma 3.4. Consider arbitrary $\{\{f_n\}, f\} \subset \mathcal{M} \text{ and let } \mu \text{ be double} asymptotic null-additive. If <math>f_n \xrightarrow{\mu} f$, $\inf_{t \in T} |f(t)| > 0$ and $\sup_{t \in T} |f(t)| < \infty$, then $f_n^2 \xrightarrow{\mu} f^2$.

Proof. Obviously, since $f_n \xrightarrow{\mu} f$, then $f_n - f \xrightarrow{\mu} 0$, so $(f_n - f)^2 \xrightarrow{\mu} 0$. On the other hand, by Lemma 3.3, $f_n f \xrightarrow{\mu} f^2$, so, by Proposition 3.2 i), $2f_n f \xrightarrow{\mu} 2f^2$. Applying now Theorem 3.1, we get that $f_n^2 \xrightarrow{\mu} f^2$. \Box

Lemma 3.4'. Suppose μ is (S)-continuous and pseudo-autocontinuous from below. Consider arbitrary $\{\{f_n\}, f\} \subset \mathcal{M}$ so that $\sup_{t \in T} |f(t)| < \infty$ and $f_n \xrightarrow{p,\mu} f$. Then $f_n^2 \xrightarrow{p,\mu} f^2$.

Proof. Because $f_n \xrightarrow{p.\mu} f$, then $f_n - f \xrightarrow{p.\mu} 0$, whence $(f_n - f)^2 \xrightarrow{p.\mu} 0$. By Lemma 3.3', $f_n f \xrightarrow{p.\mu} f^2$, so, according to Proposition 3.2 i), $2f_n f \xrightarrow{p.\mu} 2f^2$. By Theorem 3.1', the conclusion follows.

Theorem 3.5. Consider arbitrary $\{\{f_n\}, \{g_n\}, f, g\} \subset \mathcal{M}$. Suppose that $\inf_{t \in T} |f(t)| > 0$, $\inf_{t \in T} |g(t)| > 0$, $\inf_{t \in T} |f(t) + g(t)| > 0$, $\sup_{t \in T} |f(t)| < \infty$ and $\sup_{t \in T} |g(t)| < \infty$.

Then μ is double asymptotic null-additive if and only if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ imply $f_n g_n \xrightarrow{\mu} f g$.

Proof. Necessity. Suppose μ is double asymptotic null-additive, $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. By Lemma 3.4, we have $f_n^2 \xrightarrow{\mu} f^2, g_n^2 \xrightarrow{\mu} g^2$ and $(f_n + g_n)^2 \xrightarrow{\mu} (f + g)^2$, which imply by Theorem 3.1 that $f_n g_n \xrightarrow{\mu} fg$.

Sufficiency. Suppose that, on the contrary, μ is not double asymptotic null-additive. Consequently, there exist $(A_n)_n, (B_n)_n \subset \mathcal{A}$ and $\varepsilon_0 > 0$ so that $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \{0\}$ and $\lim_{n\to\infty} \mu(A_n \cup B_n) \supseteq \{0\}$. For every $t \in T$ and $n \in \mathbb{N}$, consider $f_n(t) = \begin{cases} 1, & t \notin A_n \\ \frac{1}{2}, & t \in A_n \end{cases}$ and $g_n(t) =$

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 $\begin{cases} 2, & t \notin B_n \\ 1, & t \in B_n \end{cases}. \text{ We observe that } \{\{f_n\}, \{g_n\}\} \subset \mathcal{M}, f_n \xrightarrow{\mu} f \equiv 1, g_n \xrightarrow{\mu} \end{cases}$ $\begin{array}{ll} (1, & t \in B_n \\ g \equiv 2, \ f_n(t)g_n(t) = \begin{cases} 2, & t \notin A_n \cup B_n \\ 1, & t \in (A_n \cup B_n) \setminus (A_n \cap B_n) \ , \ f(t)g(t) = 2. \\ \frac{1}{2}, & t \in A_n \cap B_n \end{cases}$ Since $t \in A_n \cup B_n$ if and only if $|f_n(t)g_n(t) - f(t)g(t)| \ge \frac{1}{2}$, we have

$$\mu(A_n \cup B_n) = \mu(\{t \in T; |f_n(t)g_n(t) - f(t)g(t)| \ge \frac{1}{2}\}).$$

By the hypothesis, $f_n g_n \xrightarrow{\mu} fg$, so $\lim_{n \to \infty} \mu(\{t \in T; |f_n(t)g_n(t) - f(t)g(t)|$ $\geq \frac{1}{2}$ = {0}, which implies $\lim_{n\to\infty} \mu(A_n \cup B_n) = \{0\}$ and this is a contradiction.

Theorem 3.5'. Suppose μ is (S)-continuous and pseudo-autocontinuous from below. Let $\{\{f_n\}, \{g_n\}, f, g\} \subset \mathcal{M}$ so that $\sup_{t \in T} |f(t)| < \infty$ and $\sup_{t \in T} |g(t)| < \infty. \text{ If } f_n \xrightarrow{p.\mu} f \text{ and } g_n \xrightarrow{p.\mu} g, \text{ then } f_n g_n \xrightarrow{p.\mu} fg.$

Proof. The statement is straightforward applying Lemma 3.4' and Theorem 3.1'.

4. Fundamental (pseudo-) convergence

In this section, several results concerning fundamental convergence and pseudo-convergence are established.

Theorem 4.1. μ is double asymptotic null-additive if and only if for every fixed $A \in \mathcal{A}$, every sequence $\{f_n\} \subset \mathcal{M}$ on A so that $f_n \xrightarrow{\mu} f \in \mathcal{M}$ is μ -fundamental.

Proof. Without any loss of generality, we may assume that A = T. *Necessity.* By the hypothesis, we have $\lim_{n\to\infty} \mu(\{t \in T; |f_n(t) - f(t)| \geq t\})$ $\left\{\frac{\varepsilon}{2}\right\} = \{0\}.$

For every $\varepsilon > 0$, since μ is double asymptotic null-additive and

$$\{t \in T; |f_n(t) - f_m(t)| \ge \varepsilon\} \subset \{t \in T; |f_n(t) - f(t)| \ge \frac{\varepsilon}{2}\}$$
$$\cup \{t \in T; |f_m(t) - f(t)| \ge \frac{\varepsilon}{2}\},\$$

by Lemma 2.7 we get that $\lim_{m,n\to\infty} \mu(\{t \in T; |f_m(t) - f_n(t)| \ge \frac{\varepsilon}{2}\}) = \{0\}$, i.e., $\{f_n\}$ is μ -fundamental.

Sufficiency. Suppose by the contrary that there exist $(A_n)_n, (B_n)_n \subset \mathcal{A}$, with $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \{0\}$ and $\lim_{n\to\infty} \mu(A_n \cup B_n) \supseteq \{0\}$.

We consider the sequence $\{f_n\}$, defined by

$$f_n(t) = \begin{cases} \chi_{A_k}(t), & \text{if } n = 2k \\ \chi_{B_k \setminus A_k}(t), & \text{if } n = 2k+1 \end{cases}$$

Let also be f(t) = 0. We observe that $\{f_n\} \subset \mathcal{M}, f \in \mathcal{M}$ and for every $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\{t \in T; |f_n(t) - f(t)| \ge \varepsilon\} \subseteq A_n \text{ or } \{t \in T; |f_n(t) - f(t)| \ge \varepsilon\} \subseteq B_n.$$

Consequently, by Lemma 2.7, $\lim_{n\to\infty} \mu(\{t \in T; |f_n(t) - f(t)| \ge \varepsilon\}) = \{0\}$, i.e., $f_n \xrightarrow{\mu} f$. By the hypothesis, $\{f_n\}$ is then μ -fundamental, so, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu(\{t \in T; |f_{2n}(t) - f_{2n+1}(t)| \ge \varepsilon\}) = \{0\}.$$

On the other hand, for every $\varepsilon \in (0, 1)$, we have $A_n \cup B_n = \{t \in T; |f_{2n}(t) - f_{2n+1}(t)| \ge \varepsilon\}$.

Consequently, $\lim_{n\to\infty} \mu(A_n \cup B_n) = \{0\}$ and this is a contradiction. \Box

Theorem 4.2. If μ is pseudo-autocontinuous from above and $A \in \mathcal{A}$ is fixed, then every sequence $\{f_n\} \subset \mathcal{M}$ on A so that $f_n \xrightarrow{p,\mu}{A} f \in \mathcal{M}$ is $p.\mu$ -fundamental.

Proof. By the hypothesis, $\lim_{n\to\infty} e(\mu(T), \mu(\{t \in T; |f_n(t) - f(t)| < \frac{\varepsilon}{2}\})) = 0$ and $\lim_{m\to\infty} e(\mu(T), \mu(\{t \in T; |f_m(t) - f(t)| < \frac{\varepsilon}{2}\})) = 0$. For every $m, n \in \mathbb{N}$, let us denote $B_n = \{t \in T; |f_n(t) - f(t)| < \frac{\varepsilon}{2}\}$ and $C_{m,n} = \{t \in T; |f_n(t) - f_m(t)| < \varepsilon\}$. We observe that $B_n \cap B_m \subset C_{m,n}$, whence $(T \setminus B_n) \cup (T \setminus B_m) \cup C_{m,n} = T$. Because μ is pseudo-autocontinuous from above, then for every $m \in \mathbb{N}$, $\lim_{n\to\infty} e(\mu((T \setminus B_n) \cup (T \setminus B_m) \cup C_{m,n}), \mu((T \setminus B_m) \cup C_{m,n})) = 0$ and for every $n \in \mathbb{N}$, $\lim_{m\to\infty} e(\mu((T \setminus B_m) \cup C_{m,n}), \mu(C_{m,n})) = 0$.

Consequently, $\lim_{m,n\to\infty} e(\mu(T),\mu(C_{m,n})) = 0$, which says that (f_n) is $p.\mu$ -fundamental.

Theorem 4.3. If μ is uniformly pseudo-autocontinuous from below and $\{f_n\} \subset \mathcal{M}$ is $p.\mu$ -fundamental, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ so that $\{f_{n_k}\}_k$ is pseudo almost uniformly convergent.

Proof. Since $\{f_n\}_n$ is $p.\mu$ -fundamental, then $\lim_{m,n\to\infty} e(\mu(T), \mu(\{t \in T; |f_n(t) - f_m(t)| < \frac{1}{2}\})) = 0$, so, there exists $n_1 \in \mathbb{N}$ such that, for every $n \ge n_1$,

$$e(\mu(T), \mu(\{t \in T; |f_n(t) - f_{n_1}(t)| < \frac{1}{2}\})) < \frac{1}{2}.$$

Because μ is uniformly pseudo-autocontinuous from below, there exists $n_2 > n_1$ so that for every $n \ge n_2$,

$$e(\mu(T), \mu(\{t \in T; |f_n(t) - f_{n_2}(t)| < \frac{1}{2^2}\})) < \frac{1}{2^2}$$

and

$$\begin{split} &e(\mu(\{t\in T; |f_n(t)-f_{n_2}(t)|<\frac{1}{2^2}\}), \mu(\{t\in T; |f_{n_2}(t)-f_{n_1}(t)|<\frac{1}{2}\}\\ &\cap \{t\in T; |f_n(t)-f_{n_2}(t)|<\frac{1}{2^2}\}))<\frac{1}{2}, \end{split}$$

whence

$$e(\mu(T), \mu(\{t \in T; |f_{n_2}(t) - f_{n_1}(t)| < \frac{1}{2}\} \cap \{t \in T; |f_n(t) - f_{n_2}(t)| < \frac{1}{2^2}\})) < \frac{1}{2} + \frac{1}{2^2}.$$

Analogously, there exists $n_3 > n_2$ so that for every $n \ge n_3$,

$$\begin{split} e(\mu(T), \mu(\{t \in T; |f_n(t) - f_{n_3}(t)| < \frac{1}{2^3}\})) < \frac{1}{2^3}, \\ e(\mu(T), \mu(\{t \in T; |f_{n_3}(t) - f_{n_2}(t)| < \frac{1}{2^3}\}) \\ &\cap \{t \in T; |f_n(t) - f_{n_3}(t)| < \frac{1}{2^2}\})) < \frac{1}{2^2} + \frac{1}{2^3} \end{split}$$

and

$$e(\mu(T), \mu(\{t \in T; |f_{n_3}(t) - f_{n_2}(t)| < \frac{1}{2^2}\})$$

$$\cap \{t \in T; |f_{n_2}(t) - f_{n_1}(t)| < \frac{1}{2}\}$$

$$\cap \{t \in T; |f_n(t) - f_{n_3}(t)| < \frac{1}{2^3}\}) < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$$

Inductively, there exists a subsequence $(n_k)_k$ so that for every $l \in \mathbb{N}^*$,

$$e(\mu(T), \mu(\{t \in T; \bigcap_{k=l}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < \frac{1}{2^k}\})) < \frac{1}{2^l} + \frac{1}{2^{l+1}} + \dots$$

If for every $l \in \mathbb{N}^*$, we denote $C_l = \{t \in T; \bigcap_{k=l}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < \frac{1}{2^k}\},\$ then the sequence $(\tilde{C}_l)_l = (cC_l)_l$ is decreasing and $e(\mu(T), \mu(T \setminus \tilde{C}_l)) <$ $\sum_{n=0}^{\infty} \frac{1}{2^{l+n}} = \frac{1}{2^{l-1}}$, which implies that $\lim_{l\to\infty} e(\mu(T), \mu(T\setminus \widetilde{C}_l)) = 0$.

In consequence, $\{f_{n_k}\}_k$ is pseudo-almost uniformly convergent.

Corollary 4.4. Suppose μ is (S)-continuous and uniformly pseudoautocontinuous from below. Then $\{f_n\} \subset \mathcal{M}$ is p.µ-fundamental if and only if there exists $f \in \mathcal{M}$ so that $f_n \xrightarrow{p,\mu} f$.

Proof. Necessity. If $\{f_n\} \subset \mathcal{M}$ is $p.\mu$ -fundamental, by Theorem 4.3, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ so that $\{f_{n_k}\}_k$ is pseudo-almost uniformly convergent. Because μ is continuous from below, one can easily observe that it is also pseudo-almost everywhere convergent, i.e., there exists $A \in \mathcal{A}$ such that $\mu(T) = \mu(T \setminus A)$ and $\{f_{n_k}\}_k$ is convergent on $T \setminus A$.

We consider $f(t) = \begin{cases} \lim_{k} f_{n_k}(t), & \text{if } t \in T \setminus A \\ 0, & \text{if } t \in A \end{cases}$. Then $f \in \mathcal{M}$. Evidently,

for every $\varepsilon > 0$,

$$\begin{aligned} \{t \in T; |f_n(t) - f(t)| < \varepsilon \} \\ \supset \{t \in T; |f_n(t) - f_{n_k}(t)| < \frac{\varepsilon}{2} \} \cap \{t \in T; |f_{n_k}(t) - f(t)| < \frac{\varepsilon}{2} \}. \end{aligned}$$

Then one can easily observe that $\lim_{n\to\infty} e(\mu(T), \mu(\{t \in T; |f_n(t) - f(t)| < t\})$ $\varepsilon\})) = 0$, so $f_n \xrightarrow{p.\mu} f$.

Sufficiency. The statement results from Theorem 4.2 and Remark 2.4 I)-iii). \square

The following result is straightforward applying the definitions.

Theorem 4.5. Suppose $A \in \mathcal{A}$ and $\{f_n\} \subset \mathcal{F}$. If $\{f_n\}$ converges a.u. (p.a.u., respectively) on A, then $\{f_n\}$ is fundamental a.u. (p.a.u., respectively) on A.

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5. Concluding remarks

In this paper, the studies [22, 23] concerning convergences and pseudoconvergences of sequences of measurable functions on monotone set multifunction spaces are furthered in order to obtain results concerning operations and uniqueness of the limit of such convergences. As application, important asymptotic structural properties of the monotone set multifunction are characterized.

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