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ON THE UNIQUENESS AND CONTINUOUS DEPENDENCE IN THE LINEAR THEORY OF THERMO-MICROSTRETCH ELASTICITY BACKWARD IN TIME

BY

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Abstract. We study the uniqueness and the continuous dependence problems for the thermo-microstretch elastic processes backward in time. The data are given for the final time $t = 0$ and we want to study the solution at the previous moments. We transform the problem in a boundary-initial value problem by an appropriate change of variables. The uniqueness theorems presented in this article extend in a particular case the uniqueness theorem of Passarella and Tibullo (2010) and we also discuss a different class of problems than the one considered by them. We find some estimates that prove the continuous dependence of solution with respect to the final data.

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Key words: thermo-microstretch elasticity, backward in time process, uniqueness theorem, continuous dependence.

1. Introduction

The theory of thermo-microstretch elastic solids was introduced by Eringen [6]. The particles of the solid with microstretch can expand and contract independent of translations and the rotations which they execute.

In the class of nonstandard problems, the backward in time problems have been initially considered by Serrin [10], who studied the uniqueness of solutions for the Navier-Stokes equations backward in time. An important study was made by Ames and Straughan [2] for this class of nonstandard and improperly posed problems. Ciarletta [5] studied the uniqueness and continuous dependence problems for the thermoelastic processes backward in time. At this hour, the nonstandard problems are intensively studied and we mention only some of the researchers in this domain: Ames [1,2], Payne [1,7], Knops [7], Ciarletta [5,4], Chiriţă [3,4], Quintanilla [9], Straughan [2, 9] and Passarella [8].

In this article we consider the boundary final value problem for the linear theory of thermo-microstretch elastic solids. The data are given for the final time $t = 0$ and we want to study the solution at the previous moments. By an appropriate change of variable, we transform this problem into a boundary initial value problem. Using the Lagrange-Brun identities, we deduce some preliminary results that combined with a method based on Gronwall's inequality will be the principal ingredients in obtaining the uniqueness and the continuous dependence results.

Passarella and Tibullo [8] have demonstrated the uniqueness of solutions for the backward in time problem of the linear theory of thermo-microstretch elastic materials and the impossibility of the localization in time of the solution of the corresponding forward in time problem. Our results concerning the uniqueness of solution extend in a particular case the uniqueness theorem of Passarella and Tibullo [8] and we also discuss a different class of problems than the one considered by them. Some estimates that prove the continuous dependence of solution with respect to the final data are obtained.

2. The boundary final value problem

In this article, we shall denote by $\Omega \subset \mathbb{R}^3$ the domain occupied by an anisotropic and inhomogeneous thermo-microstretch elastic solid, whose boundary surface is $\partial\Omega$. The standard convention of summation over repeated suffixes is adopted and a subscript comma denotes the spatial partial differentiation with respect to the corresponding cartesian coordinate and a superposed dot denotes differentiation with respect to time. Greek subscripts vary over $\{1,2\}$ and Latin subscripts vary over $\{1,2,3\}$.

We consider the boundary final value problems on the time interval $(-T,0], T > 0$ and T may be infinite. The fundamental system of field equations consists [6] of the geometric relations

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \gamma_i = \psi_{,i} \quad \text{on } \overline{\Omega} \times (-T,0], \tag{2.1}$$

the constitutive equations

$$t_{ij} = A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + D_{ijr}\gamma_r + A_{ij}\psi - \beta_{ij}\theta,$$

$$
\begin{aligned}
m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + E_{ijr}\gamma_r + B_{ij}\psi - C_{ij}\theta, \\
3\pi_i &= D_{rsi}e_{rs} + E_{rsi}\kappa_{rs} + D_{ij}\gamma_j + d_i\psi - \xi_i\theta, \\
3\sigma &= A_{rs}e_{rs} + B_{rs}\kappa_{rs} + d_i\gamma_i + m\psi - \zeta\theta, \\
\rho\eta &= \beta_{rs}e_{rs} + C_{rs}\kappa_{rs} + \xi_i\gamma_i + \zeta\psi + a\theta, \\
q_i &= k_{ij}\theta_{,j} \quad \text{in } \overline{\Omega} \times (-T, 0],
\end{aligned} \tag{2.2}
$$

the equations of motion

$$
\begin{aligned}
&t_{ji,j} + f_i = \rho\ddot{u}_i, \\
&m_{ji,j} + \varepsilon_{irs}t_{rs} + g_i = I_{ij}\ddot{\varphi}_j, \\
&\pi_{i,i} - \sigma + h = J\ddot{\psi} \quad \text{in } \Omega \times (-T, 0)
\end{aligned} \tag{2.3}
$$

and the energy equation

$$
\rho T_0 \dot{\eta} = q_{i,i} + r \quad \text{in } \Omega \times (-T, 0). \tag{2.4}
$$

In the above equations we have used the following notations: t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, π_i is the microstress vector, σ is the scalar microstress function, η is the specific entropy, ρ is the mass density (in the reference configuration), f_i is the body force, g_i is the body couple, h is the (scalar) body load, r is the heat source density, q_i is the heat flux vector, I_{ij} is the microinertia tensor, J is the microstretch inertia and ε_{ijk} is the alternating symbol. The variables of this theory are: u_i the components of the displacement vector, φ_i the components of the microrotation vector, ψ the microstretch function and θ the variation of temperature from the uniform reference absolute temperature T_0. We assume the following symmetries for the constitutive coefficients and the microinertia tensor

$$
A_{ijrs} = A_{rsij}, \quad C_{ijrs} = C_{rsij}, \quad D_{ij} = D_{ji}, \quad k_{ij} = k_{ji}, \quad I_{ij} = I_{ji}. \tag{2.5}
$$

Considering (2.2), the energy equation can be rewritten in the following form

$$
-\beta_{ij}\dot{e}_{ij} - C_{ij}\dot{\kappa}_{ij} - \xi_i\dot{\gamma}_i - \zeta\dot{\psi} + \frac{1}{T_0}q_{i,i} + \frac{1}{T_0}r = a\dot{\theta} \quad \text{in } \Omega \times (-T, 0). \tag{2.6}
$$

We consider the boundary-final value problem $(\mathcal{P})$ defined by relations

(2.1)–(2.3) and (2.6), the final conditions

$$u_i(\mathbf{x},0)=u_i^0(\mathbf{x}),\quad \dot{u}_i(\mathbf{x},0)=\dot{u}_i^0(\mathbf{x}),\quad \varphi_i(\mathbf{x},0)=\varphi_i^0(\mathbf{x}),$$
$$\dot{\varphi}_i(\mathbf{x},0)=\dot{\varphi}_i^0(\mathbf{x}),\quad \psi(\mathbf{x},0)=\psi^0(\mathbf{x}),\quad \dot{\psi}(\mathbf{x},0)=\dot{\psi}^0(\mathbf{x}),\tag{2.7}$$
$$\theta(\mathbf{x},0)=\theta^0(\mathbf{x}),\qquad \mathbf{x}\in\overline{\Omega}$$

and the boundary conditions

$$\begin{aligned}
&u_i(\mathbf{x},t)=\widetilde{u}_i(\mathbf{x},t)\quad \text{on } \Sigma_1\times(-T,0],\\
&t_i(\mathbf{x},t)\equiv t_{ji}(\mathbf{x},t)n_j(\mathbf{x})=\widetilde{t}_i(\mathbf{x},t)\quad \text{on } \Sigma_2\times(-T,0],\\
&\varphi_i(\mathbf{x},t)=\widetilde{\varphi}_i(\mathbf{x},t)\quad \text{on } \Sigma_3\times(-T,0],\\
&m_i(\mathbf{x},t)\equiv m_{ji}(\mathbf{x},t)n_j(\mathbf{x})=\widetilde{m}_i(\mathbf{x},t)\quad \text{on } \Sigma_4\times(-T,0],\\
&\psi(\mathbf{x},t)=\widetilde{\psi}(\mathbf{x},t)\quad \text{on } \Sigma_5\times(-T,0],\\
&\pi(\mathbf{x},t)\equiv \pi_i(\mathbf{x},t)n_i(\mathbf{x})=\widetilde{\pi}(\mathbf{x},t)\quad \text{on } \Sigma_6\times(-T,0],\\
&\theta(\mathbf{x},t)=\widetilde{\theta}(\mathbf{x},t)\quad \text{on } \Sigma_7\times(-T,0],\\
&q(\mathbf{x},t)\equiv q_i(\mathbf{x},t)n_i(\mathbf{x})=\widetilde{q}(\mathbf{x},t)\quad \text{on } \Sigma_8\times(-T,0],
\end{aligned}\tag{2.8}$$

where $u_i^0,\dot{u}_i^0,\varphi_i^0,\dot{\varphi}_i^0,\psi^0,\dot{\psi}^0,\theta^0,\widetilde{u}_i,\widetilde{t}_i,\widetilde{\varphi}_i,\widetilde{m}_i,\widetilde{\psi},\widetilde{\pi},\widetilde{\theta}$ and $\widetilde{q}$ are prescribed functions, n_i are the components of the outward unit normal vector to the boundary surface and $\Sigma_i, i=1,2,...,8$ are subsurfaces of $\partial\Omega$, such that $\Sigma_1\cup\overline{\Sigma}_2=\Sigma_3\cup\overline{\Sigma}_4=\Sigma_5\cup\overline{\Sigma}_6=\Sigma_7\cup\overline{\Sigma}_8=\partial\Omega$ and $\Sigma_1\cap\Sigma_2=\Sigma_3\cap\Sigma_4=\Sigma_5\cap\Sigma_6=\Sigma_7\cap\Sigma_8=\emptyset$.

We assume that ρ, I_{ij}, J and the constitutive coefficients are continuous and bounded fields on $\overline{\Omega}$, the constitutive coefficients are continuously differentiable functions on $\overline{\Omega}$ and

$$\rho(\mathbf{x})\geq\rho_0>0,\quad J(\mathbf{x})\geq J_0>0,\quad I(\mathbf{x})\geq I_0>0,\tag{2.9}$$

where $I(\mathbf{x})$ denotes the minimum eigenvalue of $I_{ij}(\mathbf{x})$ and ρ_0, J_0, I_0 are constants. We say that the internal energy density per unit of volume is a positive semidefinite quadratic form if

$$W(\varkappa)\geq 0,\tag{2.10}$$

for all $\varkappa=\{e_{ij},\kappa_{ij},\gamma_i,\psi\}$, where

$$\begin{aligned}2W(\varkappa)=&A_{ijrs}e_{ij}e_{rs}+C_{ijrs}\kappa_{ij}\kappa_{rs}+D_{ij}\gamma_i\gamma_j+m\psi^2+2B_{ijrs}e_{ij}\kappa_{rs}\\
&+2D_{ijr}e_{ij}\gamma_r+2E_{ijr}\kappa_{ij}\gamma_r+2A_{ij}e_{ij}\psi+2B_{ij}\kappa_{ij}\psi+2d_i\gamma_i\psi.\end{aligned}\tag{2.11}$$

We denote by $W(t) = W(\{e_{ij}(\mathbf{x},t), \kappa_{ij}(\mathbf{x},t), \gamma_i(\mathbf{x},t), \psi(\mathbf{x},t)\})$. We define

$$
\begin{aligned}
2\mathcal{E}(\tau_1,\tau_2)= & A_{ijrs}e_{ij}(\tau_1)e_{rs}(\tau_2)+C_{ijrs}\kappa_{ij}(\tau_1)\kappa_{rs}(\tau_2)+D_{ij}\gamma_i(\tau_1)\gamma_j(\tau_2)\\
& +m\psi(\tau_1)\psi(\tau_2) + B_{ijrs}[e_{ij}(\tau_1)\kappa_{rs}(\tau_2) + e_{ij}(\tau_2)\kappa_{rs}(\tau_1)]\\
& +D_{ijr}[e_{ij}(\tau_1)\gamma_r(\tau_2)+e_{ij}(\tau_2)\gamma_r(\tau_1)]+A_{ij}[e_{ij}(\tau_1)\psi(\tau_2)+e_{ij}(\tau_2)\psi(\tau_1)]\\
& +E_{ijr}[\kappa_{ij}(\tau_1)\gamma_r(\tau_2)+\kappa_{ij}(\tau_2)\gamma_r(\tau_1)]+B_{ij}[\kappa_{ij}(\tau_1)\psi(\tau_2)+\kappa_{ij}(\tau_2)\psi(\tau_1)]\\
& +d_i[\gamma_i(\tau_1)\psi(\tau_2) + \gamma_i(\tau_2)\psi(\tau_1)],
\end{aligned}
\tag{2.12}
$$

where, for convenience, the dependence on $\mathbf{x}$ was suppressed. We can remark that $\mathcal{E}(t,t) = W(t)$. If k_{ij} is a positive definite tensors, we have

$$
k_m\theta_{,i}\theta_{,i} \le k_{ij}\theta_{,i}\theta_{,j} \le k_M\theta_{,i}\theta_{,i}, \tag{2.13}
$$

where k_m and k_M are positive constants, related to the minimum and the maximum eigenvalue (conductivity moduli) for k_{ij}.

By using the relations (2.1), (2.2) and (2.11), we obtain

$$
t_{ij}\dot{e}_{ij} + m_{ij}\dot{\kappa}_{ij} + 3\pi_i\dot{\gamma}_i + 3\sigma\dot{\psi} = \dot{W} - \beta_{ij}\dot{e}_{ij}\theta - C_{ij}\dot{\kappa}_{ij}\theta - \xi_i\dot{\gamma}_i\theta - \zeta\dot{\psi}\theta. \tag{2.14}
$$

For future convenience, we set

$$
\delta = \sup_{\overline{\Omega}}\left[\frac{1}{\rho_0}\beta_{ij}\beta_{ij} + \frac{1}{I_0}C_{ij}C_{ij} + \frac{1}{3J_0}\xi_i\xi_i\right]^{\frac{1}{2}} > 0, \tag{2.15}
$$

$$
\begin{aligned}
\delta^* = \sup_{\overline{\Omega}}\Bigg[&\frac{1}{\rho_0}\beta_{ji,j}\beta_{ki,k} + \frac{1}{I_0}(\varepsilon_{jik}\beta_{kj} + C_{ki,k})(\varepsilon_{jil}\beta_{lj} + C_{li,l})\\
&+\frac{2}{3J_0}\xi_{i,i}\xi_{i,i} + \frac{2}{3J_0}\zeta^2\Bigg]^{\frac{1}{2}} > 0.
\end{aligned}
\tag{2.16}
$$

3. The transformed boundary-initial value problem

The aim of this article is to study the uniqueness and the continuous dependence of the solutions of the boundary-final value problem $(\mathcal{P})$ with respect to the final data. To do this, we transform $(\mathcal{P})$ into a boundary-initial value problem using the change of variables: $t \rightsquigarrow -t$. Thus, the boundary-initial value problem $(\overline{\mathcal{P}})$ is defined by the geometric equations

$$
e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \gamma_i = \psi_{,i} \quad \text{on } \overline{\Omega}\times[0,T), \tag{3.1}
$$

the constitutive equations

$$
\begin{aligned}
t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + D_{ijr}\gamma_r + A_{ij}\psi - \beta_{ij}\theta,\\
m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + E_{ijr}\gamma_r + B_{ij}\psi - C_{ij}\theta,\\
3\pi_i &= D_{rsi}e_{rs} + E_{rsi}\kappa_{rs} + D_{ij}\gamma_j + d_i\psi - \xi_i\theta,\\
3\sigma &= A_{rs}e_{rs} + B_{rs}\kappa_{rs} + d_i\gamma_i + m\psi - \zeta\theta,\\
\rho\eta &= \beta_{rs}e_{rs} + C_{rs}\kappa_{rs} + \xi_i\gamma_i + \zeta\psi + a\theta,\\
q_i &= k_{ij}\theta_{,j} \quad \text{in } \overline{\Omega}\times[0,T),
\end{aligned} \tag{3.2}
$$

the equations of motion

$$
\begin{aligned}
&t_{ji,j} + f_i = \rho\ddot{u}_i,\\
&m_{ji,j} + \varepsilon_{irs}t_{rs} + g_i = I_{ij}\ddot{\varphi}_j,\\
&\pi_{i,i} - \sigma + h = J\ddot{\psi} \quad \text{in } \Omega\times(0,T),
\end{aligned} \tag{3.3}
$$

the energy equation

$$
\beta_{ij}\dot{e}_{ij} + C_{ij}\dot{\kappa}_{ij} + \xi_i\dot{\gamma}_i + \zeta\dot{\psi} + \frac{1}{T_0}q_{i,i} + \frac{1}{T_0}r = -a\dot{\theta} \quad \text{in } \Omega\times(0,T), \tag{3.4}
$$

with the initial conditions

$$
\begin{gathered}
u_i(\mathbf{x},0)=u_i^0(\mathbf{x}), \dot{u}_i(\mathbf{x},0)=\dot{u}_i^0(\mathbf{x}), \varphi_i(\mathbf{x},0)=\varphi_i^0(\mathbf{x}), \dot{\varphi}_i(\mathbf{x},0)=\dot{\varphi}_i^0(\mathbf{x}),\\
\psi(\mathbf{x},0)=\psi^0(\mathbf{x}),\ \dot{\psi}(\mathbf{x},0)=\dot{\psi}^0(\mathbf{x}),\ \theta(\mathbf{x},0)=\theta^0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}
\end{gathered} \tag{3.5}
$$

and the boundary conditions

$$
\begin{aligned}
&u_i(\mathbf{x},t)=\widetilde{u}_i(\mathbf{x},t) \text{ on } \Sigma_1\times[0,T),\ t_i(\mathbf{x},t)=\widetilde{t}_i(\mathbf{x},t) \text{ on } \Sigma_2\times[0,T),\\
&\varphi_i(\mathbf{x},t)=\widetilde{\varphi}_i(\mathbf{x},t) \text{ on } \Sigma_3\times[0,T),\ m_i(\mathbf{x},t)=\widetilde{m}_i(\mathbf{x},t) \text{ on } \Sigma_4\times[0,T),\\
&\psi(\mathbf{x},t)=\widetilde{\psi}(\mathbf{x},t) \text{ on } \Sigma_5\times[0,T),\ \pi(\mathbf{x},t)=\widetilde{\pi}(\mathbf{x},t) \text{ on } \Sigma_6\times[0,T),\\
&\theta(\mathbf{x},t)=\widetilde{\theta}(\mathbf{x},t) \text{ on } \Sigma_7\times[0,T),\ q(\mathbf{x},t)=\widetilde{q}(\mathbf{x},t) \text{ on } \Sigma_8\times[0,T).
\end{aligned} \tag{3.6}
$$

By a solution of the boundary-initial value problem $(\overline{\mathcal{P}})$ we mean an ordered array $\varpi = [u_i, \varphi_i, \psi, e_{ij}, \kappa_{ij}, \gamma_i, t_{ij}, m_{ij}, \pi_i, \sigma, \theta, \theta_{,i}, q_i]$ with its components continuous on $\overline{\Omega}\times[0,T)$ and satisfying equations (3.1)–(3.6).

In the rest of this section we establish some auxiliary identities concerning the solutions of the boundary-initial value problem $(\overline{\mathcal{P}})$ with the external given data $\mathcal{D} = [f_i, g_i, h, r, u_i^0, \dot{u}_i^0, \varphi_i^0, \dot{\varphi}_i^0, \psi^0, \dot{\psi}^0, \theta^0, \widetilde{u}_i, \widetilde{t}_i, \widetilde{\varphi}_i, \widetilde{m}_i, \widetilde{\psi}, \widetilde{\pi}, \widetilde{\theta}, \widetilde{q}]$.

Lemma 3.1. *Let ϖ be a solution of the boundary-initial value problem $(\overline{\mathcal{P}})$ corresponding to the external given data $\mathcal{D}$. Then, for all $t \in [0, T)$, we have*

$$\begin{aligned}
&\int_\Omega \Big[\rho \dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) + 2W(t) + a\theta^2(t)\Big]\, dv \\
&- 2\int_0^t \int_\Omega \frac{1}{T_0} k_{ij}\theta_{,i}(s)\theta_{,j}(s) dvds \\
&= \int_\Omega \Big[\rho \dot{u}_i(0)\dot{u}_i(0) + I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(0) + 3J\dot{\psi}^2(0) + 2W(0) + a\theta^2(0)\Big]\, dv \\
&+ 2\int_0^t \int_\Omega \Big[f_i(s)\dot{u}_i(s) + g_i(s)\dot{\varphi}_i(s) + 3h(s)\dot{\psi}(s) - \frac{1}{T_0} r(s)\theta(s)\Big]\, dvds \\
&+ 2\int_0^t \int_{\partial\Omega} \Big[t_i(s)\dot{u}_i(s) + m_i(s)\dot{\varphi}_i(s) + 3\pi(s)\dot{\psi}(s) - \frac{1}{T_0} q(s)\theta(s)\Big]\, dads.
\end{aligned} \tag{3.7}$$

Proof. We deduce from (2.5), (2.14), (3.1) and (3.3) that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial s}\Big[\rho \dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s) + 2W(s)\Big] = f_i(s)\dot{u}_i(s) \\
&+ g_i(s)\dot{\varphi}_i(s) + 3h(s)\dot{\psi}(s) + \Big[t_{ji}(s)\dot{u}_i(s) + m_{ji}(s)\dot{\varphi}_i(s) + 3\pi_j(s)\dot{\psi}(s)\Big]_{,j} \\
&+ \beta_{ij}\dot{e}_{ij}(s)\theta(s) + C_{ij}\dot{\kappa}_{ij}(s)\theta(s) + \xi_i\dot{\gamma}_i(s)\theta(s) + \zeta\dot{\psi}(s)\theta(s).
\end{aligned} \tag{3.8}$$

If we integrate the above relation over $\Omega \times [0, t], t \in [0, T)$ and then use the divergence theorem and equivalences from (2.8), we have

$$\begin{aligned}
&\int_\Omega [\rho \dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) + 2W(t)] dv \\
&= \int_\Omega [\rho \dot{u}_i(0)\dot{u}_i(0) + I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(0) + 3J\dot{\psi}^2(0) + 2W(0)] dv \\
&+ 2\int_0^t \int_\Omega [f_i(s)\dot{u}_i(s) + g_i(s)\dot{\varphi}_i(s) + 3h(s)\dot{\psi}(s)] dvds \\
&+ 2\int_0^t \int_{\partial\Omega} [t_i(s)\dot{u}_i(s) + m_i(s)\dot{\varphi}_i(s) + 3\pi(s)\dot{\psi}(s)] dads \\
&+ 2\int_0^t \int_\Omega [\beta_{ij}\dot{e}_{ij}(s)\theta(s) + C_{ij}\dot{\kappa}_{ij}(s)\theta(s) + \xi_i\dot{\gamma}_i(s)\theta(s) + \zeta\dot{\psi}(s)\theta(s)] dvds.
\end{aligned} \tag{3.9}$$

Further, by using the relations (3.2) and (3.4) we deduce

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial s}\left[a\theta^2(s)\right] = -\,\beta_{ij}\dot{e}_{ij}(s)\theta(s) - C_{ij}\dot{\kappa}_{ij}(s)\theta(s) - \xi_i\dot{\gamma}_i(s)\theta(s) \\ &- \zeta\dot{\psi}(s)\theta(s) - \left[\frac{1}{T_0}q_i(s)\theta(s)\right]_{,i} + \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s) - \frac{1}{T_0}r(s)\theta(s) \end{aligned} \tag{3.10}$$

which by integrating over $\Omega\times[0,t], t\in[0,T)$ and by using the divergence theorem and the relation (2.8), gives us the identity

$$\begin{aligned} &\int_\Omega a\theta^2(t)dv - 2\int_0^t\int_\Omega \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds = \int_\Omega a\theta^2(0)dv \\ &-2\int_0^t\int_\Omega \frac{1}{T_0}r(s)\theta(s)dvds - 2\int_0^t\int_{\partial\Omega}\frac{1}{T_0}q(s)\theta(s)dads \\ &-2\int_0^t\int_\Omega[\beta_{ij}\dot{e}_{ij}(s)\theta(s) + C_{ij}\dot{\kappa}_{ij}(s)\theta(s) + \xi_i\dot{\gamma}_i(s)\theta(s) + \zeta\dot{\psi}(s)\theta(s)]dvds. \end{aligned} \tag{3.11}$$

Summing the relation (3.9) with (3.11) we obtain the identity (3.7), and so the proof is complete. □

Lemma 3.2. *Let ϖ be a solution of the boundary-initial value problem $(\overline{\mathcal{P}})$ corresponding to the external given data $\mathcal{D}$. Then, for all $t\in[0,\frac{T}{2})$, we have*

$$\begin{aligned} &\int_\Omega\Big\{\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) - \big[2W(t) + a\theta^2(t)\big]\Big\}\,dv \\ &= \int_\Omega\big\{\rho\dot{u}_i(0)\dot{u}_i(2t) + I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(2t) + 3J\dot{\psi}(0)\dot{\psi}(2t) \\ &- [2\mathcal{E}(0,2t) + a\theta(0)\theta(2t)]\big\}dv \\ &+ \int_0^t\int_\Omega\Big\{[f_i(t-s)\dot{u}_i(t+s) - f_i(t+s)\dot{u}_i(t-s)] \\ &+ [g_i(t-s)\dot{\varphi}_i(t+s) - g_i(t+s)\dot{\varphi}_i(t-s)] \\ &+ 3\Big[h(t-s)\dot{\psi}(t+s) - h(t+s)\dot{\psi}(t-s)\Big] \\ &+ \frac{1}{T_0}[r(t-s)\theta(t+s) - r(t+s)\theta(t-s)]\Big\}dvds \\ &+ \int_0^t\int_{\partial\Omega}\Big\{[t_i(t-s)\dot{u}_i(t+s) - t_i(t+s)\dot{u}_i(t-s)] \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&+ [m_i(t-s)\dot{\varphi}_i(t+s) - m_i(t+s)\dot{\varphi}_i(t-s)] \\
&+ 3\left[\pi(t-s)\dot{\psi}(t+s) - \pi(t+s)\dot{\psi}(t-s)\right] \\
&+ \frac{1}{T_0}\left[q(t-s)\theta(t+s) - q(t+s)\theta(t-s)\right]\Big\} da ds.
\end{aligned}
$$

Proof. Considering $t \in [0, \frac{T}{2})$, from (2.5) we have

$$
\begin{aligned}
&\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) \\
&= \rho\dot{u}_i(0)\dot{u}_i(2t) + I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(2t) + 3J\dot{\psi}(0)\dot{\psi}(2t) \\
&+ \int_0^t \Big\{\rho[\dot{u}_i(t+s)\ddot{u}_i(t-s) - \dot{u}_i(t-s)\ddot{u}_i(t+s)] \\
&+ I_{ij}[\dot{\varphi}_i(t+s)\ddot{\varphi}_j(t-s) - \dot{\varphi}_i(t-s)\ddot{\varphi}_j(t+s)] \\
&+ 3J[\dot{\psi}(t+s)\ddot{\psi}(t-s) - \dot{\psi}(t-s)\ddot{\psi}(t+s)]\Big\}\, ds.
\end{aligned}
\tag{3.13}
$$

Further, using the relations (2.5), (2.12) and (3.1)–(3.4), we deduce that

$$
\begin{aligned}
&\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) - \left[2W(t) + a\theta^2(t)\right] \\
&= \rho\dot{u}_i(0)\dot{u}_i(2t) + I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(2t) + 3J\dot{\psi}(0)\dot{\psi}(2t) \\
&- [2\mathcal{E}(0,2t) + a\theta(0)\theta(2t)] \\
&+ \int_0^t \Big\{ f_i(t-s)\dot{u}_i(t+s) - f_i(t+s)\dot{u}_i(t-s) \\
&+ g_i(t-s)\dot{\varphi}_i(t+s) - g_i(t+s)\dot{\varphi}_i(t-s) \\
&+ 3\left[h(t-s)\dot{\psi}(t+s) - h(t+s)\dot{\psi}(t-s)\right] \\
&+ \frac{1}{T_0}\left[r(t-s)\theta(t+s) - r(t+s)\theta(t-s)\right]\Big\}\, ds \\
&+ \int_0^t \Big\{ t_{ji}(t-s)\dot{u}_i(t+s) - t_{ji}(t+s)\dot{u}_i(t-s) \\
&+ m_{ji}(t-s)\dot{\varphi}_i(t+s) - m_{ji}(t+s)\dot{\varphi}_i(t-s) \\
&+ 3\left[\pi_j(t-s)\dot{\psi}(t+s) - \pi_j(t+s)\dot{\psi}(t-s)\right] \\
&+ \frac{1}{T_0}\left[q_j(t-s)\theta(t+s) - q_j(t+s)\theta(t-s)\right]\Big\}_{,j} ds,\ t \in \left[0, \frac{T}{2}\right).
\end{aligned}
\tag{3.14}
$$

To complete the proof of this lemma, we integrate relation (3.14) over Ω, use the divergence theorem and the equivalences from (2.8). $\square$

The identities obtained in this section constitute the essential ingredients in deducing the uniqueness and continuous dependence results in the next three sections, for the boundary-initial value problem $(\overline{\mathcal{P}})$ with respect to the external given data $\mathcal{D}$.

4. Uniqueness results

The aim of this section is to establish the uniqueness of the solution of the boundary-initial value problem $(\overline{\mathcal{P}})$. Passarella and Tibullo [8] have demonstrated a uniqueness result for the problem we have considered. The theorems given in this section extend their theorem in the particular case when meas $\Sigma_8 = 0$. In what follows, we assume that the symmetry relations (2.5) are satisfied. We will use the following hypotheses:

(H_1) the relation (2.9) holds true and k_{ij} is a positive definite tensor (i.e. the relation (2.13) holds true);

(H_2) W is a positive semidefinite quadratic form;

(H_3) $a(\mathbf{x}) \leq a_0 < 0,$ where a_0 is a constant.

These assumptions are reasonable and characterize a certain state supported by the thermo-microstretch elastic body.

Theorem 4.1. *Assume that* (H_1) *and* (H_2) *hold true. Then the boundary-initial value problem* $(\overline{\mathcal{P}})$ *has at most one solution.*

Proof. We consider $\varpi^{(\alpha)} = [u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \psi^{(\alpha)}, e_{ij}^{(\alpha)}, \kappa_{ij}^{(\alpha)}, \gamma_i^{(\alpha)}, t_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}, \pi_i^{(\alpha)}, \sigma^{(\alpha)}, \theta^{(\alpha)},\ \theta_{,i}^{(\alpha)}, q_i^{(\alpha)}](\alpha = 1,2)$ two solutions of the boundary-initial value problem $(\overline{\mathcal{P}})$ corresponding to the same external given data $\mathcal{D}$. The difference $\varpi = \varpi^{(1)} - \varpi^{(2)} = [u_i, \varphi_i, \psi, e_{ij}, \kappa_{ij}, \gamma_i, t_{ij}, m_{ij}, \pi_i, \sigma, \theta, \theta_{,i}, q_i]$ is a solution for the boundary-initial value problem $(\overline{\mathcal{P}})$ corresponding to the null external given data.

Relations (3.7) and (3.12), in the context of this theorem, provides us

$$\begin{aligned}
&\int_\Omega \left[\rho \dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t)\right] dv \\
&= \int_0^t \int_\Omega \frac{1}{T_0} k_{ij}\theta_{,i}(s)\theta_{,j}(s) dv ds, \quad t \in \left[0, \frac{T}{2}\right).
\end{aligned} \tag{4.1}$$

The assumption that meas $\Sigma_8 = 0$ and the homogeneous external given data implies that $\theta(\mathbf{x},t) = 0$ on $\partial\Omega \times [0,T)$ and hence we obtain

$$\begin{aligned}
&2\int_0^t\int_\Omega \left[\beta_{ij}\dot{e}_{ij}(s)\theta(s)+C_{ij}\dot{\kappa}_{ij}(s)\theta(s)+\xi_i\dot{\gamma}_i(s)\theta(s)+\zeta\dot{\psi}(s)\theta(s)\right] dvds\\
&= -2\int_0^t\int_\Omega \Big\{\left[\beta_{ij}\theta(s)\right]_{,i}\dot{u}_j(s) - \varepsilon_{jik}\beta_{ij}\dot{\varphi}_k(s)\theta(s) + \left[C_{ij}\theta(s)\right]_{,i}\dot{\varphi}_j(s)\\
&+ \left[\xi_i\theta(s)\right]_{,i}\dot{\psi}(s) - \zeta\dot{\psi}(s)\theta(s)\Big\} dvds.
\end{aligned}\tag{4.2}$$

The Poincaré inequality (see [5]) gives us

$$\int_\Omega \theta_{,i}(t)\theta_{,i}(t)dv \geq \lambda\int_\Omega \theta^2(t)dv,\tag{4.3}$$

where $\lambda > 0$ is the minimum eigenvalue of the clamped membrane problem.

On the other hand, by using Schwarz's inequality, arithmetic-geometric mean inequality and the relations (2.15)-(2.16), we deduce

$$\begin{aligned}
&2\int_0^t\int_\Omega \left[\beta_{ij}\dot{e}_{ij}(s)\theta(s)+C_{ij}\dot{\kappa}_{ij}(s)\theta(s)+\xi_i\dot{\gamma}_i(s)\theta(s)+\zeta\dot{\psi}(s)\theta(s)\right] dvds\\
&\leq \int_0^t\int_\Omega \Big\{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\left[\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s)\right]\\
&+ \varepsilon_1\delta^{*2}\theta^2(s) + \varepsilon_2\delta^2\theta_{,i}(s)\theta_{,i}(s)\Big\}dvds, \quad \forall\,\varepsilon_1,\varepsilon_2 > 0.
\end{aligned}\tag{4.4}$$

Supposing that hypothesis (H_1) holds true and considering the relations (2.9), (2.13), (4.3) and (4.4), we obtain

$$\begin{aligned}
&2\int_0^t\int_\Omega \left[\beta_{ij}\dot{e}_{ij}(s)\theta(s)+C_{ij}\dot{\kappa}_{ij}(s)\theta(s)+\xi_i\dot{\gamma}_i(s)\theta(s)+\zeta\dot{\psi}(s)\theta(s)\right] dvds\\
&\leq \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\int_0^t\int_\Omega \left[\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s)\right] dvds\\
&+ \mu\int_0^t\int_\Omega \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds, \quad \forall\,\varepsilon_1,\varepsilon_2 > 0, t\in[0,T),
\end{aligned}\tag{4.5}$$

where $\mu = \frac{T_0}{k_m}(\varepsilon_1 \lambda^{-1} \delta^{*2} + \varepsilon_2 \delta^2)$ and λ is defined by (4.3). Moreover, from (4.1) and (4.5), we deduce

$$\begin{aligned}
&2\int_0^t \int_\Omega \left[\beta_{ij}\dot{e}_{ij}(s)\theta(s) + C_{ij}\dot{\kappa}_{ij}(s)\theta(s) + \xi_i\dot{\gamma}_i(s)\theta(s) + \zeta\dot{\psi}(s)\theta(s)\right] dvds \\
&\le \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}\int_0^t \int_\Omega \left[\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s)\right] dvds \\
&+ \mu \int_\Omega \left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t)\right] dv, \quad t \in \left[0, \frac{T}{2}\right).
\end{aligned} \tag{4.6}$$

Assuming that hypothesis (H_2) holds true, we can conclude from relations (3.9) and (4.6), with null external given data that

$$\begin{aligned}
&\int_\Omega \left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t) + 2W(t)\right] dv \\
&\le \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}\int_0^t \int_\Omega \left[\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s)\right] dvds \\
&+ \mu \int_\Omega \left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t)\right] dv, \ t \in \left[0, \frac{T}{2}\right).
\end{aligned} \tag{4.7}$$

Choosing the parameters $\varepsilon_1, \varepsilon_2$ sufficiently small (e.g. $\varepsilon_1 = \frac{\lambda k_m}{3\delta^{*2} T_0}$ and $\varepsilon_2 = \frac{k_m}{3\delta^2 T_0}$) we have $1 - \mu > 0$ and hence, from (2.10) and (4.7), we deduce

$$\begin{aligned}
&\int_\Omega \left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t)\right] dv \le \frac{\varepsilon_1 + \varepsilon_2}{(1-\mu)\varepsilon_1\varepsilon_2} \\
&\times \int_0^t \int_\Omega \left[\rho\dot{u}_i(s)\dot{u}_i(s) + I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 3J\dot{\psi}^2(s)\right] dvds, \ t \in \left[0, \frac{T}{2}\right).
\end{aligned} \tag{4.8}$$

By means of Gronwall's inequality, the above relation gives us

$$\int_\Omega \left[\rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + 3J\dot{\psi}^2(t)\right] dv = 0, \quad t \in \left[0, \frac{T}{2}\right). \tag{4.9}$$

Considering the relation (2.9), we have that all the terms in the integral are positive and hence, we deduce that

$$\dot{u}_i(\mathbf{x}, t) = 0, \quad \dot{\varphi}_i(\mathbf{x}, t) = 0, \quad \dot{\psi}(\mathbf{x}, t) = 0 \quad \text{in } \Omega \times \left[0, \frac{T}{2}\right) \tag{4.10}$$

and by taking into account the null initial conditions for ϖ, the relations (2.13) and (4.1) and the assumption that $\theta(\mathbf{x},t)=0$ on $\partial\Omega\times[0,T)$, we get

$$u_i(\mathbf{x},t)=0,\ \varphi_i(\mathbf{x},t)=0,\ \psi(\mathbf{x},t)=0,\ \theta(\mathbf{x},t)=0\ \text{ in }\Omega\times\left[0,\frac{T}{2}\right). \tag{4.11}$$

For the case when $T=\infty$, the relations (4.11) give us the uniqueness of solutions of the boundary-initial value problem $(\overline{\mathcal{P}})$. If $T<\infty$, we repeat the procedure of proof on the time interval $\left[\frac{T}{2},T\right)$ and we get the relations (4.11) for $\Omega\times\left[\frac{T}{2},\frac{3T}{4}\right)$. Continuing this algorithm for extension of the set on witch we have the uniqueness of solution $(\overline{\mathcal{P}})$, we obtain the validity of relations (4.11) over $\Omega\times[0,T)$. □

Theorem 4.2. *Assume that* (H$_1$) *and* (H$_3$) *hold true. Then the boundary-initial value problem* $(\overline{\mathcal{P}})$ *has at most one solution.*

Proof. The relations (4.1)-(4.6) from the proof of the previous theorem remain valid because they are obtained under the only hypothesis (H_1). From relation (3.11) with null external given data, we obtain that

$$\begin{aligned}\int_0^t\int_\Omega \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds \le \int_0^t\int_\Omega \big[\beta_{ij}\dot{e}_{ij}(s)\theta(s)+C_{ij}\dot{\kappa}_{ij}(s)\theta(s)\\ +\xi_i\dot{\gamma}_i(s)\theta(s)+\zeta\dot{\psi}(s)\theta(s)\big]dvds,\quad t\in[0,T).\end{aligned} \tag{4.12}$$

Further, using relations (4.1) and (4.5), the above inequality becomes

$$\begin{aligned}2\int_0^t\int_\Omega \frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds\le\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\int_0^t\int_0^s\int_\Omega\frac{1}{T_0}k_{ij}\theta_{,i}(z)\theta_{,j}(z)dvdzds\\ +\mu\int_0^t\int_\Omega\frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds,\quad \forall\,\varepsilon_1,\varepsilon_2>0,\ t\in\left[0,\frac{T}{2}\right).\end{aligned} \tag{4.13}$$

If we choose parameters $\varepsilon_1,\varepsilon_2$ sufficiently small (e.g $\varepsilon_1=\frac{2\lambda k_m}{3\delta^{*2}T_0}$ and $\varepsilon_2=\frac{2k_m}{3\delta^2T_0}$) such that $2-\mu>0$, from relation (4.13) we have that

$$\begin{aligned}\int_0^t\int_\Omega\frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds\le\frac{1}{2-\mu}\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\\ \times\int_0^t\int_0^s\int_\Omega\frac{1}{T_0}k_{ij}\theta_{,i}(z)\theta_{,j}(z)dvdzds,\quad\forall\,\varepsilon_1,\varepsilon_2>0,\ t\in\left[0,\frac{T}{2}\right).\end{aligned} \tag{4.14}$$

By applying Gronwall's lemma to the above relation, we deduce

$$\int_0^t \int_\Omega \frac{1}{T_0} k_{ij}\theta_{,i}(s)\theta_{,j}(s)dvds = 0, \quad t \in \left[0, \frac{T}{2}\right). \tag{4.15}$$

Having that k_{ij} is a positive definite tensor, the assumption meas $\Sigma_8 = 0$ and the null external given data, from relations (2.9), (4.1) and (4.15) we have that

$$u_i(\mathbf{x}, t) = 0,\ \varphi_i(\mathbf{x}, t) = 0,\ \psi(\mathbf{x}, t) = 0,\ \theta(\mathbf{x}, t) = 0 \ \text{ in } \Omega \times \left[0, \frac{T}{2}\right). \tag{4.16}$$

Finally, we extend relations (4.16) to $\Omega \times [0, T)$ using the same procedure as at the end of the proof of the previous theorem and so, for hypotheses (H_1) and (H_3), we have the uniqueness of solutions of the boundary-initial value problem $(\overline{\mathcal{P}})$. □

Remark 4.1. If we assume that (H_1), (H_2) and (H_3) hold true, we obtain the uniqueness of the solution of the boundary-initial value problem $(\overline{\mathcal{P}})$ in $\Omega \times [0, T)$, without any procedure of extension.

We can now compare our results with the uniqueness theorem of Passarella and Tibullo [8]:

- in Theorem 4.1 we extend their result in a particular case: W is considered a positive semidefinite quadratic form and we had no condition for a, but we supplementary imposed that meas $\Sigma_8 = 0$;
- in Theorem 4.2 we discussed a different class of problems than the one considered by Passarella and Tibullo because we don't impose any restriction on W and the condition considered for a is complementary to the one they considered.

5. Continuous dependence with respect to the final data

In this section, we obtain the continuous dependence with respect to the final data in the context of the hypotheses (H_1), (H_2) and (H_3). This result is obtained without imposing any constraining restriction upon the solution.

Theorem 5.1. *If* $\varpi = [u_i, \varphi_i, \psi, e_{ij}, \kappa_{ij}, \gamma_i, t_{ij}, m_{ij}, \pi_i, \sigma, \theta, \theta_{,i}, q_i]$ *is a solution of the boundary-initial value problem* $(\overline{\mathcal{P}})$ *corresponding to the external given data* $\mathcal{D}_0 = \left[0, 0, 0, 0, u_i^0, \dot{u}_i^0, \varphi_i^0, \dot{\varphi}_i^0, \psi^0, \dot{\psi}^0, \theta^0, 0, 0, 0, 0, 0, 0, 0\right]$, *assuming that the hypotheses* $(\mathrm{H}_1), (\mathrm{H}_2)$ *and* (H_3) *hold true, we have the estimate*

$$\mathcal{E}(t) + \int_0^t \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) dv ds \leq \mathcal{E}(0) \exp(Mt), \qquad \forall\, t \in [0, T) \tag{5.1}$$

where $M > 0$ *is a constant which depends on* T_0, k_m, λ *and* δ^2 *defined in the previous sections and*

$$\mathcal{E}(t) = \int_\Omega [\rho \dot{u}_i(t) \dot{u}_i(t) + I_{ij} \dot{\varphi}_i(t) \dot{\varphi}_j(t) + 3J \dot{\psi}^2(t) + 2W(t) - a\theta^2(t)] dv. \tag{5.2}$$

Proof. To study the continuous dependence with respect to the initial data means that the solution of the boundary-initial values problem $(\overline{\mathcal{P}})$ do not change significantly when we introduce small perturbations of initial data. Because the boundary-initial values problem $(\overline{\mathcal{P}})$ is linear, then the continuous dependence with respect to the initial data is equivalent to the stability of the null solution. Therefore, we consider $\varpi = [u_i, \varphi_i, \psi, e_{ij}, \kappa_{ij}, \gamma_i, t_{ij}, m_{ij}, \pi_i, \sigma, \theta, \theta_{,i}, q_i]$ a solution of the boundary-initial value problem $(\overline{\mathcal{P}})$ corresponding to the external given data $\mathcal{D}_0 = [0, 0, 0, 0, u_i^0, \dot{u}_i^0, \varphi_i^0, \dot{\varphi}_i^0, \psi^0, \dot{\psi}^0, \theta^0, 0, 0, 0, 0, 0, 0, 0, 0]$. Combining (3.9) and (3.11) in the context of the external given data $\mathcal{D}_0$ with (5.2), we have

$$\begin{aligned} &\mathcal{E}(t) + 2\int_0^t \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) dv ds = \mathcal{E}(0) + 4\int_0^t \int_\Omega \big[\beta_{ij} \dot{e}_{ij}(s) \theta(s) \\ &+ C_{ij} \dot{\kappa}_{ij}(s) \theta(s) + \xi_i \dot{\gamma}_i(s) \theta(s) + \zeta \dot{\psi}(s) \theta(s)\big] dv ds, \quad \forall\, t \in [0, T). \end{aligned} \tag{5.3}$$

Using Schwarz's inequality, the arithmetic-geometric mean inequality, the divergence theorem combined with relations (2.5), (2.9), (2.13), (2.15)–(2.16) and (4.3), we obtain a convenient approximation for the integral in the right-hand side of (5.3)

$$\begin{aligned} &4\int_0^t \int_\Omega [\beta_{ij} \dot{e}_{ij}(s) \theta(s) + C_{ij} \dot{\kappa}_{ij}(s) \theta(s) + \xi_i \dot{\gamma}_i(s) \theta(s) + \zeta \dot{\psi}(s) \theta(s)] dv ds \\ &\leq 2\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \varepsilon_2} \int_0^t \int_\Omega \left[\rho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + 3J \dot{\psi}^2(s)\right] dv ds \\ &+ \frac{2T_0}{k_m} \left(\varepsilon_1 \delta^{*2} \lambda^{-1} + \varepsilon_2 \delta^2\right) \int_0^t \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) dv ds. \end{aligned} \tag{5.4}$$

for all $\varepsilon_1, \varepsilon_2 > 0, t \in [0,T)$, and λ defined by (4.3). To simplify the above relation, we choose $\varepsilon_1, \varepsilon_2$ and M to be

$$\varepsilon_1 = \frac{k_m \lambda}{4T_0 \delta^{*2}}, \quad \varepsilon_2 = \frac{k_m}{4T_0 \delta^2}, \quad M = \frac{8T_0}{k_m \lambda}\left(\delta^{*2} + \lambda \delta^2\right) \tag{5.5}$$

and thus, substituting relation (5.4) in the integral in the right-hand side of (5.3), we get

$$\begin{aligned} &\mathcal{E}(t) + \int_0^t \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) dv ds \leq \mathcal{E}(0) \\ &+ M \int_0^t \int_\Omega \left[\rho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + 3J \dot{\psi}^2(s)\right] dv ds, \qquad \forall\, t \in [0,T). \end{aligned} \tag{5.6}$$

Because we work under the hypotheses (H_1), (H_2) and (H_3), we can write relation (5.6) in the form

$$\begin{aligned} &\mathcal{E}(t) + \int_0^t \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) dv ds \leq \mathcal{E}(0) \\ &+ M \int_0^t \left\{ \mathcal{E}(s) + \int_0^s \int_\Omega \frac{1}{T_0} k_{ij} \theta_{,i}(z) \theta_{,j}(z) dv dz \right\} ds, \quad \forall\, t \in [0,T). \end{aligned} \tag{5.7}$$

We can easily observe that we are in the hypotheses of the Gronwall's lemma and so the estimate (5.1) holds true and hence the proof is complete. □

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REFERENCES

1. Ames, K.A.; Payne, L.E. – *Stabilizing solutions of the equations of dynamical linear thermoelasticity backward in time*, Stability Appl. Anal. Contin. Media, 1 (1991), 243–260.

2. Ames, K.A.; Straughan, B. – *Non-Standard and Improperly Posed Problems*, Mathematics in Science and Engineering Series, Academic Press, 194, 1997.

3. Chiriţă, S. – *Uniqueness and continuous dependence of solutions to the incompressible micropolar flows forward and backward in time*, Internat. J. Engrg. Sci., 39 (2001), 1787–1802.

4. Chiriţă, S.; Ciarletta, M. – *Spatial behavior in dynamical thermoelasticity backward in time*, The Fourth International Congress on Thermal Stresses THERMAL STRESSES 2001, Osaka, Japan, 485–488.

5. Ciarletta, M. – *On the uniqueness and continuous dependence of solutions in dynamical thermoelasticity backward in time*, J. Thermal Stresses, 25 (2002), 969–984.

6. Eringen, A.C. – *Theory of thermomicrostretch elastic solids*, Internat. J. Engrg. Sci., 28 (1990), 1291–1301.

7. Knops, R.J.; Payne, L.E. – *On the uniqueness and continuous data dependence in dynamical problems of linear thermoelasticity*, Internat. J. Solids Structs., 6 (1970), 1173–1184.

8. Passarella, F.; Tibullo, V. – *Some results in linear theory of thermoelasticity backward in time for microstretch materials*, J. Therm. Stresses, 33 (2010), 559–576.

9. Quintanilla, R.; Straughan, B. – *Energy bounds for some non-standard problems in thermoelasticity*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), 1147–1162.

10. Serrin, J. – *The initial value problem for the Navier-Stokes equations*, 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), pp. 69–98 Univ. of Wisconsin Press, Madison, Wis.

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