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A TOPOLOGICAL VERSION OF ITERATED FUNCTION SYSTEMS

BY

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Abstract. The aim of the paper is to give a generalization of the notion of iterated function system in a topological setting, namely to define a topological iterated function system. We will also give some examples of compact metric spaces which are not attractors of iterated function systems but are attractors of topological iterated function systems. In some examples this spaces are homeomorphic with attractors of classical iterated function systems and in others are not.

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1. Introduction

Iterated function systems (IFS) are one of the most common and general ways to generate fractals. IFS were conceived by Hutchinson ([7]) and were popularized by Barnsley ([2]). There is a current effort to extend Hutchinson's classical framework for fractals to more general spaces and infinite IFSs and to study them. For example, in [9], it was provided such a general framework where attractors are non-empty closed and bounded subsets of a complete metric spaces and where the IFSs may be infinite, in contrast with the classical theory ([2, 4, 5, 15]), where only attractors that are compact metric spaces and IFSs that are finite were considered. Some extensions of IFSs come from replacing contractions from the definition of an IFS with more general contractive conditions (see [14], for example). Other extensions of IFSs can be found in [1, 4, 10, 12, 13]. There are examples of compact metric spaces which are homeomorphic with an attractor of an IFS but are not attractors of IFSs. We define the notion of topological iterated function systems (TIFSs) to see this spaces as attractors of a generalized kind of IFSs and we will give an example of such a space. We will also give an example of a metrizable compact space which is the attractor of a TIFS but is not the attractor of an IFS with any metric which induces the initial topology of the space. The notion of the shift space of an IFS has an important place in the definition of a TIFS. A generalization of the shift space for an infinite IFS can be found in [11].

2. Preliminaries

For a set X, $\mathcal{P}(X)$ denotes the subsets of X. For a subset A of $\mathcal{P}(X)$, by A^* we mean $A - \{\emptyset\}$. For a topological space (X, τ), $\mathcal{K}(X)$ denotes the set of compact subsets of X.

Definition 2.1. Let (X, d) be a metric space. A function $f : X \to X$ is a Lipschitz function if $Lip(f) < +\infty$ and a contraction if $Lip(f) < 1$, where $Lip(f) = \sup_{x,y \in X; x \neq y} \frac{d(f(x),f(y))}{d(x,y)}$.

Definition 2.2. Let (X, d) be a metric space. The *generalized Hausdorff-Pompeiu semidistance* is an application $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, +\infty]$ defined by $h(A, B) = \max(d(A, B), d(B, A))$, where $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}(\inf_{y \in B} d(x, y))$.

It is well-known that if (X, d) is a complete metric space, then $(\mathcal{K}^*(X), h)$ is also a complete metric space (see for example [2], Ch. 2, Sec. 7, Theorem 1, [4, 5, 6] or [15] Ch. 1, Sec. 2, Theorem 1.3).

Concerning the Hausdorff-Pompeiu semidistance we have the following important properties:

Proposition 2.1 ([2], Ch. 3, Sec. 7, [5, 6] Sec. 3 proof of Lemma 1, [15], Ch. 1, Sec. 1, Theorem 1.1)**.** *Let (X, d) be a metric space. Then*:

1) *If H and K are two nonempty subsets of X, then $h(H, K) = h(\overline{H}, \overline{K})$.*

2) *If $(H_i)_{i \in I}$ and $(K_i)_{i \in I}$ are two families of nonempty subsets of X, then*

$$h(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i) = h(\overline{\bigcup_{i \in I} H_i}, \overline{\bigcup_{i \in I} K_i}) \leq \sup_{i \in I} h(H_i, K_i).$$

3) *If H and K are two nonempty subsets of X and $f : X \to X$ is a Lipschitz function, then $h(f(K), f(H)) \leq Lip(f) \cdot h(K, H)$.*

Definition 2.3. An iterated function system on a metric space (X, d) consists in a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and it is denoted by $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$.

For an IFS $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$, one can consider the function $F_{\mathcal{S}} : \mathcal{K}^*(X) \to \mathcal{K}^*(X)$ defined by $F_{\mathcal{S}}(B) = \bigcup_{k=1}^{n} f_k(B)$.

The function $F_{\mathcal{S}}$ is a contraction with $Lip(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} Lip(f_k)$ (see [2, 4, 5, 15]).

Using Banach contraction theorem one can prove that there exists, for an IFS $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$ defined on a complete metric space, a unique compact nonvoid set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. More precisely we have the following well-known result (see [2, 4, 5, 15]).

Theorem 2.1 ([2], Ch. 3, Sec. 7, Theorem 1, [4, 5, 15] Ch. 3, Sec. 1, Theorem 3.1)**.** *Let (X, d) be a complete metric space and $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$ be an IFS with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Then there exists a unique $A(\mathcal{S}) \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. Moreover, for any $H_0 \in \mathcal{K}^*(X)$ the sequence $(H_m)_{m\geq 0}$ defined recursively by $H_{m+1} = F_{\mathcal{S}}(H_m)$ is convergent to $A(\mathcal{S})$. Concerning the speed of the convergence we have the following estimation $h(H_m, A(\mathcal{S})) \leq \frac{c^m}{1-c} h(H_0, H_1)$, for every $m \in \mathbb{N}^*$.*

Definition 2.3. The set $A(\mathcal{S})$ from the above theorem is called the attractor of the IFS $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$.

More generally we introduce the notion of a topological iterated function system.

Definition 2.4. A topologically iterated function system (TIFS) on a topological Hausdorff space (X, τ) consists in a finite family of continuous functions $(f_k)_{k=\overline{1,n}}$, where $f_k : X \to X$, such that:

1) For every $K \in \mathcal{K}^*(X)$, there exists $H_K \in \mathcal{K}^*(X)$ such that:
   i) $K \subset H_K$,
   ii) $\bigcup_{k=1}^{n} f_k(H_K) \subset H_K$.
2) For every sequence $(\alpha_l)_{l\geq 1}$ with $\alpha_l \in \{1, 2, \ldots, n\}$ and every $K \in \mathcal{K}^*(X)$ such that $\bigcup_{k=1}^{n} f_k(K) \subset K$, the set $\bigcap_{l\geq 1} f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_l}(K)$ has at most one point.

A TIFS is denoted by $\mathcal{S} = ((X, \tau), (f_k)_{k=\overline{1,n}})$.

As in the case of an IFS, for a TIFS $\mathcal{S} = ((X, \tau), (f_k)_{k=\overline{1,n}})$, one can consider the function $F_{\mathcal{S}} : \mathcal{K}^*(X) \to \mathcal{K}^*(X)$ defined by $F_{\mathcal{S}}(B) = \bigcup_{k=1}^{n} f_k(B)$.

Remark 2.1. Let $\mathcal{S} = ((X, \tau), (f_k)_{k=\overline{1,n}})$ be a TIFS and $K \in \mathcal{K}^*(X)$ be such that $F_{\mathcal{S}}(K) \subset K$. We remark that the sequence of sets $(f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_l}(K))_l$ is decreasing since $f_k(K) \subset K$, for every $k \in \{1, 2, \ldots, n\}$ and the set $\bigcap_{l \geq 1} f_{\alpha_1} \circ f_{\alpha_2} \circ \ldots \circ f_{\alpha_l}(K)$ has exactly one point as we will see in the proof of Theorem 3.1.

We now present some notations used in the definition of the shift space of an (T)IFS: $\mathbb{R}$ denotes the real numbers, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}_n^* = \{1, 2, \ldots, n\}$. For two non-empty sets A and B, B^A denotes the set of functions from A to B.

By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as words $\omega = \omega_1\omega_2 \ldots \omega_n$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n \geq 1} \Lambda_n(B)$. By $|\omega|$ we will understand the length of the word ω. If $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ or if $\omega = \omega_1\omega_2 \ldots \omega_n$ and $n \geq m$ then $[\omega]_m$ denotes the word $\omega_1\omega_2 \ldots \omega_m$. For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B) \cup \Lambda(B)$ by $\alpha\beta$ we will understand the concatenation of the words α and β, namely $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m\beta_{m+1} \ldots$. For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ we say that $\alpha \prec \beta$ if $\alpha = \beta$ or there exists $\gamma \in \Lambda^*(B) \cup \Lambda(B)$ such that $\beta = \alpha\gamma$. For $\alpha \in \Lambda^*(B)$ $\dot{\alpha} = \alpha\alpha \ldots \alpha \ldots$.

On $\Lambda = \Lambda(\mathbb{N}_n^*) = (\mathbb{N}_n^*)^{\mathbb{N}^*}$ one can consider the metric $d_s(\alpha, \beta) = \sum_{k=1}^{\infty}(1 - \delta_{\alpha_k}^{\beta_k})/3^k$, where $\delta_x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$.

Definition 2.5. The pair $(\Lambda(\mathbb{N}_n^*), d_s)$ is a compact metric space and it is called the shift space with n letters.

Let $F_k : \Lambda(\mathbb{N}_n^*) \to \Lambda(\mathbb{N}_n^*)$ be defined by $F_k(\omega) = k\omega$ for $k = \overline{1,n}$. The functions F_k are continuous and they are called the right shift functions. Let us note that

$$d_s(F_k(\alpha), F_k(\beta)) = \frac{d_s(\alpha, \beta)}{3}.$$

Remark 2.2. Since $\Lambda = \bigcup_{k=1}^{n} F_k(\Lambda)$, Λ is the attractor of the IFS $\mathcal{S} = ((\Lambda, d_s), (F_k)_{k=\overline{1,n}})$.

Notation 2.1. Let (X, τ) be a topological space, $\mathcal{S} = ((X, \tau), (f_k)_{k=\overline{1,n}})$ be a TIFS on X and A be a set in X. For $\omega = \omega_1\omega_2 \dots \omega_m \in \Lambda_m(\mathbb{N}_n^*)$ one can consider $f_\omega \overset{def}{=} f_{\omega_1} \circ f_{\omega_2} \circ \dots \circ f_{\omega_m}$ and $A_\omega \overset{def}{=} f_\omega(A)$. In particular, if (X, τ) is (Λ, d_s), we have $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ \dots \circ F_{\omega_m}$ and $\Lambda_\omega = F_\omega(\Lambda)$.

Notation 2.2. Let X be a set, $n \in \mathbb{N}^*$ and $f : X \to X$ be a function. By $f^{[n]}$ we will understand the function $f \circ f \circ \dots \circ f$, where f is taken for n-times.

3. The main result

The main result of the paper is the following Theorem 3.1. It contains the generalization for TIFS of the results concerning the existence of the attractor of an IFS and of the relation between the shift space associated with an IFS and the attractor of the IFS.

Theorem 3.1. *Let (X, τ) be a topological space and $\mathcal{S}=((X, \tau), (f_k)_{k=\overline{1,n}})$ be a TIFS on X. Then:*

1) *There exists a unique nonvoid compact set $A = A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$.*

2) *For every $\omega \in \Lambda = \Lambda(\mathbb{N}_n^*)$ and every $K \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(K) \subset K$, there exists a unique $a_\omega(K)$ such that $\bigcap_{n\geq 1} f_{[\omega]_n}(K) = \{a_\omega(K)\}$. Moreover, if $H \in \mathcal{K}^*(X)$ has also the property that $F_{\mathcal{S}}(H) \subset H$, then $a_\omega(K) = a_\omega(H)$. So we can define $a_\omega = a_\omega(K)$, where K is an arbitrary set from $\mathcal{K}^*(X)$ having the property that $F_{\mathcal{S}}(K) \subset K$.*

3) *$A = A(\mathcal{S}) = \bigcup_{\omega\in\Lambda}\{a_\omega\}$ and $A_\alpha = A(\mathcal{S})_\alpha = \bigcup_{\omega\in\Lambda}\{a_{\alpha\omega}\}$, for every $\alpha \in \Lambda^*$.*

4) *For every $K_0 \in \mathcal{K}^*(X)$, let us consider the sequence $(K_m)_m$ defined recursively by $K_{m+1} = F_{\mathcal{S}}(K_m)$. Then for every open sets $(D_i)_{i=\overline{1,l}}$ such that $A(\mathcal{S}) \subset \bigcup_{i=1}^{l} D_i$ and $A(\mathcal{S}) \cap D_i \neq \emptyset$, for every $i \in \{1, 2, \dots, l\}$, there exists a m_0 such that, for every $m \geq m_0$, we have $K_m \subset \bigcup_{i=1}^{l} D_i$ and $K_m \cap D_i \neq \emptyset$, for every $i \in \{1, 2, \dots, l\}$.*

5) *$\pi \circ F_k = f_k \circ \pi$ for $k \in \{1, 2, \dots, n\}$, where π is the function $\pi : \Lambda \to A$ defined by $\pi(\omega) = a_\omega$.*

6) *The function* $\pi : \Lambda = \Lambda(\mathbb{N}_n^*) \to A$ *defined by* $\pi(\omega) = a_\omega$ *is a continuous and surjective function. We also have* $\pi(\Lambda) = A(\mathcal{S})$ *and* $\pi(\Lambda_\omega) = A(\mathcal{S})_\omega$, *for every* $\omega \in \Lambda^*$.

7) *The function* f_ω, *for* $\omega \in \Lambda^*$ *and* $m \in \mathbb{N}^*$, *has a unique fixed point denoted by* e_ω *and the set* $\{e_\omega : \omega \in \Lambda^*\}$ *is dense in* A. *In fact* $e_\omega = a_{\omega\omega\ldots\omega\ldots}$.

8) *For every* $x \in X$ *and every* $\omega \in \Lambda$, $\lim_{m\to\infty} f_{[\omega]_m}(x) = a_\omega$.

Definition 3.1. The set $A(\mathcal{S})$, defined in Theorem 3.1, is called the attractor of the TIFS $\mathcal{S} = ((X,\tau),(f_k)_{k=\overline{1,n}})$.

The metric space $(\Lambda = \Lambda(\mathbb{N}_n^*), d_s)$ is called the shift space of the TIFS $\mathcal{S}$=$((X,\tau),(f_k)_{k=\overline{1,n}})$. The function $\pi : \Lambda$=$\Lambda(\mathbb{N}_n^*) \to A$ is called the canonical projection between the shift space of the TIFS $\mathcal{S} = ((X,\tau),(f_k)_{k=\overline{1,n}})$ and the attractor of the TIFS $\mathcal{S}$.

Remark 3.1. The set $A(\mathcal{S})$ from the above Theorem is a topological self-similar set (see [8]).

Proof of Theorem 3.1. We start with the proof of point 2). Let $\omega \in \Lambda$ and $H, K \in \mathcal{K}^*(X)$ be such that $F_{\mathcal{S}}(H) \subset H$ and $F_{\mathcal{S}}(K) \subset K$. We remark that the sequence of sets $(f_{[\omega]_n}(K))_n$ is decreasing since $f_k(K) \subset K$, for every $k \in \{1,2,\ldots,n\}$. The fact that $\bigcap_{n\geq 1} f_{[\omega]_n}(K)$ is nonvoid results from the fact that $f_{[\omega]_n}(K)$ are compact sets since K is compact and $f_{[\omega]_n}$ are continuous. The fact that $\bigcap_{n\geq 1} f_{[\omega]_n}(K)$ has at most one point results from the definition of a TIFS. If $H \subset K$ then $f_{[\omega]_n}(H) \subset f_{[\omega]_n}(K)$ and $\{a_\omega(H)\} = \bigcap_{n\geq 1} f_{[\omega]_n}(H) \subset \bigcap_{n\geq 1} f_{[\omega]_n}(K) = \{a_\omega(K)\}$. Thus $a_\omega(H) = a_\omega(K)$. In general, for arbitrary $H, K \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(H) \subset H$ and $F_{\mathcal{S}}(K) \subset K$ we have $a_\omega(H) = a_\omega(H \cup K) = a_\omega(K)$. The proof of point 2) is finished.

We also remark that if $\omega \in \Lambda$ and $\alpha \in \Lambda^*$, then

$$f_\alpha(a_\omega) = a_{\alpha\omega}. \tag{1}$$

Indeed, for a set $K \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(K) \subset K$, we have $f_\alpha(a_\omega) \in f_\alpha(\bigcap_{n\geq 1} f_{[\omega]_n}(K)) \subset \bigcap_{n\geq 1} f_\alpha \circ f_{[\omega]_n}(K) \subset \bigcap_{n\geq 1} f_{[\alpha\omega]_n}(K) = \{a_{\alpha\omega}\}$. Let $\tilde{A} = \{a_\omega : \omega \in \Lambda\}$. We want to prove that $\tilde{A} = A(\mathcal{S})$. For this we remark first that $F_{\mathcal{S}}(\tilde{A}) = \tilde{A}$. Indeed, using (1), we have

$$F_{\mathcal{S}}(\tilde{A}) = \bigcup_{i=1}^{n} f_i(\tilde{A}) = \bigcup_{i=1}^{n} f_i(\{a_\omega : \omega \in \Lambda\}) = \bigcup_{i=1}^{n} \{f_i(a_\omega) : \omega \in \Lambda\}$$
$$= \{a_{i\omega} : \omega \in \Lambda, i = \overline{1,n}\} = \{a_\omega : \omega \in \Lambda\} = \tilde{A}.$$

We now prove that $\tilde{A}$ is a compact set. For a set $H \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(H) \subset H$, we have $H_{m+1} \subset H_m$, where $(H_m)_m$ is the sequence defined by $H_{m+1} = F_{\mathcal{S}}(H_m)$ and $H_0 = H$. Let us consider $A_H = \bigcap_{m\geq 1} H_m$. We remark that A_H is a nonvoid compact set, being a intersection of nonvoid compact sets. We also remark that $A_H \subset A_K$ if $H, K \in \mathcal{K}^*(X)$ are such that $F_{\mathcal{S}}(H) \subset H$, $F_{\mathcal{S}}(K) \subset K$ and $H \subset K$.

We want to prove that $\tilde{A} = A_H$, for every $H \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(H) \subset H$. This will show that $\tilde{A}$ is a nonvoid compact set.

We start with the inclusion $\tilde{A} \subset A_H$. Indeed, for $\omega \in \Lambda$ we have $a_\omega \in \bigcap_{n\geq 1} f_{[\omega]_n}(H) \subset \bigcap_{n\geq 1} F_{\mathcal{S}}^{[n]}(H) = A_H$.

Let us prove the inclusion $A_H \subset \tilde{A}$. Let x_0 be an arbitrary element of $A_H = \bigcap_{m\geq 1} H_m$. Then $x_0 \in H_m = F_{\mathcal{S}}^{[m]}(H) = \bigcup_{\omega\in\Lambda_m} f_\omega(H)$, for all $m \geq 1$. Let $\tilde{T}_m = \{\omega \in \Lambda_m : x_0 \in f_\omega(H)\}$. The set $\tilde{T}_m$ is nonvoid and so the set $T = \bigcup_{m\geq 1} \tilde{T}_m$ is infinite. Let $T_\omega = \{\alpha \in \Lambda : \alpha \in T \text{ and } \omega \prec \alpha\}$ for $\omega \in \Lambda^*$. We remark first that if T_ω is nonvoid then $\omega \in T_\omega$. Indeed, we have $f_\alpha(H) \subset f_\omega(H)$ if $\omega \prec \alpha$ since $f_\alpha(H) = f_{\omega\beta}(H) = f_\omega(f_\beta(H)) \subset f_\omega(F_{\mathcal{S}}^{[|\beta|]}(H)) \subset f_\omega(H)$, where β is defined by $\alpha = \omega\beta$. As $x_0 \in f_\alpha(H)$ we conclude $x_0 \in f_\omega(H)$.

Since $T = \bigcup_{\omega\in\Lambda_1} T_\omega$, there exists $\alpha_1 \in \Lambda_1$ such that T_{α_1} is infinite. Since $T_{\alpha_1}\backslash\{\alpha_1\} = \bigcup_{\omega\in\Lambda_1} T_{\alpha_1\omega}$, there exists $\alpha_2 \in \Lambda_1$ such that $T_{\alpha_1\alpha_2}$ is infinite. By induction one can find $\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots$ such that $T_{\alpha_1\alpha_2\ldots\alpha_m}$ is infinite, for every $m \in \mathbb{N}^*$. Let $\alpha = \alpha_1\alpha_2\ldots\alpha_m\ldots$. From the above considerations $\alpha_1\alpha_2\ldots\alpha_m \in T_{\alpha_1\alpha_2\ldots\alpha_m}$ and so $x_0 \in H_{\alpha_1\alpha_2\ldots\alpha_m}$, for all $m \geq 1$. Therefore $x_0 \in \bigcap_{m\geq 1} H_{\alpha_1\alpha_2\ldots\alpha_m} = \{a_\alpha\}$. Thus $x_0 = a_\alpha \in \tilde{A}$.

This finishes the proof of the fact that $\tilde{A} = A_H$, for every $H \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(H) \subset H$.

We remark that there exists a set $H \in \mathcal{K}^*(X)$ such that $F_{\mathcal{S}}(H) \subset H$ (this results from point 1) ii) of the Definition 2.5 and from the fact that a finite set is compact). Therefore $\tilde{A} = A_H$ is a nonvoid compact set.

To finish the proof of the first point it is enough to remark that if $H \in \mathcal{K}^*(X)$ is such that $F_{\mathcal{S}}(H) = H$, then $H = A_H = \tilde{A}$.

Therefore there exists a unique nonvoid compact set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$ and $A(\mathcal{S}) = \tilde{A}$. With these we have also proved the first part of the point 3). For the second we have $A_\alpha = f_\alpha(A(\mathcal{S})) = f_\alpha(\{a_\omega : \omega \in \Lambda\}) = \{f_\alpha(a_\omega) : \omega \in \Lambda\} = \bigcup_{\omega\in\Lambda}\{a_{\alpha\omega}\}$, for every $\alpha \in \Lambda^*$.

4) Let $K \in \mathcal{K}^*(X)$ and $H_K \in \mathcal{K}^*(X)$ be such that $K \subset H_K$ and $F_{\mathcal{S}}(H_K) \subset H_K$. Let $(K_m)_m$ and $(H_m)_m$ be defined recursively by $K_0 = K$, $H_0 = H_K$, $K_{m+1} = F_{\mathcal{S}}(K_m)$ and $H_{m+1} = F_{\mathcal{S}}(H_m)$. It is obvious that $K_m \subset H_m$.

Let D be an open set in X such that $A(\mathcal{S}) \subset D$. Then $A(\mathcal{S}) = A_{H_K} = \bigcap_{m\geq 1} H_m \subset D$. Since H_m are compact sets, $H_{m+1} \subset H_m$, for every m and D is open it follows that there exists m_0 such that $H_{m_0} \subset D$. Then, for $m \geq m_0$, we have $K_m \subset H_m \subset H_{m_0} \subset D$.

Let $(D_i)_{i=\overline{1,l}}$ be open sets such that $A(\mathcal{S}) \subset \bigcup_{i=1}^l D_i$ and $A(\mathcal{S}) \cap D_i \neq \emptyset$, for every $i \in \{1,..,l\}$. Taking $D = \bigcup_{i=1}^l D_i$, from the above considerations, there exists m_0 such that $H_m \subset D$, for all $m \geq m_0$.

On the other side, taking into account 3), there are $a_{\omega_i} \in D_i$, for every $i \in \{1,..,l\}$. Since $\{a_{\omega_i}\} = \bigcap_{n\geq 1} f_{[\omega_i]_n}(H_K) \subset D_i$, $f_{[\omega_i]_n}(H_K)$ are compact sets and D_i is open, it follows that, for every $i \in \{1,..,l\}$, there exists m_i such that $f_{[\omega_i]_{m_i}}(H_K) \subset D_i$. Thus $f_{[\omega_i]_m}(K) \subset f_{[\omega_i]_m}(H_K) \subset f_{[\omega_i]_{m_i}}(H_K) \subset D_i$, for every $m \geq m_i$ and every $i \in \{1,..,l\}$. Let $m_0 = \max_{i=1}^l m_i$. Then, for $m \geq m_0$, we have $\emptyset \neq f_{[\omega_i]_m}(K) \subset D_i \cap F_{\mathcal{S}}^{[m]}(K) = D_i \cap K_m$, for every $i \in \{1,..,l\}$.

Point 5), $\pi \circ F_k = f_k \circ \pi$ for $k \in \{1,..,n\}$, is a particular case of (1).

6) The surjectivity of the function $\pi : \Lambda \to A$ results from point 3). We also have $\pi(\Lambda_\omega) = A(\mathcal{S})_\omega$, for every $\omega \in \Lambda^*$, from point 5) and point 3).

We now prove the continuity of π. Let $\omega \in \Lambda$, $a_\omega = \pi(\omega)$ and D be an open set such that $a_\omega \in D$. With a similar argument as above there exists a m such that $A(\mathcal{S})_{[\omega]_m} \subset D$. Then $B(\omega, \frac{1}{3^m}) = \Lambda_{[\omega]_m} \subset \pi^{-1}(A(\mathcal{S})_{[\omega]_m}) \subset \pi^{-1}(D)$.

7) Let us consider $\omega \in \Lambda^*$ and $\dot{\omega} = \omega\omega\ldots\omega\ldots$. Let $e_\omega = a_{\dot{\omega}} = \pi(\dot{\omega})$. Then $f_\omega(e_\omega) = f_\omega(\pi(\dot{\omega})) = \pi(F_\omega(\dot{\omega})) = \pi(\dot{\omega}) = e_\omega$ and so e_ω is a fixed point for f_ω. Let e be another fixed point for f_ω. We want to prove that $e = e_\omega$. The set $\{e\}$ is compact and so there exists $H_e \in \mathcal{K}^*(X)$ such that $\{e\} \subset H_e$ and $F_{\mathcal{S}}(H_e) \subset H_e$. Then $e = f_\omega^{[m]}(e) \in f_\omega^{[m]}(H_e) = f_{[\dot{\omega}]_{m|\omega|}}(H_e)$ and so $e \in \bigcap_{m\geq 1} f_{[\dot{\omega}]_{m|\omega|}}(H_e) = \bigcap_{m\geq 1} f_{[\dot{\omega}]_m}(H_e) = \{e_\omega\}$. Therefore $e = e_\omega$.

Now we will prove that the set $\{e_\omega : \omega \in \Lambda^*\}$ is dense in A. To this end let D be an open set in X such that $A(\mathcal{S}) \cap D \neq \emptyset$. Taking into account 3), there exists $\omega \in \Lambda$ such that $a_\omega \in D$. Then with a similar argument as above there exists a m such that $A(\mathcal{S})_{[\omega]_m} \subset D$. Then $e_{[\omega]_m} \in A(\mathcal{S})_{[\omega]_m} \subset D$.

8) The set $\{x\}$ is compact. Therefore there exists $H_{\{x\}} \in \mathcal{K}^*(X)$ such that $x \in H_{\{x\}}$ and $F_{\mathcal{S}}(H_{\{x\}}) \subset H_{\{x\}}$. Let $\omega \in \Lambda$. We remark that $f_{[\omega]_m}(x) \in f_{[\omega]_m}(H_{\{x\}})$. Since for every open neighborhood D of a_ω, there exists a m_0 such that, for every $m \geq m_0$ $f_{[\omega]_m}(H_{\{x\}}) \subset D$, it follows that $f_{[\omega]_m}(x) \in D$, for every $m \geq m_0$. It results that $\lim_{m\to\infty} f_{[\omega]_m}(x) = a_\omega$. □

4. The relation between IFSs and TIFSs

In this section we discuss the relation between IFSs and TIFSs. We will prove first that every system of functions defined on a topological space, for which there exists an IFS and a homeomorphism between the topological space and the space where the IFS is defined such that the homeomorphism moves the functions of the system of functions into the functions of the IFS is a TIFS. After that we present an example of a compact metric space which is homeomorphic with an attractor of the IFS (in fact it is homeomorphic with a closed compact interval), and so is the attractor of a TIFS, but is not an attractor of an IFS (Example 4.1 and Remark 4.2). We also give an example of a compact metrizable space which is the attractor of a TIFS but is not the attractor of an IFS, for every distance on the compact space whose associated topology is the topology of the compact space (Example 4.4).

Proposition 4.1. *Let (X, d) be a complete metric space and $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$ be an IFS. Then $\mathcal{S} = ((X, \tau_d), (f_k)_{k=\overline{1,n}})$ is a TIFS.*

Proof. Let $A(\mathcal{S})$ be the attractor of the IFS $\mathcal{S}$, $K \in \mathcal{K}^*(X)$ and $(K_n)_{n\geq 0}$ be the sequence defined by $K_0 = K$ and $K_{n+1} = F_{\mathcal{S}}(K_n)$. Let $H_K = A(\mathcal{S}) \cup (\bigcup_{n\geq 0} K_n)$. We have $F_{\mathcal{S}}(H_K) = F_{\mathcal{S}}(A(\mathcal{S})) \cup (\bigcup_{n\geq 0} F_{\mathcal{S}}(K_n)) = A(\mathcal{S}) \cup (\bigcup_{n\geq 1} K_n) \subset H_K$. We also remark that $H_K \in \mathcal{K}^*(X)$. Indeed, the fact that K_H is nonvoid is obvious. Since the sequence $(K_n)_{n\geq 0}$ is convergent to $A(\mathcal{S})$ it follows that the sequence $(\bigcup_{i\geq 0}^n K_i)_{n\geq 0}$ is convergent to K_H. To see this we notice that

$$h(\bigcup_{i\geq 0}^{n} K_i, K_H) = h(\bigcup_{i=0}^{n} K_i, A(\mathcal{S}) \cup (\bigcup_{i\geq 0} K_i))$$

$$= h(K_n \cup (\bigcup_{i=0}^{n} K_i) \cup (\bigcup_{i\geq n+1} K_n), A(\mathcal{S}) \cup (\bigcup_{i=0}^{n} K_i) \cup (\bigcup_{i\geq n+1} K_i))$$

$$\leq \max\{h(K_n, A(\mathcal{S})), \max_{i=0}^{n} h(K_i, K_i), \sup_{i>n} h(K_n, K_i)\}$$
$$\leq \sup\{h(K_n, A(\mathcal{S})), h(K_1, K_1), \ldots, h(K_n, K_n), h(K_n, K_{n+1}), \ldots\}$$
$$= \sup\{h(K_n, A(\mathcal{S})), h(K_n, K_{n+1}), h(K_n, K_{n+2}), \ldots\}.$$

Since $(K^*(X), h)$ is complete it follows that $H_K \in \mathcal{K}^*(X)$. With this we have proved point 1) from the definition of a TIFS (Definition 2.5). The requirement of point 2) result from the fact that if $H \in \mathcal{K}^*(X)$ is such that $F_{\mathcal{S}}(H) \subset H$ then $H_{[\omega]_{m+1}} \subset H_{[\omega]_m}$ and $\delta(H_{[\omega]_m}) \leq (\max_{i=i}^{n} Lip(f_i))^m \delta(H)$, for every $\omega \in \Lambda$ and $m \in \mathbb{N}^*$, where $\delta(H)$ denotes the diameter of H, i.e. $\delta(H) = \sup_{x,y \in H} d(x, y)$. □

Remark 4.1. Let (X, τ), (Y, τ') be two topological Hausdorff spaces, $(g_k)_{k=\overline{1,n}}$ be a family of continuous functions, $g_k : Y \to Y$, and $\mathcal{S} = ((X, \tau), (f_k)_{k=\overline{1,n}})$ be a TIFS on X such that, there exists a homeomorphism $\phi : X \to Y$ with the property $g_k \circ \phi = \phi \circ f_k$ that for every $k \in \{1, .., n\}$. Then $\mathcal{S}' = ((Y, \tau'), (g_k)_{k=\overline{1,n}})$ is a TIFS.

Corollary 4.1. *Let (X, d) be a complete metric space and $\mathcal{S} = ((X, d), (f_k)_{k=\overline{1,n}})$ be an IFS. If (Y, τ) is a topological Hausdorff space and $(g_k)_{k=\overline{1,n}}$ is a family of continuous functions on Y such that, there exists a homeomorphism $\phi : X \to Y$ with the property $g_k \circ \phi = \phi \circ f_k$ that for every $k \in \{1, 2, \ldots, n\}$ then $\mathcal{S}' = ((Y, \tau), (g_k)_{k=\overline{1,n}})$ is a TIFS.*

Notation 4.1. 1) We denote by p the function $p : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = x$.

2) For a function $\phi : [a, b] \to \mathbb{R}$ by $\bigvee_a^b(f)$ we understand the variation of f on $[a, b]$ that is the number

$$\sup_{\Delta}\{\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| : \Delta = (a = x_0 < x_1 < \ldots < x_n = b)\}.$$

3) If $b < a$ by $[a, b]$ we mean $[b, a]$ and by $\bigvee_a^b(f)$ we mean $\bigvee_b^a(f)$.

Example 4.1. Let us consider a continuous function $\lambda : [0, 1] \to [0, 1]$ such that, $\lambda(0) = 0$, $\bigvee_{\varepsilon}^1(\lambda) < +\infty$, for every $\varepsilon > 0$ and $\bigvee_0^1(\lambda) = +\infty$. We denote by G_λ the graph of λ and consider the metric space (G_λ, d_2), where d_2 is the euclidean distance induced from $\mathbb{R}^2$. We claim that there does not exist any IFS $\mathcal{S}$ on (G_λ, d_2), such that $G_\lambda = A(\mathcal{S})$.

To see this let us suppose by reduction ad absurdum that there exists an IFS $\mathcal{S} = ((G_\lambda, d_2), (f_i)_{i\in I})$ such that $G_\lambda = A(\mathcal{S})$, where I is a finite set.

Let us note first that:

1) $f_i(G_\lambda) = G_{\lambda|[a_i,b_i]}$, where $a_i, b_i \in [0,1]$.

2) There exists an $i \in I$, such that $(0,0) \in f_i(G_\lambda)$ and $f_i(G_\lambda) \neq \{(0,0)\}$.

3) If there exists an $i \in I$ and $a \in (0,1]$, such that $f_i(a, \lambda(a)) = (0,0)$, then $f_i(G_\lambda) = \{(0,0)\}$.

4) There exists an $i_0 \in I$, such that $(0,0) = f_{i_0}((0,0))$ and $f_{i_0}(a,\lambda(a)) \neq (0,0)$, for every $a \in (0,1]$.

Indeed, we have

1) $f_k(G_\lambda) = G_{\lambda|A}$, where $A = p \circ f_k(G_\lambda) \subset [0,1]$. Since G_λ is (arcwise) connected and compact it follows that A is (arcwise) connected and compact. Therefore A is a closed interval.

2) We consider the sets $I' = \{i \in I : f_i(G_\lambda) = \{(0,0)\}\}$, $I'' = \{i \in I : (0,0) \notin f_i(G_\lambda)\}$ and $I''' = \{i \in I : (0,0) \in f_i(G_\lambda)$ and $\{(0,0)\} \neq f_i(G_\lambda)\}$. Then, $\bigcup_{i\in I'} f_i(G_\lambda) = \{(0,0)\}$, $p(\bigcup_{i\in I'} f_i(G_\lambda)) = \{0\}$ and $0 \notin p(\bigcup_{i\in I''} f_i(G_\lambda))$. Because $0 \notin p(\bigcup_{i\in I''} f_i(G_\lambda))$ and $p(\bigcup_{i\in I''} f_i(G_\lambda))$ is a compact set it follows that there exists $\mu > 0$, such that $p(\bigcup_{i\in I''} f_i(G_\lambda)) \subset [\mu, 1]$. Thus $p(\bigcup_{i\in I'} f_i(G_\lambda)) \cup p(\bigcup_{i\in I''} f_i(G_\lambda)) \subset \{0\} \cup [\mu,1]$. But

$$\begin{aligned}[0,1] = p(\bigcup_{i\in I} f_i(G_\lambda)) &= p(\bigcup_{i\in I'} f_i(G_\lambda)) \cup p(\bigcup_{i\in I''} f_i(G_\lambda)) \cup p(\bigcup_{i\in I'''} f_i(G_\lambda)) \\ &\subset \{0\} \cup [\mu,1] \cup p(\bigcup_{i\in I'''} f_i(G_\lambda))\end{aligned}$$

and so I''' is nonvoid.

3) Let us suppose by reduction ad absurdum that there exist $i \in I$ and $a, b \in (0,1]$, such that $f_i(a,\lambda(a)) = (0,0)$ and $f_i(b,\lambda(b)) \neq (0,0)$. Then $G_{\lambda|_{[0,\lambda(b)]}} \subset f_i(G_{\lambda|_{[a,b]}})$. It follows that $+\infty = \bigvee_0^{\lambda(b)}(\lambda) \leq \bigvee_a^b(f_i \circ \lambda) \leq Lip(f_i) \bigvee_a^b(\lambda) < +\infty$, which is a contradiction. If, there exists an $i \in I$ such that there exist $a \in (0,1]$ such that $f_i(a,\lambda(a)) = (0,0)$, using the above considerations, it follows that $f_i(G_\lambda - (0,0)) = \{(0,0)\}$ and so $f_i(G_\lambda) = \{(0,0)\}$.

4) It results from 2) and 3).

Therefore, using 4), one can consider $i_0 \in I$ be such that $(0,0) = f_{i_0}((0,0))$ and $f_{i_0}(a, \lambda(a)) \neq (0,0)$ for every $a \in (0,1]$.

Let us consider the sequence $(x_n)_n$ defined by $x_0 = 1$ and $(x_{n+1}, \lambda(x_{n+1})) = f_{i_0}(x_n, \lambda(x_n)) = f_{i_0}^{[n+1]}(x_0, \lambda(x_0))$.

It is easy to see by induction that $x_n \neq 0$.

We remark first that, because $(0,0)$ is the fixed point of the contraction f_{i_0}, $(x_n, \lambda(x_n)) \to (0,0)$ when $n \to 0$ and so $x_n \to 0$. Then $G_{\lambda|_{[x_{n+2},x_{n+1}]}} \subset f_{i_0}(G_{\lambda|_{[x_{n+1},x_n]}})$ and so

$$\bigvee_{x_{n+2}}^{x_{n+1}} (\lambda) \leq \bigvee_{x_{n+1}}^{x_n} (f_{i_0} \circ \lambda) \leq Lip(f_{i_0}) \bigvee_{x_{n+1}}^{x_n} (\lambda) \leq Lip^n(f_{i_0}) \bigvee_{x_1}^{1} (\lambda).$$

Thus

$$\bigvee_{x_n}^{1} (\lambda) \leq \bigvee_{x_n}^{x_{n-1}} (\lambda) + \bigvee_{x_{n-1}}^{x_{n-2}} (\lambda) + \ldots + \bigvee_{x_1}^{1} (\lambda) \leq \frac{1}{1 - Lip(f_{i_0})} \bigvee_{x_1}^{1} (\lambda).$$

Since λ is continuous in 0 it results that $+\infty = \bigvee_0^1(\lambda) \leq \limsup_n \bigvee_{x_n}^1(\lambda) \leq \frac{1}{1-Lip(f_{i_0})} \bigvee_{x_1}^1(\lambda) < +\infty$, which is a contradiction.

Remark 4.2. In the above framework (Example 4.1) G_λ is homeomorphic with the interval $[0,1]$. Since $[0,1]$ is the attractor of an IFS (see Example 4.2 below), from Proposition 4.1 it follows that G_λ is the attractor of a TIFS.

Example 4.2. The interval $[0,1]$ is the attractor of the IFS $\mathcal{S} = ((\mathbb{R}, d), (f, g))$, where the functions $f, g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = \frac{x}{2}$ and by $f(x) = \frac{x}{2} + \frac{1}{2}$ and d is the usual distance on $\mathbb{R}$.

Example 4.3. An example of a function with the properties from Example 4.1.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\phi(x) = d(x, \mathbb{Z})$. Let us also consider the functions $\phi_a : \mathbb{R} \to \mathbb{R}$ and $\psi_a : (0,1] \to \mathbb{R}$ defined by $\phi_a(x) = a\phi(x)$ and by $\psi_a(x) = \phi_a(1/x)$, where $a > 0$. Then $\bigvee_n^{n+p}(\phi) = p$, $\bigvee_n^{(n+p)}(\phi_a) = ap$ and $\bigvee_{1/(n+p)}^{1/n}(\psi_a) = ap$, where $a > 0$ and n, p are natural numbers.

Let us consider the function $\lambda : [0,1] \to [0,1]$ defined by

$$\lambda(x) = \begin{cases} 0, & \text{if } x = 0 \\ \psi_{1/(n+1)}(x), & \text{if } x \in (1/2^{n+1}, 1/2^n]. \end{cases}$$

It follows that λ is a continuous function, $\bigvee_{1/2^{n+1}}^{1/2^n}(\lambda) = \frac{2^n}{n+1}$, $\bigvee_{1/2^n}^{1}(\lambda) = \sum_{k=0}^{n-1} \frac{2^k}{k+1}$ and $\bigvee_0^1(\lambda) = +\infty$.

For the next example we need some preparations. The next result is well-known.

Lemma 4.1. *Let (X, d) and (Y, d_1) be two metric spaces, $f : X \to Y$ be a continuous function and $(K_n)_{n\geq 1}$ a sequence of compact subsets of X such that $K_{n+1} \subset K_n$. Then $f(\bigcap_{n\geq 1} K_n) = \bigcap_{n\geq 1} f(K_n)$.*

Notation 4.2 (for Example 4.4)**.** 1) We denote by l_2 the space of all sequences $x = (x_n)_{n\geq 1}$ such that $\sum_{n\geq 1} x_n^2 < +\infty$. l_2 is a Hilbert space with the norm $\|x\| = (\sum_{n\geq 1} x_n^2)^{1/2}$. The associated distance with the norm $\|.\|$ is denoted by d.

2) We consider the functions $i : \mathbb{R} \to l_2$, $\pi : l_2 \to \mathbb{R}$, $\overline{f}_1 : \mathbb{R} \to \mathbb{R}$ and $\overline{f}_2 : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = (x, 0, 0, \ldots, 0, \ldots)$, $\pi((x_n)_{n\geq 1}) = x_1$,

$$\overline{f}_1(x) = \begin{cases} 2, & \text{if } x \leq 2 \\ \frac{x+2}{2}, & \text{if } 2 \leq x \leq 4 \\ 3, & \text{if } x \geq 4 \end{cases} \text{ and } \overline{f}_2(x) = \begin{cases} 3, & \text{if } x \leq 2 \\ \frac{x+4}{2}, & \text{if } 2 \leq x \leq 4 \\ 4, & \text{if } x \geq 4 \end{cases}.$$

3) We denote by C the set $\times_{n\geq 1}[0, \frac{1}{2^{n-1}}] \subset l_2$ and by A the set $C \cup i([1, 4])$.

4) Let $f_1, f_2 : A \to A$ be the functions defined by $f_1 = i \circ \overline{f}_1 \circ \pi|_A$ and $f_2 = i \circ \overline{f}_2 \circ \pi|_A$.

We remark that C is a Peano space (the image of continuous function defined on a closed interval) (paragraph 6.314 page 461,462 [3]) since it is a convex compact space and so it is a compact arcwise connected and a local arcwise connected space (with the distance from l_2). Also A is a Peano space since it is the union of two Peano spaces, namely C and $i([1, 4])$, with a common point, namely $i(1)$. Therefore, there exists a continuous and surjective function $\varphi : [0, 1] \to C$, such that $\varphi(0) = i(1)$.

Let us consider the function $f_3 : A \to A$ defined by:

$$f_3(x) = \begin{cases} i(2), & \text{if } x \in C \cup i([1, 2]) \\ i(-\pi(x) + 4), & \text{if } x \in i([2, 3]) \\ \varphi(\pi(x) - 3), & \text{if } x \in i([3, 4]) \end{cases}.$$

It is obvious that the set A has infinite topological dimension and so it has infinite Hausdorff dimension. But, for every IFS $\mathcal{S}$, the Hausdorff dimension of the attractor of $\mathcal{S}$ is finite. Indeed the proof of the above claim is similar with the one from Proposition 9.6, page 135, from [4]. It follows that the set A is not an attractor of an IFS with any metric which induces the same topology on A as the distance d from l_2.

Example 4.4. Let us consider the metric space (A, d) and the TIFS $\mathcal{S} = ((A, d), (f_1, f_2, f_3))$. The attractor of the TIFS $\mathcal{S}$ is A. We claim that the attractor of an IFS has a finite topological dimension, but the set A is infinite dimensional. It follows that the space (A, τ_d) is the attractor of an TIFS but is not the attractor of an IFS for any distance defined on A which induces the same topology.

We prove now that $\mathcal{S} = (A, (f_1, f_2, f_3))$ is a TIFS and that A is the attractor of the TIFS $\mathcal{S}$. Let us remark first that, since A is a compact set then the first conditions from the definition of a TIFS is fulfilled.

Let us notice that $f_1(A) = i([2, 3])$, $f_2(A) = i([3, 4])$, $f_3(C \cup i([1, 2])) = \{i(2)\}$ and $f_3(i([2, 4])) = f_3(A) = C \cup i([1, 2])$. Therefore $A \subset f_1(A) \cup f_2(A) \cup f_3(A) \subset A$ and so $A = f_1(A) \cup f_2(A) \cup f_3(A)$.

We also remark that $f_{33}(A) = f_3 \circ f_3(A) = f_3(C \cup i([1, 2])) = \{i(2)\}$ and $f_{\alpha 3\beta}(A) = f_\alpha \circ f_3 \circ f_\beta(A) \subset f_\alpha \circ f_3(i([2, 4])) = f_\alpha(C \cup i([1, 2]))$, for every $\alpha, \beta \in \{1, 2\}$. Thus $f_{13\beta}(A) = \{i(2)\}$ and $f_{23\beta}(A) = \{i(3)\}$.

We want to prove that for every $\omega \in \Lambda = \Lambda(\{1, 2, 3\})$ and every $K \in \mathcal{K}^*(A)$, such that $F_{\mathcal{S}}(K) = \bigcup_{k=1}^{3} f_k(K) \subset K$, the set $\bigcap_{n\geq 1} f_{[\omega]_n}(K)$ has at most one point. Since A is a compact space it is enough to prove that the set $\bigcap_{n\geq 1} f_{[\omega]_n}(A)$ has at most one point, for every $\omega \in \Lambda$ (this is point 2) from the definition of a TIFS). Let us consider an $\omega = i_1 i_2 \ldots i_n \ldots \in \Lambda = \Lambda(\{1, 2, 3\})$. We have two cases:

1) $3 \in \{i_2, i_3, \ldots, i_n, \ldots\}$;

2) $3 \notin \{i_2, \ldots, i_n, \ldots\}$.

In the first case let n be such that $i_n = 3$. There are two subcases:

i) $3 \in \{i_{n-1}, i_{n+1}\}$;

ii) $3 \notin \{i_{n-1}, i_{n+1}\}$.

In the first subcase we have $f_{i_{n-1}i_n}(A) = \{i(2)\}$ or $f_{i_n i_{n+1}}(A) = \{i(2)\}$ and so $f_{i_{n-1}i_n i_{n+1}}(A)$ has one point. The same conclusions holds in the second subcase since in this subcase $f_{i_{n-1}i_n i_{n+1}}(A)$ is $\{i(2)\}$ or $\{i(3)\}$. Therefore $f_{i_1 i_2 \dots i_{n-1} i_n i_{n+1}}(A)$ has at most one point and so has $\bigcap_{n\geq 1} f_{[\omega]_n}(A)$.

In the second case we consider first the case when $3 \neq i_1$. Then

$$\begin{aligned} f_{i_1 i_2 \dots i_{n-1} i_n i_{n+1}}(A) &= f_{i_1 i_2 \dots i_{n-1} i_n} \circ f_{i_{n+1}}(A) \\ &\subset f_{i_1 i_2 \dots i_{n-1} i_n}(i([2,4])) = i(\overline{f}_{i_1 i_2 \dots i_{n-1} i_n}([2,4])) \end{aligned}$$

and so

$$\begin{aligned} \bigcap_{n\geq 1} f_{[\omega]_n}(A) = \bigcap_{n\geq 2} f_{[\omega]_n}(A) &\subset \bigcap_{n\geq 2} i(\overline{f}_{i_1 i_2 \dots i_{n-1} i_n}([2,4])) \\ &= i(\bigcap_{n\geq 2} \overline{f}_{i_1 i_2 \dots i_{n-1} i_n}([2,4])). \end{aligned}$$

But $\bigcap_{n\geq 2} \overline{f}_{i_1 i_2 \dots i_{n-1} i_n}([2,4])$ contains one point since $\mathcal{S} = ([2,4],(\overline{f}_1,\overline{f}_2))$ is a classical IFS.

In the case that $3 = i_1$, we have $\bigcap_{n\geq 1} f_{[\omega]_n}(A) = \bigcap_{n\geq 2} f_{i_1 i_2 \dots i_{n-1} i_n}(A) = f_3(\bigcap_{n\geq 2} f_{i_2 i_3 \dots i_{n-1} i_n}(A))$ and so $\bigcap_{n\geq 1} f_{[\omega]_n}(A)$ has one point since $\bigcap_{n\geq 2} f_{i_2 i_3 \dots i_{n-1} i_n}(A)$ has one point, taking into account the previous case.

Since A is a compact space and $A = f_1(A) \cup f_2(A) \cup f_3(A)$ it follows that A is the attractor of the TIFS $\mathcal{S} = (A,(f_1,f_2,f_3))$.

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