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ON $\mathcal{A}_I^*$-SETS, $\mathcal{C}_I$-SETS, $\mathcal{C}_I^*$-SETS AND DECOMPOSITIONS OF CONTINUITY IN IDEAL TOPOLOGICAL SPACES

BY

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Abstract. The aim of this paper is to introduce and study the notions of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces. Properties of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets are investigated. Moreover, decompositions of continuous functions and decompositions of $\mathcal{A}_I^*$-continuous functions via $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are established.

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Key words: $\mathcal{A}_I^*$-set, $\mathcal{C}_I$-set, $\mathcal{C}_I^*$-set, pre-I-regular set, ideal topological space, decomposition, $\star$-extremally disconnected ideal space, $\star$-hyperconnected ideal space, I-submaximal ideal space.

1. Introduction and preliminaries

In this paper, $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are introduced and studied. The relationships and properties of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are investigated. Futhermore, decompositions of continuous functions and decompositions of $\mathcal{A}_I^*$-continuous functions via $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are provided.

In the present paper, (X, τ) or (Y, σ) will denote topological spaces with no separation properties assumed. For a subset V of X, let $Cl(V)$ and $Int(V)$ denote the closure and the interior of V, respectively, with respect to the topological space (X, τ).

An ideal I on a set X is a nonempty collection of subsets of X which satisfies

(1) $V \in I$ and $G \subset V$ implies $G \in I$.

(2) $V \in I$ and $G \in I$ implies $V \cup G \in I$ [13].

If I is an ideal on X and $X \notin I$, then $\mathcal{F} = \{X \backslash G : G \in I\}$ is a filter [11].

For an ideal I on (X, τ), (X, τ, I) is said to be an ideal topological space or briefly an ideal space. Let $P(X)$ be the set of all subsets of X. For an ideal topological space (X, τ, I), a set operator $(.)^* : P(X) \to P(X)$, which will be said the local function [13] of $G \subset X$ with respect to τ and I, is defined as follows: $G^*(I, \tau) = \{x \in X : H \cap G \notin I \text{ for every } H \in \tau(x)\}$ where $\tau(x) = \{H \in \tau : x \in H\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, said to be the $\star$-topology, finer than τ, is defined by $Cl^*(G) = G \cup G^*(I, \tau)$ [11]. We will briefly write G^* for $G^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$.

Remark 1. The $\star$-topology is generated by τ and by the filter $\mathcal{F}$. Also, the family $\{H \cap G : H \in \tau, G \in \mathcal{F}\}$ is a basis for this topology [11].

Lemma 2 ([10])**.** *Let K be a subset of an ideal topological space (X, τ, I). If N is open, then $N \cap Cl^*(K) \subset Cl^*(N \cap K)$.*

Definition 3. A subset K of an ideal topological space (X, τ, I) is called pre-I-open [3] (resp. semi-I-open [8], α-I-open [8], strongly β-I-open [9], $\star$-dense [4], t-I-set [8], semi*-I-open [5, 6]) if $K \subset Int(Cl^*(K))$ (resp. $K \subset Cl^*(Int(K))$, $K \subset Int(Cl^*(Int(K)))$, $K \subset Cl^*(Int(Cl^*(K)))$, $Cl^*(K) = X$, $Int(K) = Int(Cl^*(K))$, $K \subset Cl(Int^*(K))$).

Lemma 4 ([6])**.** *Every semi-I-open set is semi*-I-open in an ideal topological space.*

Remark 5. The reverse implication of Lemma 4 is not true in general as shown in [5, 6].

Definition 6. The complement of a pre-I-open (resp. semi-I-open, α-I-open, semi*-I-open) set is called pre-I-closed [3] (resp. semi-I-closed [8], α-I-closed [8], semi*-I-closed [5, 6]).

Definition 7. The pre-I-closure of a subset K of an ideal topological space (X, τ, I), denoted by $p_I Cl(K)$, is defined as the intersection of all pre-I-closed sets of X containing K [6].

Lemma 8 ([6])**.** *For a subset K of an ideal topological space (X, τ, I), $p_I Cl(K) = K \cup Cl(Int^*(K))$.*

Definition 9. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be pre-I-continuous [3] (resp. α-I-continuous [8]) if $f^{-1}(K)$ is pre-I-open (resp. α-I-open) in X for each open set K in Y.

Definition 10. A subset K of an ideal topological space (X, τ, I) is called

(1) an $\eta\zeta$-set [14] if $K = L \cap M$, where L is open and M is clopen in X.

(2) locally closed [2] if $K = L \cap M$, where L is open and M is closed in X.

(3) a B_I-set [8] if $K = L \cap M$, where L is open and M is a t-I-set in X.

(4) semi-I-regular [12] if K is a t-I-set and semi-I-open in X.

(5) an AB_I-set [12] if $K = L \cap M$, where L is open and M is a semi-I-regular set in X.

2. $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets, $\mathcal{C}_I^*$-sets in ideal topological spaces

A subset K of an ideal topological space (X, τ, I) is called pre-I-regular if K is pre-I-open and pre-I-closed in (X, τ, I).

Definition 11. Let (X, τ, I) be an ideal topological space and $K \subset X$. K is said to be a $\mathcal{C}_I^*$-set if $K = L \cap M$, where L is an open set and M is a pre-I-regular set in X.

Theorem 12. *Let (X, τ, I) be an ideal topological space. Then each $\mathcal{C}_I^*$-set in X is a pre-I-open set.*

Proof. Let K be a $\mathcal{C}_I^*$-set in X. It follows that $K = L \cap M$, where L is an open set and M is a pre-I-regular set in X. Since M is a pre-I-open set, then by Proposition 2.10 of [3], $K = L \cap M$ is a pre-I-open set in X.□

Remark 13. The converse of Theorem 12 need not be true in general as shown in the following example.

Example 14. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{a, b, c\}$ is a pre-I-open set but it is not a $\mathcal{C}_I^*$-set.

Remark 15. In an ideal topological space, every open set and every pre-I-regular set is a $\mathcal{C}_I^*$-set. The converse of this implication is not true in general as shown in the following example.

Example 16. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{a, b, c\}$ is a $\mathcal{C}_I^*$-set but it is not pre-I-regular. The set $L = \{a, c, d\}$ is a $\mathcal{C}_I^*$-set but it is not open.

Remark 17. By Remark 15 and Theorem 12, the following diagram holds for a subset K of an ideal topological space (X, τ, I):

$$\begin{array}{ccc} & & \text{pre-}I\text{-open} \\ & & \Uparrow \\ \text{pre-}I\text{-regular} & \Longrightarrow & \mathcal{C}_I^*\text{-set} \end{array}$$

Definition 18. A subset K of an ideal topological space (X, τ, I) is said to be

(1) a $\mathcal{C}_I$-set if $K = L \cap M$, where L is an open set and M is a pre-I-closed set in X.

(2) an η_I-set if $K = L \cap M$, where L is an open set and M is an α-I-closed set in X.

(3) an $\mathcal{A}_I^*$-set if $K = L \cap M$, where L is an open set and $M = Cl(Int^*(M))$.

Remark 19. Let (X, τ, I) be an ideal topological space and $K \subset X$. The following diagram holds for K:

$$\begin{array}{ccc} \mathcal{C}_I^*\text{-set} & \Longrightarrow & \mathcal{C}_I\text{-set} \\ & & \Uparrow \\ \mathcal{A}_I^*\text{-set} & \Longrightarrow & \eta_I\text{-set} \end{array}$$

The following examples show that these implications are not reversible in general.

Example 20. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{b, c, d\}$ is a $\mathcal{C}_I$-set and an $\mathcal{A}_I^*$-set but it is not a $\mathcal{C}_I^*$-set. The set $L = \{a, b, d\}$ is a $\mathcal{C}_I$-set but it is not an η_I-set.

Example 21. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{d\}$ is an η_I-set but it is not an $\mathcal{A}_I^*$-set. The set $L = \{a, b, d\}$ is a $\mathcal{C}_I^*$-set but it is not an η_I-set.

Theorem 22. *For a subset K of an ideal topological space (X, τ, I), the following properties are equivalent*:

(1) *K is a $\mathcal{C}_I$-set and a semi*-I-open set in X.*

(2) *$K = L \cap Cl(Int^*(K))$ for an open set L.*

Proof. (1) $\Rightarrow$ (2): Suppose that K is a $\mathcal{C}_I$-set and a semi*-I-open set in X. Since K is a $\mathcal{C}_I$-set, then we have $K = L \cap M$, where L is an open set and M is a pre-I-closed set in X. We have $K \subset M$, so $Cl(Int^*(K)) \subset Cl(Int^*(M))$. Since M is a pre-I-closed set in X, we have $Cl(Int^*(M)) \subset M$. Since K is a semi*-I-open set in X, we have $K \subset Cl(Int^*(K))$. It follows that $K = K \cap Cl(Int^*(K)) = L \cap M \cap Cl(Int^*(K)) = L \cap Cl(Int^*(K))$.

(2) $\Rightarrow$ (1): Let $K = L \cap Cl(Int^*(K))$ for an open set L. We have $K \subset Cl(Int^*(K))$. It follows that K is a semi*-I-open set in X. Since $Cl(Int^*(K))$ is a closed set, then $Cl(Int^*(K))$ is a pre-I-closed set in X. Hence, K is a $\mathcal{C}_I$-set in X. □

Theorem 23. *For a subset K of an ideal topological space (X, τ, I), the following properties are equivalent*:

(1) *K is an $\mathcal{A}_I^*$-set in X.*

(2) *K is an η_I-set and a semi*-I-open set in X.*

(3) *K is a $\mathcal{C}_I$-set and a semi*-I-open set in X.*

Proof. (1) $\Rightarrow$ (2): Suppose that K is an $\mathcal{A}_I^*$-set in X. It follows that $K = L \cap M$, where L is an open set and $M = Cl(Int^*(M))$. This implies $K = L \cap M = L \cap Cl(Int^*(M)) \subset Cl(L \cap Int^*(M)) = Cl(Int^*(L \cap M)) = Cl(Int^*(K))$. Thus, $K \subset Cl(Int^*(K))$ and hence K is a semi*-I-open set in X. Moreover, by Remark 19, K is an η_I-set in X.

(2) $\Rightarrow$ (3): It follows from the fact that every η_I-set is a $\mathcal{C}_I$-set in X by Remark 19.

(3) $\Rightarrow$ (1): Suppose that K is a $\mathcal{C}_I$-set and a semi*-I-open set in X. By Theorem 22, $K = L \cap Cl(Int^*(K))$ for an open set L. We have $Cl(Int^*(Cl(Int^*(K))))=Cl(Int^*(K))$. It follows that K is an $\mathcal{A}_I^*$-set in X.□

Theorem 24 ([5])**.** *A subset K of an ideal topological space (X, τ, I) is semi*-I-open if and only if $Cl(K) = Cl(Int^*(K))$.*

Theorem 25. *A subset K of an ideal topological space (X, τ, I) is semi*-I-closed if and only if K is a t-I-set.*

Proof. Let K be a semi*-I-closed set in X. Then $X \backslash K$ is semi*-I-open. By Theorem 24, we have $Cl(X \backslash K) = Cl(Int^*(X \backslash K))$. It follows that $Cl(X \backslash K) = X \backslash Int(K) = Cl(Int^*(X \backslash K)) = X \backslash Int(Cl^*(K))$. Thus, $Int(K) = Int(Cl^*(K))$ and hence K is a t-I-set in X. The converse is similar. □

Theorem 26. *Let (X, τ, I) be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *K is an open set.*

(2) *K is an α-I-open set and an $\mathcal{A}_I^*$-set.*

(3) *K is a pre-I-open set and an $\mathcal{A}_I^*$-set.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every open set is an α-I-open set and an $\mathcal{A}_I^*$-set.

(2) $\Rightarrow$ (3): It follows from the fact that every α-I-open set is pre-I-open.

(3) $\Rightarrow$ (1): Suppose that K is a pre-I-open set and an $\mathcal{A}_I^*$-set. Since K is an $\mathcal{A}_I^*$-set, then we have $K = L \cap M$, where L is an open set and $M = Cl(Int^*(M))$. It follows that $Int(Cl^*(M)) \subset Cl^*(M) \subset Cl(M) = Cl(Int^*(M)) = M$. Since $Int(Cl^*(M)) \subset M$, then M is a semi*-I-closed set. By Theorem 25, M is a t-I-set. Hence, K is a B_I-set. Since K is a B_I-set and a pre-I-open set, then by Proposition 3.3 of [8], K is an open set in X. □

Theorem 27. *Let (X, τ, I) be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *K is an open set.*

(2) *K is a $\mathcal{C}_I^*$-set and a semi*-I-open set.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every open set is a $\mathcal{C}_I^*$-set and a semi*-I-open set.

(2) $\Rightarrow$ (1): Let K be a $\mathcal{C}_I^*$-set and a semi*-I-open set. Since K is a $\mathcal{C}_I^*$-set, then K is a $\mathcal{C}_I$-set. Since K is a $\mathcal{C}_I$-set and a semi*-I-open set in X, then by Theorem 23, K is an $\mathcal{A}_I^*$-set. Moreover, since K is a $\mathcal{C}_I^*$-set, then K is a pre-I-open by Theorem 12. Hence, by Theorem 26, K is an open set in X. □

Theorem 28. *Let (X, τ, I) be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *K is an open set.*

(2) *K is an α-I-open set and a $\mathcal{C}_I^*$-set.*

(3) *K is an α-I-open set and a $\mathcal{C}_I$-set.*

Proof. (1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): It follows from the fact that every $\mathcal{C}_I^*$-set is a $\mathcal{C}_I$-set.

(3) $\Rightarrow$ (1): Let K be an α-I-open set and a $\mathcal{C}_I$-set. It follows that K is a semi*-I-open set and a $\mathcal{C}_I$-set. By Theorem 23, K is an $\mathcal{A}_I^*$-set. Since K

is an α-I-open set and an $\mathcal{A}_I^*$-set, then by Theorem 26, K is an open set in X. □

Definition 29. A subset K of an ideal topological space (X, τ, I) is said to be *gp*-I-open [15] if $N \subset p_I Int(K)$ whenever $N \subset K$ and N is a closed set in X, where $p_I Int(K) = K \cap Int(Cl^*(K))$.

Definition 30. A subset K of an ideal topological space (X, τ, I) is said to be generalized pre-I-closed (gp_I-closed) in (X, τ, I) if $X \backslash K$ is *gp*-I-open.

Theorem 31. *For a subset K of an ideal topological space (X, τ, I), K is gp_I-closed if and only if $p_I Cl(K) \subset N$ whenever $K \subset N$ and N is an open set in (X, τ, I).*

Proof. Let K be a gp_I-closed set in X. Suppose that $K \subset N$ and N is an open set in (X, τ, I). Then $X \backslash K$ is *gp*-I-open and $X \backslash N \subset X \backslash K$ where $X \backslash N$ is closed. Since $X \backslash K$ is *gp*-I-open, then we have $X \backslash N \subset p_I Int(X \backslash K)$, where $p_I Int(X \backslash K) = (X \backslash K) \cap Int(Cl^*(X \backslash K))$. Since $(X \backslash K) \cap Int(Cl^*(X \backslash K)) = (X \backslash K) \cap (X \backslash Cl(Int^*(K))) = X \backslash (K \cup Cl(Int^*(K)))$, then by Lemma 8, $(X \backslash K) \cap Int(Cl^*(X \backslash K)) = X \backslash (K \cup Cl(Int^*(K))) = X \backslash p_I Cl(K)$. It follows that $p_I Int(X \backslash K) = X \backslash p_I Cl(K)$. Thus, $p_I Cl(K) = X \backslash p_I Int(X \backslash K) \subset N$ and hence $p_I Cl(K) \subset N$. The converse is similar.□

Theorem 32. *Let (X, τ, I) be an ideal topological space and $V \subset X$. Then V is a $\mathcal{C}_I$-set in X if and only if $V = G \cap p_I Cl(V)$ for an open set G in X.*

Proof. If V is a $\mathcal{C}_I$-set, then $V = G \cap M$ for an open set G and a pre-I-closed set M. But then $V \subset M$ and so $V \subset p_I Cl(V) \subset M$. It follows that $V = V \cap p_I Cl(V) = G \cap M \cap p_I Cl(V) = G \cap p_I Cl(V)$. Conversely, it is enough to prove that $p_I Cl(V)$ is a pre-I-closed set. But $p_I Cl(V) \subset M$, for any pre-I-closed set M containing V. So, $Cl(Int^*(p_I Cl(V))) \subset Cl(Int^*(M)) \subset M$. It follows that $Cl(Int^*(p_I Cl(V))) \subset \cap_{V \subset M,\ M \text{ is pre-}I\text{-closed}} M = p_I Cl(V)$.□

Theorem 33. *Let (X, τ, I) be an ideal topological space and $N \subset X$. The following properties are equivalent:*
(1) *N is a pre-I-closed set in X.*
(2) *N is a $\mathcal{C}_I$-set and a gp_I-closed set in X.*

Proof. (1) ⇒ (2): It follows from the fact that any pre-I-closed set in X is a $\mathcal{C}_I$-set and a gp_I-closed set in X.

(2) $\Rightarrow$ (1): Suppose that N is a $\mathcal{C}_I$-set and a gp_I-closed set in X. Since N is a $\mathcal{C}_I$-set, then by Theorem 32, $N = G \cap p_ICl(N)$ for an open set G in (X, τ, I). Since $N \subset G$ and N is a gp_I-closed set in X, then $p_ICl(N) \subset G$. It follows that $p_ICl(N) \subset G \cap p_ICl(N) = N$. Thus, $N = p_ICl(N)$ and hence N is pre-I-closed. □

Theorem 34. *Let (X, τ, I) be an ideal topological space and $K \subset X$. If K is a $\mathcal{C}_I$-set in X, then $p_ICl(K)\backslash K$ is a pre-I-closed set and $K \cup (X\backslash p_ICl(K))$ is a pre-I-open set in X.*

Proof. Suppose that K is a $\mathcal{C}_I$-set in X. By Theorem 32, we have $K = L \cap p_ICl(K)$ for an open set L in X. It follows that $p_ICl(K)\backslash K = p_ICl(K)\backslash(L \cap p_ICl(K)) = p_ICl(K) \cap (X\backslash(L \cap p_ICl(K))) = p_ICl(K) \cap ((X\backslash L) \cup (X\backslash p_ICl(K))) = (p_ICl(K) \cap (X\backslash L)) \cup (p_ICl(K) \cap (X\backslash p_ICl(K))) = (p_ICl(K) \cap (X\backslash L)) \cup \emptyset = p_ICl(K) \cap (X\backslash L)$. Thus, $p_ICl(K)\backslash K = p_ICl(K) \cap (X\backslash L)$ and hence $p_ICl(K)\backslash K$ is pre-I-closed. Moreover, since $p_ICl(K)\backslash K$ is a pre-I-closed set in X, then $X\backslash(p_ICl(K)\backslash K) = X\backslash(p_ICl(K) \cap (X\backslash K)) = (X\backslash p_ICl(K)) \cup K$ is a pre-I-open set.
Thus, $X\backslash(p_ICl(K)\backslash K) = (X\backslash p_ICl(K)) \cup K$ is a pre-I-open set in X. □

3. Further properties

Definition 35. Let (X, τ, I) be an ideal topological space. (X, τ, I) is said to be pre-I-connected if X can not be expressed as the disjoint union of two nonvoid pre-I-open sets.

Theorem 36. *Let (X, τ, I) be an ideal topological space. The following properties are equivalent:*

(1) *(X, τ, I) is pre-I-connected.*

(2) *(X, τ, I) can not be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets.*

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ, I) can be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets. Since any $\mathcal{C}_I^*$-set is a pre-I-open set, then (X, τ, I) can be expressed as the disjoint union of two nonvoid pre-I-open sets. So, (X, τ, I) is not pre-I-connected. This is a contradiction.

(2) $\Rightarrow$ (1): Suppose that (X, τ, I) is not pre-I-connected. Then, X can be expressed as the disjoint union of two nonvoid pre-I-open sets. It follows that X has a nontrivial pre-I-regular subset A. Moreover, A and

$B = X \backslash A$ are pre-I-regular. Then A and B are $\mathcal{C}_I^*$-sets. Hence, (X, τ, I) can be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets. This is a contradiction. □

Definition 37. An ideal topological space (X, τ, I) is called I-submaximal [1, 7] if every $\star$-dense subset of X is open.

Theorem 38 ([7])**.** *For an ideal topological space* (X, τ, I)*, the following properties are equivalent*:
(1) *X is I-submaximal.*
(2) *Every pre-I-open set is open.*
(3) *Every pre-I-open set is semi-I-open and every α-I-open set is open.*

Theorem 39. *In an I-submaximal ideal space (X, τ, I), the following properties hold*:
(1) *Any $\mathcal{C}_I^*$-set is an $\eta\zeta$-set and an AB_I-set.*
(2) *Any η_I-set is a locally closed set.*

Proof. (1): Suppose that K is a $\mathcal{C}_I^*$-set in X. It follows that $K = L \cap M$, where L is an open set and M is a pre-I-regular set in X. By Theorem 38, M is semi-I-open and semi-I-closed. It follows from Lemma 4 that M is semi-I-open and semi*-I-closed. By Theorem 25, M is semi-I-open and a t-I-set in X. Thus, K is an AB_I-set in X. Furthermore, by Theorem 38, K is an $\eta\zeta$-set in X.
(2): It follows from Theorem 38. □

Definition 40. An ideal topological space (X, τ, I) is said to be $\star$-hyperconnected [6] if A is $\star$-dense for every open subset $A \neq \emptyset$ of X.

Theorem 41 ([6])**.** *The following properties are equivalent for an ideal topological space* (X, τ, I):
(1) *X is $\star$-hyperconnected.*
(2) *A is $\star$-dense for every strongly β-I-open subset $\emptyset \neq A \subset X$.*

Theorem 42. *For an ideal topological space (X, τ, I), the following properties are equivalent*:
(1) *(X, τ, I) is $\star$-hyperconnected.*
(2) *any $\mathcal{C}_I^*$-set in X is $\star$-dense.*

Proof. (1) $\Rightarrow$ (2): Let K be a $\mathcal{C}_I^*$-set in X. By Theorem 12, K is pre-I-open. Since (X, τ, I) is a $\star$-hyperconnected ideal topological space, then by Theorem 41, K is $\star$-dense.

(2) $\Rightarrow$ (1): Suppose that any $\mathcal{C}_I^*$-set in (X, τ, I) is $\star$-dense in X. Since an open set K in X is a $\mathcal{C}_I^*$-set, then K is $\star$-dense. Thus, (X, τ, I) is $\star$-hyperconnected. □

4. Decompositions of continuity and $\mathcal{A}_I^*$-continuity

Definition 43. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) $\mathcal{C}_I^*$-continuous if $f^{-1}(A)$ is a $\mathcal{C}_I^*$-set in X for every open set A in Y.

(2) PR_I-continuous if $f^{-1}(A)$ is a pre-I-regular set in X for every open set A in Y.

Remark 44. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following diagram holds. The reverses of these implications are not true in general as shown in the following example

$$\begin{array}{ccc} & & \text{pre-}I\text{-continuous} \\ & & \Uparrow \\ PR_I\text{-continuous} & \Longrightarrow & \mathcal{C}_I^*\text{-continuous} \end{array}$$

Example 45. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. The function $f : (X, \tau, I) \to (X, \tau)$, defined by $f(a) = c$, $f(b) = d$, $f(c) = a$, $f(d) = b$ is pre-I-continuous but it is not $\mathcal{C}_I^*$-continuous. The identity function $i : (X, \tau, I) \to (X, \tau)$ is $\mathcal{C}_I^*$-continuous but it is not PR_I-continuous.

Definition 46. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) $\mathcal{C}_I$-continuous if $f^{-1}(A)$ is a $\mathcal{C}_I$-set in X for every open set A in Y.

(2) $\mathcal{A}_I^*$-continuous if $f^{-1}(A)$ is an $\mathcal{A}_I^*$-set in X for every open set A in Y.

(3) η_I-continuous if $f^{-1}(A)$ is an η_I-set in X for every open set A in Y.

Remark 47. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following diagram holds. The reverses of these implications are not true in general as shown in the following example

$$\begin{array}{ccc} \mathcal{C}_I^*\text{-continuous} & \Longrightarrow & \mathcal{C}_I\text{-continuous} \\ & & \Uparrow \\ \mathcal{A}_I^*\text{-continuous} & \Longrightarrow & \eta_I\text{-continuous} \end{array}$$

Example 48. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. The function $f : (X, \tau, I) \to (X, \tau)$, defined by

$f(a) = b$, $f(b) = c$, $f(c) = c$, $f(d) = a$ is η_I-continuous but it is not $\mathcal{A}_I^*$-continuous. The function $g : (X, \tau, I) \to (X, \tau)$, defined by $g(a) = b$, $g(b) = c$, $g(c) = a$, $g(d) = c$ is $\mathcal{C}_I^*$-continuous but it is not η_I-continuous.

Example 49. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. The function $f : (X, \tau, I) \to (X, \tau)$, defined by $f(a)=b, f(b)=a, f(c)=c, f(d)=d$ is $\mathcal{C}_I$-continuous and $\mathcal{A}_I^*$-continuous but it is not $\mathcal{C}_I^*$-continuous. The function $g : (X, \tau, I) \to (X, \tau)$, defined by $g(a) = a$, $g(b) = a$, $g(c) = b$, $g(d) = a$ is $\mathcal{C}_I$-continuous but it is not η_I-continuous.

Definition 50. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be semi*-I-continuous if $f^{-1}(V)$ is a semi*-I-open set in X for every open set V in Y.

Theorem 51. *The following properties are equivalent for a function* $f : (X, \tau, I) \to (Y, \sigma)$:
(1) f *is* $\mathcal{A}_I^*$*-continuous.*
(2) f *is* η_I*-continuous and semi*-I*-continuous.*
(3) f *is* $\mathcal{C}_I$*-continuous and semi*-I*-continuous.*

Proof. It follows from Theorem 23. □

Theorem 52. *The following properties are equivalent for a function* $f : (X, \tau, I) \to (Y, \sigma)$:
(1) f *is continuous.*
(2) f *is* α-I*-continuous and* $\mathcal{A}_I^*$*-continuous.*
(3) f *is pre-*I*-continuous and* $\mathcal{A}_I^*$*-continuous.*
(4) f *is semi*-I*-continuous and* $\mathcal{C}_I^*$*-continuous.*
(5) f *is* α-I*-continuous and* $\mathcal{C}_I^*$*-continuous.*
(6) f *is* α-I*-continuous and* $\mathcal{C}_I$*-continuous.*

Proof. It follows from Theorem 26, 27 and 28. □

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REFERENCES

1. Açikgöz, A.; Yüksel, Ş.; Noiri, T. – *α-I-preirresolute functions and β-I-preirressolute functions*, Bull. Malays. Math. Sci. Soc., 28 (2005), 1–8.

2. Bourbaki, N. – *Elements of Mathematics. General Topology*, Part 1, Hermann, Paris, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

3. Dontchev, J. – *Idealization of Ganster-Reilly decomposition theorems*, arxiv:math.GN/9901017v1 (1999).

4. Dontchev, J.; Ganster, M.; Rose, D. – *Ideal resolvability*, Topology Appl., 93 (1999), 1–16.

5. Ekici, E.; Noiri, T. – *⋆-extremally disconnected ideal topological spaces*, Acta Math. Hungar., 122 (2009), 81–90.

6. Ekici, E.; Noiri, T. – *⋆-hyperconnected ideal topological spaces*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 58 (2012), 121–129.

7. Ekici, E.; Noiri, T. – *Properties of I-submaximal ideal topological spaces*, Filomat, 24 (2010), 87–94.

8. Hatir, E.; Noiri, T. – *On decompositions of continuity via idealization*, Acta Math. Hungar., 96 (2002), 341–349.

9. Hatir, E.; Keskin, A.; Noiri, T. – *On a new decomposition of continuity via idealization*, JP J. Geom. Topol., 3 (2003), 53–64.

10. Hatir, E.; Keskin, A.; Noiri, T. – *A note on strong β-I-sets and strongly β-I-continuous functions*, Acta Math. Hungar., 108 (2005), 87–94.

11. Janković, D.; Hamlett, T.R. – *New topologies from old via ideals*, Amer. Math. Monthly, 97 (1990), 295–310.

12. Keskin, A.; Yuksel, S. – *On semi-I-regular sets, AB_I-sets and decompositions of continuity, R_IC-continuity, A_I-continuity*, Acta Math. Hungar., 113 (2006), 227–241.

13. Kuratowski, K. – *Topology*, Vol. I. New edition, revised and augmented. Translated from the French by J. Jaworowski Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966.

14. Noiri, T.; Sayed, O.R. – *On decomposition of continuity*, Acta Math. Hungar., 111 (2006), 1–8.

15. Noiri, T.; Rajamani, M.; Inthumathi, V. – *On decompositions of g-continuity via idealization*, Bull. Calcutta Math. Soc., 99 (2007), 415–424.

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