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$\star$-HYPERCONNECTED IDEAL TOPOLOGICAL SPACES

BY

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Abstract. The aim of this paper is to introduce and study $\star$-hyperconnected ideal topological spaces. Characterizations and properties of $\star$-hyperconnected ideal topological spaces are investigated.

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1. Introduction

Several notions which are equivalent to hyperconnectedness were investigated in the literature such as the notions of D-spaces, semi-connected spaces, s-connected spaces, irreducible spaces. On the other hand, the notion of ideal topological spaces was studied by Kuratowski [7] and Vaidyanathaswamy [9]. In 1990, Janković and Hamlett [6] investigated further properties of ideal topological spaces. In this paper, the notion of $\star$-hyperconnected ideal topological spaces are introduced and studied. Characterizations and properties of $\star$-hyperconnected ideal topological spaces are investigated.

2. Preliminaries

Throughout the present paper, (X,τ) or (Y,σ) will denote a topological space with no separation properties assumed. For a subset A of a topological space (X,τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X,τ), respectively. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (1) $A \in I$ and $B \subset A$ implies

$B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and, if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function ([7]) of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : G \cap A \notin I$ for every $G \in \tau(x)\}$ where $\tau(x) = \{G \in \tau : x \in G\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*(I, \tau)$ ([6]). When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. For any ideal space (X, τ, I), the collection $\{U \backslash J : U \in \tau$ and $J \in I\}$ is a basis for τ^*. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or simply an ideal space.

Definition 1. A subset A of an ideal space (X, τ, I) is said to be:

(1) pre-I-open ([1]) if $A \subset Int(Cl^*(A))$;

(2) semi-I-open ([3]) if $A \subset Cl^*(Int(A))$;

(3) strongly β-I-open ([4]) if $A \subset Cl^*(Int(Cl^*(A)))$;

(4) $\star$-dense ([2]) if $Cl^*(A) = X$;

(5) $\star$-nowhere dense if $Int(Cl^*(A)) = \emptyset$.

The complement of a pre-I-open (resp. semi-I-open, strongly β-I-open) set is called pre-I-closed (resp. semi-I-closed, strongly β-I-closed). A topological space X is said to be hyperconnected ([8]) if every pair of nonempty open sets of X has nonempty intersection. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be semi-I-continuous ([3]) if, for every open set A of Y, $f^{-1}(A)$ is semi-I-open in X.

3. Characterizations of $\star$-hyperconnected spaces

Definition 2. An ideal space (X, τ, I) is said to be:

(1) $\star$-hyperconnected if A is $\star$-dense for every open subset $A \neq \emptyset$ of X;

(2) $\star$-connected if X can not be written as the union of nonempty and disjoint an open set and a $\star$-open set of X.

Remark 3. (1) Generally, it is known that every hyperconnected topological space is connected, but not conversely.

(2) For an ideal space (X, τ, I), $\tau \subset \tau^*$ and we have the following properties:

$$\begin{array}{ccc} (X,\tau,I) \text{ is } \star\text{-hyperconnected} & \Rightarrow & (X,\tau) \text{ is hyperconnected} \\ \Downarrow & & \Downarrow \\ (X,\tau,I) \text{ is } \star\text{-connected} & \Rightarrow & (X,\tau) \text{ is connected} \end{array}$$

The implications in the diagram are not reversible as shown in the following examples:

Example 4. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{a\}\}$. Then the space (X, τ) is hyperconnected but (X, τ, I) is not $\star$-connected.

Example 5. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{b\}\}$. Then, the ideal space (X, τ, I) is $\star$-connected but it is not hyperconnected.

Definition 6. A subset A of an ideal space (X, τ, I) is called

(1) semi*-I-open if $A \subset Cl(Int^*(A))$;

(2) semi*-I-closed if its complement is semi*-I-open.

Lemma 7. *Every semi-I-open set is semi*-I-open.*

Proof. Let A be a semi-I-open subset in an ideal space (X, τ, I). Then $A \subset Cl^*(Int(A))$. We have $A \subset Cl^*(Int(A)) \subset Cl(Int^*(A))$. Thus, A is semi*-I-open. □

The implication in Lemma 7 is not reversible as shown in the following example:

Example 8. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then, the set $A = \{b, c, d\}$ is semi*-I-open but it is not semi-I-open.

Lemma 9 ([5])**.** *A subset A of an ideal space (X, τ, I) is semi-I-open, if and only if there exists $B \in \tau$, such that $B \subset A \subset Cl^*(B)$.*

Lemma 10. *A subset A of an ideal space (X, τ, I) is semi*-I-open, if and only if there exists $B \in \tau^*$, such that $B \subset A \subset Cl(B)$.*

Proof. Let A be semi*-I-open. Then $A \subset Cl(Int^*(A))$. Take $B = Int^*(A)$. We have $B \subset A \subset Cl(B)$.

Conversely, let $B \subset A \subset Cl(B)$ for a $B \in \tau^*$. Since $B \subset A$, then $B \subset Int^*(A)$. Thus, $Cl(B) \subset Cl(Int^*(A))$ and $A \subset Cl(Int^*(A))$. Hence, A is semi*-I-open. □

Theorem 11. *Let (X, τ, I) be an ideal space. The following properties are equivalent:*

(1) *X is $\star$-hyperconnected;*

(2) *A is $\star$-dense or $\star$-nowhere dense, for every subset $A \subset X$;*

(3) *$A \cap B \neq \emptyset$, for every nonempty open subset A and every nonempty $\star$-open subset B of X;*

(4) *$A \cap B \neq \emptyset$, for every nonempty semi-I-open subset $A \subset X$ and every nonempty semi*-I-open subset $B \subset X$.*

Proof. (1) $\Rightarrow$ (2) : Let X be $\star$-hyperconnected and $A \subset X$. Suppose that A is not $\star$-nowhere dense. Then $Cl(X \backslash Cl^*(A)) = X \backslash Int(Cl^*(A)) \neq X$. By (1), for $Int(Cl^*(A)) \neq \emptyset$, $Cl^*(Int(Cl^*(A))) = X$.

Since $Cl^*(Int(Cl^*(A))) = X \subset Cl^*(A)$, then $Cl^*(A) = X$. Thus, A is $\star$-dense.

(2) $\Rightarrow$ (3) : Suppose that $A \cap B = \emptyset$, for some nonempty sets $A \in \tau$ and $B \in \tau^*$. Then $Cl^*(A) \cap B = \emptyset$ and A is not $\star$-dense. Moreover, since $A \in \tau$, $\emptyset \neq A \subset Int(Cl^*(A))$ and A is not $\star$-nowhere dense.

(3) $\Rightarrow$ (4) : Suppose that $A \cap B = \emptyset$, for some nonempty semi-I-open set A and some nonempty semi*-I-open set B. By Lemmas 9 and 10, there exist $M \in \tau$ and $N \in \tau^*$, such that $M \subset A \subset Cl^*(M)$ and $N \subset B \subset Cl(N)$. Since A and B are nonempty, M and N are nonempty. Moreover, we have $M \cap N \subset A \cap B = \emptyset$.

(4) $\Rightarrow$ (1) : Suppose that $A \cap B \neq \emptyset$, for every nonempty semi-I-open subset $A \subset X$ and every nonempty semi*-I-open subset $B \subset X$. Since every open set is semi-I-open and every $\star$-open set is semi*-I-open, then X is $\star$-hyperconnected. □

Definition 12. The semi*-I-closure (resp. semi-I-closure, pre-I-closure, strongly β-I-closure) of a subset A of an ideal space (X, τ, I), denoted by S^*-I-$Cl(A)$ (resp. S-I-$Cl(A)$, P-I-$Cl(A)$, $S\beta$-I-$Cl(A)$), is defined by the intersection of all semi*-I-closed (resp. semi-I-closed, pre-I-closed, strongly β-I-closed) sets of X containing A.

Lemma 13. *The following properties hold for a subset A of an ideal space (X, τ, I):*

(1) $S^*\text{-}I\text{-}Cl(A) = A \cup Int(Cl^*(A))$;

(2) $S\text{-}I\text{-}Cl(A) = A \cup Int^*(Cl(A))$;

(3) $P\text{-}I\text{-}Cl(A) = A \cup Cl(Int^*(A))$;

(4) $S\beta\text{-}I\text{-}Cl(A) = A \cup Int^*(Cl(Int^*(A)))$.

Proof. (4) Since $S\beta\text{-}I\text{-}Cl(A)$ is strongly β-I-closed, then

$$Int^*(Cl(Int^*(A))) \subset Int^*(Cl(Int^*(S\beta\text{-}I\text{-}Cl(A)))) \subset S\beta\text{-}I\text{-}Cl(A).$$

Hence, $A \cup Int^*(Cl(Int^*(A))) \subset S\beta\text{-}I\text{-}Cl(A)$. Conversely, we have:

$$\begin{aligned}
&Int^*(Cl(Int^*(A \cup Int^*(Cl(Int^*(A))))))\\
&\subset Int^*(Cl(Int^*(A \cup Cl(Int^*(A)))))\\
&\subset Int^*(Cl(Int^*(A) \cup Cl(Int^*(A)))))\\
&= Int^*(Cl(Int^*(A))) \subset A \cup Int^*(Cl(Int^*(A)))).
\end{aligned}$$

Then $A \cup Int^*(Cl(Int^*(A)))$ is strongly β-I-closed containing A. Thus, $S\beta\text{-}I\text{-}Cl(A) \subset A \cup Int^*(Cl(Int^*(A)))$. Hence,

$$S\beta\text{-}I\text{-}Cl(A) = A \cup Int^*(Cl(Int^*(A))).$$

The proofs of (1), (2) and (3) are similar to that of (4). □

Theorem 14. *The following are equivalent for an ideal space (X, τ, I):*

(1) *X is ⋆-hyperconnected;*

(2) *H is ⋆-dense, for every strongly β-I-open subset $\emptyset \neq H \subset X$;*

(3) *$S^*\text{-}I\text{-}Cl(H) = X$, for every strongly β-I-open subset $\emptyset \neq H \subset X$;*

(4) *$S\beta\text{-}I\text{-}Cl(G) = X$, for every semi*-$I$-open subset $\emptyset \neq G \subset X$;*

(5) *$P\text{-}I\text{-}Cl(G) = X$, for every semi*-$I$-open subset $\emptyset \neq G \subset X$.*

Proof. (1) $\Rightarrow$ (2) : Let (X, τ, I) be a $\star$-hyperconnected ideal space. Let H be any nonempty strongly β-I-open subset of X. We have $Int(Cl^*(H)) \neq \emptyset$. Thus, $X = Cl^*(Int(Cl^*(H))) = Cl^*(H)$.

(2) $\Rightarrow$ (3) : Let H be any nonempty strongly β-I-open subset of X. Thus, by Lemma 13, S^*-I-$Cl(H) = H \cup Int(Cl^*(H)) = H \cup Int(X) = X$.

(3) $\Rightarrow$ (4) : Let G be a nonempty semi*-I-open subset of X. Thus, by Lemma 13,

$$\begin{aligned} S\beta\text{-}I\text{-}Cl(G) &= G \cup Int^*(Cl(Int^*(G))) = G \cup Int^*(Cl(G)) \\ &\supset G \cup Int(Cl^*(G)) \supset Int(G) \cup Int(Cl^*(Int(G))) \\ &= S^*\text{-}I\text{-}Cl(Int(G)) = X. \end{aligned}$$

Thus, $S\beta$-I-$Cl(G) = X$.

(4) $\Rightarrow$ (5) : Let G be a nonempty semi*-I-open subset of X. By Lemma 13, we have P-I-$Cl(G) = G \cup Cl(Int^*(G)) \supset G \cup Int^*(Cl(Int^*(G))) = S\beta$-$I$-$Cl(G) = X$. Thus, P-I-$Cl(G) = X$.

(5) $\Rightarrow$ (1) : Let G be a nonempty $\star$-open set of X. By (5), P-I-$Cl(G) = G \cup Cl(Int^*(G)) = X$. This implies that $Cl(G) = X$. By Theorem 11, X is $\star$-hyperconnected. $\square$

Corollary 15. *For an ideal space (X, τ, I), the following properties are equivalent:*

(1) *X is $\star$-hyperconnected;*

(2) *$G \cap H \neq \emptyset$, for every nonempty semi*-I-open subset $G \subset X$ and every nonempty strongly β-I-open subset $H \subset X$;*

(3) *$G \cap H \neq \emptyset$, for any nonempty semi*-I-open set G and any nonempty pre-I-open set H.*

Proof. The proof is obvious from Theorem 14. $\square$

4. $\star$-hyperconnected spaces and functions

Definition 16. The semi-I-interior of a subset A of an ideal space (X, τ, I), denoted by S-I-$Int(A)$, is defined by the union of all semi-I-open sets of X contained in A.

Definition 17. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be almost F-I-continuous if for every nonempty regular open set A of Y, $f^{-1}(A) \neq \emptyset$ implies S-I-$Int(f^{-1}(A)) \neq \emptyset$.

Theorem 18. *Every semi-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$ is almost F-I-continuous.*

Proof. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a semi-I-continuous function. Let A be any regular open subset of Y such that $f^{-1}(A) \neq \emptyset$. Then $f^{-1}(A)$ is a nonempty semi-I-open set in X and hence, $f^{-1}(A) = S$-I-$Int(f^{-1}(A))$. Thus, f is almost F-I-continuous. □

The implication in Theorem 18 is not reversible as shown in the following example:

Example 19. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Define the function $f : (X, \tau, I) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = c$, $f(c) = d$, $f(d) = c$. Then f is almost F-I-continuous but it is not semi-I-continuous.

Theorem 20. *The following properties hold for a $\star$-hyperconnected ideal space (X, τ, I):*

(1) *Every almost F-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space is constant;*

(2) *Every semi-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space is constant;*

(3) *Every semi-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a two point discrete space is constant.*

Proof. (1) : Let X be a $\star$-hyperconnected ideal space. Suppose that there exist a Hausdorff space Y and an almost F-I-continuous function $f : X \to Y$, such that f is not constant. There exist two points x and y of X, such that $f(x) \neq f(y)$. Since Y is Hausdorff, then there exist open sets A and B in Y, such that $f(x) \in A$, $f(y) \in B$ and $A \cap B = \emptyset$. Take $M = Int(Cl(A))$ and $N = Int(Cl(B))$. This implies that M and N are nonempty regular open and $M \cap N = \emptyset$. Since f is almost F-I-continuous, then S-I-$Int(f^{-1}(M)) \neq \emptyset$ and S-I-$Int(f^{-1}(N)) \neq \emptyset$. We have S-I-$Int(f^{-1}(M)) \cap S$-I-$Int(f^{-1}(N)) \subset f^{-1}(M \cap N) = \emptyset$. Since S-I-$Int(f^{-1}(M))$ and S-I-$Int(f^{-1}(N))$ are semi-I-open, then by Lemma 7 and Theorem 11, X is not $\star$-hyperconnected. This is a contradiction.

(2) : Let $f : (X, \tau, I) \to (Y, \sigma)$ be a semi-I-continuous function of (X, τ, I) into a Hausdorff space (Y, σ). Since every semi-I-continuous function is almost F-I-continuous, then by (1), f is constant.

(3) : It follows from (2). □

The implication in Theorem 20 is not reversible as shown in the following example:

Example 21. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{a\}\}$. Then every semi-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$, where (Y, σ) is a two point discrete space is constant but (X, τ, I) is not $\star$-hyperconnected.

Theorem 22. *If X is a $\star$-hyperconnected ideal space and $f : (X, \tau, I) \to (Y, \sigma)$ is an almost F-I-continuous surjection, then Y is hyperconnected.*

Proof. Suppose that Y is not hyperconnected. Then there exist disjoint nonempty open sets $A \subset Y$ and $B \subset Y$. Take $M = Int(Cl(A))$ and $N = Int(Cl(B))$. Then M and N are nonempty regular open sets and $M \cap N = \emptyset$. We have $S\text{-}I\text{-}Int(f^{-1}(M)) \cap S\text{-}I\text{-}Int(f^{-1}(N)) \subset f^{-1}(M) \cap f^{-1}(N) = \emptyset$. Since f is an almost F-I-continuous surjection, then $S\text{-}I\text{-}Int(f^{-1}(M)) \neq \emptyset$ and $S\text{-}I\text{-}Int(f^{-1}(N)) \neq \emptyset$. By Lemma 7 and Theorem 11, X is not $\star$-hyperconnected. This is a contradiction. □

Corollary 23. *If X is a $\star$-hyperconnected ideal space and $f : (X, \tau, I) \to (Y, \sigma)$ is a continuous surjection, then Y is hyperconnected.*

Proof. Since every continuous function is almost F-I-continuous, it follows from Theorem 22. □

Definition 24. A function $f : (X, \tau) \to (Y, \sigma, I)$ is said to be almost F-I-open if $S\text{-}I\text{-}Int(f(A)) \neq \emptyset$, for every nonempty regular open set $A \subset X$.

Theorem 25. *If Y is a $\star$-hyperconnected ideal space and $f : (X, \tau) \to (Y, \sigma, I)$ is an almost F-I-open injection, then X is hyperconnected.*

Proof. Let A and B be any nonempty open sets of X. Take $M = Int(Cl(A))$ and $N = Int(Cl(B))$. This implies that M and N are nonempty regular open sets. Since f is almost F-I-open, then $S\text{-}I\text{-}Int(f(M)) \neq \emptyset$ and $S\text{-}I\text{-}Int(f(N)) \neq \emptyset$. Since Y is $\star$-hyperconnected, then $\emptyset \neq S\text{-}I\text{-}Int(f(M)) \cap S\text{-}I\text{-}Int(f(N)) \subset f(M) \cap f(N)$. Since f is an injective function, then $M \cap N \neq \emptyset$. Thus, $A \cap B \neq \emptyset$ and hence X is hyperconnected. □

Corollary 26. *If Y is a $\star$-hyperconnected ideal space and $f : (X, \tau) \to (Y, \sigma, I)$ is an open injection, then X is hyperconnected.*

Proof. Since every open function is almost F-I-open, it follows from Theorem 25. □

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