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ATTRACTORS OF INFINITE ITERATED FUNCTION SYSTEMS CONTAINING CONTRACTION TYPE FUNCTIONS

BY

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Abstract. We consider a complete ε-chainable metric space (X, d) and an infinite iterated function system (IIFS) formed by an infinite family of (ε, φ)-functions on X. The aim of this paper is to prove the existence and uniqueness of the attractors of such infinite iterated systems (IIFS) and to give some sufficient conditions for these attractors to be connected. Similar results are obtained in the case when the IIFS is formed by an infinite family of uniformly ε-locally strong Meir-Keeler functions.

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1. Introduction

Iterated function systems (IFS) were introduced in their present form by Hutchinson [10] and popularized by Barnsley [1] and Falconer [6], [7]. Also, infinite iterated function systems (IIFS) were first mention in [25] and the dimension of the attractors of (IIFS) were studied in [16]. Miculescu and Mihail [17] studied the shift space associated to attractors of (IIFS), which are nonempty closed and bounded subsets of complete metric spaces. Leśniak [14] presented a multivalued approach of infinite iterated function systems. Other results on infinite iterated function systems were obtained in [4], [8], [18], [20], [23]. Chiţescu and Miculescu [2] presented an example of a fractal, generated by Hutchinson's procedure, embedded in an infinite dimensional Banach space. DUMITRU and MIHAIL [3] constructed the shift space of an IFS consisting of ε-locally Meir-Keeler functions with $\varepsilon > 0$.

In this paper we give the existence and uniqueness of the attractors of infinite iterated function formed by (ε, φ)-functions, where $\varepsilon > 0$ and φ is a comparation function, or formed by a family of uniformly strong Meir-Keeler functions and we prove sufficient conditions for these attractors to become connected.

For a nonempty set X we will denote by $\mathcal{P}^*(X)$ the set of nonempty subsets of X, by $\mathcal{K}^*(X)$ the set of nonempty compact subsets of X and by $\mathcal{B}^*(X)$ the set of nonempty bounded closed subsets of X. We have the following well known definitions.

Definition 1.1. Let (X, d) be a metric space. The *generalized Hausdorff-Pompeiu semidistance* is the application $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, +\infty]$ defined by $h(A, B) = \max\{d(A, B),\, d(B, A)\} = \inf\{r \in [0, \infty] \mid A \subset B(B, r)$ and $B \subset B(A, r)\}$, where $d(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A}(\inf_{y\in B} d(x, y))$.

In this paper, by $\mathcal{K}^*(X)$ or $\mathcal{B}^*(X)$ we will refer to $(\mathcal{K}^*(X), h)$ or $(\mathcal{B}^*(X), h)$.

Definition 1.2. Let (X, d) be a metric space. For a function $f : X \to X$ let us denote by $Lip(f) \in [0, +\infty]$ the *Lipschitz constant* associated to f, which is

$$Lip(f) = \sup_{x,y\in X;\, x\neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

We say that f is a *Lipschitz function* if $Lip(f) < +\infty$ and a *contraction* if $Lip(f) < 1$.

Theorem 1.1 ([1]). *Let (X, d) be a metric space and $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, \infty]$ the Hausdorff-Pompeiu semidistance. Then:*

1) $(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are metric spaces with $(\mathcal{K}^*(X), h)$ closed in $(\mathcal{B}^*(X), h)$.

2) If (X, d) is complete, then $(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are complete metric spaces.

3) If (X, d) is compact, then $(\mathcal{K}^*(X), h)$ is compact and in this case $\mathcal{B}^*(X) = \mathcal{K}^*(X)$.

4) If (X, d) is separable, then $(\mathcal{K}^*(X), h)$ is separable.

Proposition 1.1 ([24]). *Let (X, d_X) be a metric space.*

1) If H and K are two nonempty subsets of X then $h(H, K) = h(\overline{H}, \overline{K})$.

2) *If* $(H_i)_{i\in I}$ *and* $(K_i)_{i\in I}$ *are two families of nonempty subsets of* X *then*

$$h(\bigcup_{i\in I} H_i, \bigcup_{i\in I} K_i) = h(\overline{\bigcup_{i\in I} H_i}, \overline{\bigcup_{i\in I} K_i}) \leq \sup_{i\in I} h(H_i, K_i).$$

3) *If* H *and* K *are two nonempty subsets of* X *and* $f : X \to X$ *is a Lipschitz function then* $h(f(K), f(H)) \leq Lip(f) \cdot h(K, H)$.

4) *If* $(H_n)_{n\geq 1} \subset \mathcal{P}(X)$ *is a sequence of sets and* $H \in \mathcal{P}(X)$ *is a set such that* $h_X(H, H_n) \to 0$, *then an element* $x \in X$ *is in* H *if and only if there exists* $x_n \in H_n$ *such that* $x_n \to x$.

Definition 1.3. A family of continuous functions $(f_i)_{i\in I}$, $f_i : X \to X$ for every $i \in I$, is said to be *bounded* if for every bounded set $A \subset X$ the set $\bigcup_{i\in I} f_i(A)$ is bounded.

Definition 1.4. An *infinite iterated function system* (*IIFS*) on X consists of a bounded family of continuous functions $(f_i)_{i\in I}$ on X and it is denote by $\mathcal{S} = (X, (f_i)_{i\in I})$. When I is finite we obtain the *iterated function systems* (*IFS*).

Definition 1.5. For an (IIFS) $\mathcal{S} = (X, (f_i)_{i\in I})$, the *fractal operator* $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ is the function defined by $F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$ for every $B \in \mathcal{B}^*(X)$.

Remark 1.1. Let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an (IIFS). If the functions f_i are contractions, for every $i \in I$ with $\sup_{i\in I} Lip(f_i) < 1$, then the function $F_{\mathcal{S}}$ is a contraction and verifies $Lip(F_{\mathcal{S}}) \leq \sup_{i\in I} Lip(f_i) < 1$.

Using the Banach's contraction theorem we can prove the following very known result:

Theorem 1.2 ([10])**.** *Let* (X, d) *be a complete metric space and* $S = (X, (f_i)_{i\in I})$ *an* (*IIFS*) *with* $(f_i)_{i\in I}$ *a bounded family of contractions such that* $c = \sup_{i\in I} Lip(f_i) < 1$. *Then there exists a unique set* $A(S) \in \mathcal{B}^*(X)$ *such that* $F_{\mathcal{S}}(A(S)) = A(S)$. *Moreover, for any* $H_0 \in \mathcal{B}^*(X)$ *the sequence* $(H_n)_{n\geq 0}$ *defined by* $H_{n+1} = F_{\mathcal{S}}(H_n)$ *is convergent to* $A(S)$. *For the speed of the convergence we have the following estimation:*

$$h(H_n, A(S)) \leq \frac{c^n}{1-c} h(H_0, H_1).$$

Definition 1.6. The unique set $A(\mathcal{S}) \in \mathcal{B}^*(X)$ from Theorem 1.2 is called the *attractor* of (IIFS).

Definition 1.7. Let (X, d) be a metric space and $(A_i)_{i\in I}$ be a family of nonempty subsets of X. The family $(A_i)_{i\in I}$ is said to be *connected* if for every $i, j \in I$, there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_0 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$, for every $k \in \{1, \dots, n-1\}$.

It is also well-known the following result:

Lemma 1.1. *Let (X, d) be a metric space and $(A_i)_{i\in I}$ a connected family. If A_i is a connected set for every $i \in I$, then $\bigcup_{i\in I} A_i$ is connected.*

2. Existence and uniqueness of the attractors of (IIFS) formed by contraction type functions

In this paper we will generalize the notion of infinite iterated function system by considering other contraction type functions which form the IIFS, such as (ε, φ)-functions or strong Meir-Keeler, instead of considering contraction functions. For some cases we will prove the existence and uniqueness of the attractor for these kind of IIFS.

First we start by a remark concerning Theorem 1.2.

Remark 2.1. In the conditions of Theorem 1.2, if we suppose that $c = \sup_{i\in I} Lip(f_i) = 1$ we do not obtain the uniqueness of the attractor, as one can see from the following example:

Example 2.1. We consider the countable iterated function system $S_1 = (\mathbb{R}, (f_n)_{n\in\mathbb{N}^*})$ where $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = (1 - \frac{1}{n})x$, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$. Let $a, b \in \mathbb{R}$ such that $a \leq 0 \leq b$. Then the fractal operator $F_{S_1} : \mathcal{B}^*(\mathbb{R}) \to \mathcal{B}^*(\mathbb{R})$ satisfies $F_{S_1}([a, b]) = \overline{\bigcup_{n\in\mathbb{N}^*} f_n([a, b])} = \overline{\bigcup_{n\in\mathbb{N}^*} [a - \frac{a}{n}, b - \frac{b}{n}]} = [a, b]$. Thus the intervals $[a, b]$ with $a \leq 0 \leq b$ are all attractors of S_1.

Definition 2.1. A metric space (X, d) is said to be *ε-chainable* (for $\varepsilon > 0$ fixed) if, for every $a, b \in X$, there exists an ε-chain from a to b in X, noted $x_0, x_1, \dots, x_n$, such that $x_0 = a$, $x_n = b$ and $d(x_{i-1}, x_i) < \varepsilon$, for all $i \in \{1, \dots, n\}$.

Lemma 2.1 ([3])**.** *Let (X, d) be a ε-chainable metric space, $\varepsilon > 0$. Then $(\mathcal{K}^*(X), h)$ is an ε-chainable metric space.*

Definition 2.2. A metric space (X,d) is said to be *uniformly ε-chainable* (for $\varepsilon > 0$ fixed) if, for every $M > 0$ there is $n_{M,\varepsilon} \in \mathbb{N}^*$ such that for every $a, b \in X$ with $d(a,b) < M$, there exist $x_0, x_1, \ldots, x_n \in X$, with $x_0 = a$, $x_n = b$, $n \leq n_{M,\varepsilon}$ and $d(x_{i-1}, x_i) < \varepsilon$, for every $i \in \{1, \ldots, n\}$.

Lemma 2.2. *Let (X,d) be an uniformly ε-chainable metric space where $\varepsilon > 0$. Then $(\mathcal{B}^*(X), h)$ is ε_1-chainable, for every $\varepsilon_1 > \varepsilon$.*

Proof. Let $A, B \in \mathcal{B}^*(X)$ be two bounded closed sets such that $h(A,B) < M$, where $M > 0$. For every $a \in A$ there is $b_a \in B$ such that $d(a, b_a) < M$. Thus one can choose a chain $x_0(a) = a, x_1(a), \ldots, x_{n_M}(a) = b$ such that $d(x_i(a), x_{i+1}(a)) < \varepsilon$, for every $i \in \{0, \ldots, n_M - 1\}$.

Similar, for every $b \in B$ there is $a_b \in A$ such that $d(b, a_b) < M$. Thus one can choose a chain $y_0(b) = a_b, y_1(b), \ldots, y_{n_M}(b) = b$ such that $d(y_i(b), y_{i+1}(b)) < \varepsilon$, for every $i \in \{0, \ldots, n_M - 1\}$.

We consider now the sets $A_0 = A$, $A_i = \overline{\{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}}$ for $i \in \{1, \ldots, n_M - 1\}$ and $B = A_{n_M}$. Then

$$
\begin{aligned}
&h(A_i, A_{i+1})\\
&= h\left(\overline{\{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}}, \overline{\{\bigcup_{a\in A} x_{i+1}(a)\}_{aA} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}}\right)\\
&= h\left(\{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}, \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}\right)\\
&= \max\left\{d\left(\{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}, \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}\right),\right.\\
&\qquad \left. d\left(\{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}, \{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}\right)\right\}.
\end{aligned}
$$

Hence, we have that:

$$
\begin{aligned}
\varepsilon > d(x_i(a), x_{i+1}(a)) &\geq \inf_{x \in \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}} d(x_i(a), x)\\
&= d\left(x_i(a), \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}\right)
\end{aligned}
$$

and also

$$\varepsilon > d(y_i(b), y_{i+1}(b)) \geq \inf_{x\in\{\bigcup\limits_{a\in A} x_{i+1}(a)\}\cup\{\bigcup\limits_{b\in B} y_{i+1}(b)\}} d(y_i(b), x)$$
$$= d\left(y_i(b), \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}\right), \ \forall i \in \{0,..,n_M\}.$$

Thus $d(\{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}, \{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}) \leq \varepsilon$. Similar $d(\{\bigcup_{a\in A} x_{i+1}(a)\} \cup \{\bigcup_{b\in B} y_{i+1}(b)\}, \{\bigcup_{a\in A} x_i(a)\} \cup \{\bigcup_{b\in B} y_i(b)\}) \leq \varepsilon$. Hence $h(A_i, A_{i+1}) \leq \varepsilon < \varepsilon_1$.

By definition we have that A_i is a closed subset of X, for every $i \in \{0, \dots, n_M\}$. We will prove now that A_i is also a bounded set for every $i \in \{0, \dots, n_M\}$.

Let $t_1, t_2 \in A$, $u_1, u_2 \in B$ and $i \in \{0, \dots, n_M\}$. Then:

$$\begin{aligned}
d(x_i(t_1), x_i(t_2)) &\leq d(x_i(t_1), t_1) + d(t_1, t_2) + d(t_2, x_i(t_2)) \\
&\leq i\varepsilon + \mathrm{diam}(A) + i\varepsilon < \infty, \\
d(y_i(u_1), y_i(u_2)) &\leq d(y_i(u_1), u_1) + d(u_1, u_2) + d(u_2, y_i(u_2)) \\
&\leq (n_m - i - 1)\varepsilon + \mathrm{diam}(B) + (n_M - i - 1)\varepsilon < \infty, \\
d(x_i(t_1), y_i(u_1)) &\leq d(x_i(t_1), t_1) + d(t_1, u_1) + d(u_1, y_i(u_1)) \\
&\leq i\varepsilon + h(A, B) + \mathrm{diam}(A) + \mathrm{diam}(B) + (n_M - i - 1)\varepsilon < \infty.
\end{aligned}$$

Thus A_i is a bounded set for every $i \in \{0, \dots, n_M\}$. Hence $\mathcal{B}^*(X)$ is ε_1-chainable. $\square$

Remark 2.2. If (X, d) is uniformly ε-chainable, then $(\mathcal{B}^*(X), h)$ is not necessary ε-chainable for an $\varepsilon > 0$, as one can see from the example below:

Example 2.2. We consider the Hilbert space $l^2(\mathbb{N}^*) = \{x = (x_n)_{n\geq 1} \mid \sum_{n\geq 1} x_n^2 < \infty\}$ and $\{e_1, \dots, e_n, ..\}$ an orthonormal base. By definition, the interval $[a, b] = \{ta + (1-t)b \mid t \in [0, 1]\}$. Let $x \in [e_i, e_{i+1}]$ and $y \in [e_j, e_{j+1}]$ with $j > i + 1 > 0$. Then $d(x, y) = \sqrt{t^2 + (1-t)^2 + s^2 + (1-s)^2} \geq 1$ since $\alpha^2 + (1-\alpha)^2 \geq \frac{1}{2}$, for every $\alpha > 0$. Thus, we obtain that $\inf_{t,s\in[0,1]} d(x, y) = \inf_{t,s\in[0,1]} \sqrt{t^2 + (1-t)^2 + s^2 + (1-s)^2} = 1$.

We consider now the space $X = \bigcup_{i\geq 1}[e_i, e_{i+1}] \subset l^2(\mathbb{N}^*)$ endowed with the metric induced by $l^2(\mathbb{N}^*)$, $d(x, y) = \sqrt{\sum_{n\geq 1}(x_n - y_n)^2}$. Then X is closed and bounded. We will prove now that X is uniformly 1-chainable. For that let $x, y \in X$ such that $d(x, y) < M$. We have that $x \in [e_i, e_{i+1}]$ and $y \in [e_{i+k}, e_{i+k+1}]$. Then

$$d(x, y) = d(x, e_{i+1}) + (k - 1) + d(e_{i+k}, y) < M,$$

which implies $k < M - 1$. Thus, we can take $k = [M] - 1$, where $[M]$ is the entire part of M, and, so, there exist $(x_i)_{i=\overline{0,k}} \subset X$ such that $x_0 = x$, $x_k = y$ and $d(x_i, x_{i+1}) < 1$, for every $i \in \{0, \ldots, k\}$. Thus X is uniformly 1-chainable.

We will prove now that $(\mathcal{B}^*(X), h)$ is not 1-chainable. We consider the following two closed bounded sets of X : $A = \{e_1\}$ and $B = X$. We assume that there exist the closed and bounded sets $\{A_i\}_{i=\overline{0,m}}$ such that $A_0 = A$, $A_m = B$ and $h(A_i, A_{i+1}) < 1$, which implies $d(A_i, A_{i+1}) < 1$ and $d(A_{i+1}, A_i) < 1$, for every $i \in \{0, \ldots, m - 1\}$. For $i = 0 : d(e_1, A_1) < 1$ implies $d(e_1, x) < 1$, for every $x \in A_1$. Thus $A_1 \subset [e_1, e_2]$. For $i = 1$: $d(A_1, A_2) < 1$ implies the existence of $y_x \in A_1$ such that $d(x, y_x) < 1$, for every $x \in A_2$. Thus $A_2 \subset [e_1, e_2] \cup [e_2, e_3]$.

Inductively, one can prove that $A_j \subset \bigcup_{i=1}^{j} [e_i, e_{i+1}]$, for every $j \in \{1, \ldots, m-1\}$. Then $h(A_{m-1}, A_m) = h(A_{m-1}, X) = \infty$ since for $x \in [e_{m+k}, e_{m+k+1}] \subset X$ we have that $\inf_{y \in A_{m-1}} d(x, y) > k$ which implies:

$$h(X, A_{m-1}) \geq d(X, A_{m-1}) = \sup_{x \in X} \inf_{y \in A_{m-1}} d(x, y) > k$$

Since this is true for every $k \in \mathbb{N}^*$, we have that $h(X, A_{m-1}) = \infty$. So $(\mathcal{B}^*(X), h)$ is not 1-chainable.

Definition 2.3. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparation function* if it is an increasing, continuous to the right function satisfying $\varphi(x) < x$, for every $x \in (0, \infty)$.

Remark 2.3. Let $\varphi : [0, \infty) \to [0, \infty)$ be a comparation function. Then, for every nonempty bounded set $A \subset [0, \infty)$, we have $\sup \varphi(A) \leq \varphi(\sup A)$ and $\inf \varphi(A) = \varphi(\inf A)$.

Definition 2.4. We give some contractive-type conditions. For a function $f : X \to X$ we consider the following conditions:

(1) *α-contraction condition*: there exists $\alpha \in [0, 1)$ such that for $x, y \in X$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

(2) *ε-locally contraction condition* (*where* $\varepsilon > 0$): there exists $\alpha \in [0, 1)$ such that for $x, y, \in X$ with $d(x, y) < \varepsilon$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

(3) *φ-function for a comparation function φ*: for any $x, y \in X$ we have $d(f(x), f(y)) \leq \varphi(d(x, y))$.

(4) *(ε, φ)-function for a comparation function φ (where $\varepsilon > 0$)*: for every $x, y \in X$ such that $d(x, y) < \varepsilon$ we have $d(f(x), f(y)) \leq \varphi(d(x, y))$.

(5) *contractive*: for every $x, y \in X$, $x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

(6) *non-expansive*: for every $x, y \in X$ we have $d(f(x), f(y)) \leq d(x, y)$.

(7) *strong Meir-Keeler*: for each $0 < \eta$ there is $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$.

(8) *strong ε-locally Meir-Keeler (where $\varepsilon > 0$)*: for each $0 < \eta < \varepsilon$ there is $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \eta + \delta$ we have $d(f(x), f(y) < \eta$.

Remark 2.4. a) Condition (1) is equivalent with condition (2) satisfied for every $\varepsilon > 0$.

b) Condition (3) is equivalent with condition (4) satisfied for every $\varepsilon > 0$.

c) Condition (7) is equivalent with condition (8) satisfied for every $\varepsilon > 0$.

d) Every strong Meir-Keeler function is a φ-function for a comparation function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(r) = \sup\{d(f(x), f(y))|\ d(x, y) \leq r\}$.

Definition 2.5. A family of functions $(f_i)_{i \in I}$, $f_i : X \to X$ is said to be *uniformly strong Meir-Keeler*, if for every $\eta > 0$ there is $\delta > 0$ and $\lambda > 0$ such that, for $x, y \in X$ with $d(x, y) < \eta + \delta$, we have $d(f_i(x), f_i(y)) \leq \eta - \lambda$, for every $i \in I$.

Definition 2.6. A family of functions $(f_i)_{i \in I}$, $f_i : X \to X$ is said to be *uniformly strong ε-locally Meir-Keeler*, if for every $\eta \in (0,\ \varepsilon)$ there is $\delta > 0$ and $\lambda > 0$ such that for $x, y \in X$ with $d(x, y) < \eta + \delta$ we have $d(f_i(x), f_i(y)) \leq \eta - \lambda$, for every $i \in I$.

Theorem 2.1. *Let (X, d) be a complete metric space and $\mathcal{S}=(X, (f_k)_{k \in I})$ be an (IIFS) on X and $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ is the function defined by $F_{\mathcal{S}}(B) = \overline{\bigcup_{i \in I} f_i(B)}$, for every $B \in B^*(X)$. Then the followings are true:*

1) *If the function f_k is a contraction for every $k \in I$ such that $c = \sup_{i \in I} Lip(f_i) < 1$, then $F_{\mathcal{S}}$ is a contraction with $Lip(F_{\mathcal{S}}) \leq \sup_{k \in I} Lip(f_k)$.*

2) *If the function f_k is ε-locally contraction for every $k \in I$ such that $c = \sup_{i \in I} Lip(f_i) < 1$, then $F_{\mathcal{S}}$ is a ε-locally contraction with $Lip(F_{\mathcal{S}}) \leq \sup_{k \in I} Lip(f_k)$.*

3) *If the function f_k is φ-function for every $k \in I$, for a comparation function φ, then $F_{\mathcal{S}}$ is a φ-function.*

4) If the function f_k is (ε, φ)-function for every $k \in I$, for a comparation function φ, then $F_{\mathcal{S}}$ is a (ε, φ)-function.

5) If the function f_k is contractive for every $k \in I$, then $F_{\mathcal{S}}$ is non-expansive.

Proof. 1) Let $A, B \in \mathcal{B}^*(X)$. Then:

$$h(F_S(A), F_S(B)) = h(\overline{\bigcup_{i\in I} f_i(A)}, \overline{\bigcup_{i\in I} f_i(B)}) = h(\bigcup_{i\in I} f_i(A), \bigcup_{i\in I} f_i(B))$$
$$\leq \sup_{i\in I} h(f_i(A), f_i(B)) \leq \sup_{i\in I}(Lip(f_i) \cdot h(A, B)) = c \cdot h(A, B), \text{ with } c < 1.$$

So F_S is a contraction with $Lip(F_S) \leq c = \sup_{i\in I} Lip(f_i) < 1$.

2) The point 2) is a particular case of 4).

3) Let $A, B \in \mathcal{B}^*(X)$. Then:

$$h(F_S(A), F_S(B)) = h(\overline{\bigcup_{i\in I} f_i(A)}, \overline{\bigcup_{i\in I} f_i(B)}) = h(\bigcup_{i\in I} f_i(A), \bigcup_{i\in I} f_i(B))$$
$$\leq \sup_{i\in I} h(f_i(A), f_i(B)) = \sup_{i\in I} \max\{d(f_i(A), f_i(B)), d(f_i(B), f_i(A))\}.$$

But

$$d(f_i(A), f_i(B)) = \sup_{x\in A} \inf_{y\in B} d(f_i(x), f_i(y)) \leq \sup_{x\in A} \inf_{y\in B} \varphi(d(x, y))$$
$$= \sup_{x\in A} \varphi(\inf_{y\in B} d(x, y)) \leq \varphi(\sup_{x\in A} \inf_{y\in B} d(x, y)) = \varphi(d(A, B)).$$

Similar, $d(f_i(B), f_i(A)) \leq \varphi(d(B, A))$. Thus, because φ is increasing we have

$$h(F_S(A), F_S(B)) \leq \sup_{i\in I} \max\{\varphi(d(A, B)), \varphi(d(B, A))\}$$
$$= \varphi(\max\{d(A, B)), d(B, A)\}) = \varphi(h(A, B)).$$

4) Let $A, B \in \mathcal{B}^*(X)$ such that $h(A, B) < \varepsilon$. We will prove that

$$h(F_S(A), F_S(B)) \leq \varphi(h(A, B)).$$

We have that

$$h(F_S(A), F_S(B)) = h(\overline{\bigcup_{i\in I} f_i(A)}, \overline{\bigcup_{i\in I} f_i(B)}) = h(\bigcup_{i\in I} f_i(A), \bigcup_{i\in I} f_i(B))$$
$$\leq \sup_{i\in I} h(f_i(A), f_i(B))$$
$$= \sup_{i\in I}\{\max\{\sup_{x\in A} \inf_{y\in B} d(f_i(x), f_i(y)); \sup_{x\in B} \inf_{y\in A} d(f_i(x), f_i(y))\}\}.$$

Because $h(A,B) < \varepsilon$ we have that

$$\max\{\sup_{x\in A}\inf_{y\in B} d(x,y); \sup_{x\in B}\inf_{y\in A} d(x,y)\} < \varepsilon.$$

This means that $\sup_{x\in A}\inf_{y\in B} d(x,y) < \varepsilon$ and $\sup_{x\in B}\inf_{y\in A} d(x,y) < \varepsilon$ or, $\inf_{y\in B} d(x,y) < \varepsilon, \forall x \in A$ and $\inf_{y\in A} d(x,y) < \varepsilon, \forall x \in B$. Thus, for every $x \in A$, there is an $y_x \in B$ such that $d(x,y_x) < \varepsilon$. So $d(f(x),f(y_x)) \leq \varphi(d(x,y_x))$ and $\inf_{y\in B} d(f(x),f(y)) \leq \inf_{y\in B; d(x,y)<\varepsilon} d(f(x),f(y)) \leq \inf_{y\in B; d(x,y)<\varepsilon} \varphi(d(x,y)) \leq \inf_{y\in B} \varphi(d(x,y))$, for every $x \in A$. This implies that $\sup_{x\in A}\inf_{y\in B} d(f(x),f(y)) \leq \sup_{x\in A}\inf_{y\in B} \varphi(d(x,y))$.

In the same way, we get that $\inf_{y\in A} d(f(x),f(y)) \leq \inf_{y\in A} \varphi(d(x,y))$, for every $x \in B$ and $\sup_{x\in B}\inf_{y\in A} d(f(x),f(y)) \leq \sup_{x\in B}\inf_{y\in A} \varphi(d(x,y))$. Thus we have the followings:

$$\begin{aligned} h(F_S(A),F_S(B)) &= \sup_{i\in I}\{\max\{\sup_{x\in A}\inf_{y\in B} d(f_i(x),f_i(y)); \sup_{x\in B}\inf_{y\in A} d(f_i(x),f_i(y))\} \\ &\leq \sup_{i\in I}\{\max\{\sup_{x\in A}\inf_{y\in B} \varphi(d(x,y)); \sup_{x\in B}\inf_{y\in A} \varphi(d(x,y))\}. \end{aligned}$$

Hence

$$\begin{aligned} h(F_S(A),F_S(B)) &\leq \sup_{i\in I}\{\max\{\varphi(\sup_{x\in A}\inf_{y\in B} d(x,y)); \varphi(\sup_{x\in B}\inf_{y\in A} d(x,y))\} \\ &\leq \sup_{i\in I} \varphi(\max\{\sup_{x\in A}\inf_{y\in B} d(x,y); \sup_{x\in B}\inf_{y\in A} d(x,y)\}) \\ &= \sup_{i\in I} \varphi(h(A,B)) = \varphi(h(A,B)), \end{aligned}$$

which completes the proof.

5) If f_k is contractive then $Lip(f_k) \leq 1$. But from point 1) $Lip(F_S) \leq \sup_{k\in I} Lip(f_k) \leq 1$ and thus F_S is non-expansive. □

Remark 2.5 ([21, 22])**.** In the case when X is compact we have the following results:

a) If f_k is contractive for every $k \in I$, then F_S is contractive.

b) If f_k is contractive for every $k \in I$ and I is finite, then f_k is a φ-function with common comparation function for every $k \in I$.

c) The same results also hold on Atsuji spaces. For more details, one can see [12].

Lemma 2.3 ([21, 22])**.** *Let $A, B \in \mathcal{B}^*(X)$. Then for each $\gamma > 0$ and $a \in A$ there exists $b \in B$, such that $d(a,b) \leq h(A,B) + \gamma$.*

Theorem 2.2. *Let (X, d) be a complet space, $S = (X, (f_i)_{i\in I})$ an (IIFS) and the fractal operator $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ is the function defined by $F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$, for every $B \in \mathcal{B}^*(X)$. Then the followings are true:*

1) *If the family of functions $(f_i)_{i\in I}$ is uniformly strong Meir-Keeler, then $F_{\mathcal{S}}$ is a strong Meir-Keeler function.*

2) *If the family of functions $(f_i)_{i\in I}$ is uniformly strong ε-locally Meir-Keeler, then $F_{\mathcal{S}}$ is a strong ε-locally Meir-Keeler function, $\varepsilon > 0$.*

Proof. 1) is equivalent to 2) satisfied for every $\varepsilon > 0$.

2) Let $\varepsilon > 0$ and $0 < \eta < \varepsilon$. Then there exists $\delta > 0$ such that for every $x, y \in X$ such that $d(x, y) < \eta + \delta$, we have $d(f_i(x), f_i(y)) \le \eta - \lambda(\varepsilon)$. Let $A, B \in \mathcal{B}^*(X)$ and $\gamma > 0$ such that $h(A, B) + \gamma < \eta + \delta$. We will prove that $h(F_S(A), F_S(B) < \eta$. We have that $h(F_S(A), F_S(B)) = h(\overline{\bigcup_{i\in I} f_i(A)}, \overline{\bigcup_{i\in I} f_i(B)}) = h(\bigcup_{i\in I} f_i(A), \bigcup_{i\in I} f_i(B))$

Let $z \in \bigcup_{i\in I} f_i(A)$. Then there exists $i \in I$ and $x \in A$ such that $z = f_i(x)$. Using Lemma 2.3 there exists $y \in B$ such that $d(x, y) \le h(A, B)+\gamma < \eta+\delta$. Because $d(x, y) < \eta+\delta$, from the hypothesis we get that $d(f_i(x), f_i(y)) \le \eta - \lambda(\varepsilon)$. Then $\inf_{w\in B} d(f_i(x), f_i(w)) = d(z, f_i(B)) \le \eta - \lambda(\varepsilon)$. Hence $d(z, \bigcup_{j\in I} f_j(B)) \le d(z, f_i(B)) \le \eta - \lambda(\varepsilon)$. As z was arbitrarily chosen we have $d(\bigcup_{j\in I} f_j(A), \bigcup_{j\in I} f_j(B)) = \sup_{z\in\bigcup_{j\in I} f_j(A)} d(z, \bigcup_{j\in I} f_j(B)) \le \eta-\lambda(\varepsilon)$. Interchanging the roles of $\bigcup_{j\in I} f_j(A)$ and $\bigcup_{j\in I} f_j(B)$ we also obtain that $d(\bigcup_{j\in I} f_j(B), \bigcup_{j\in I} f_j(A)) \le \eta-\lambda(\varepsilon)$, and hence $h(F_S(A), F_S(B)) \le \eta - \lambda(\varepsilon) < \eta$. □

The following two results are well-known.

Theorem 2.3 ([22])**.** *Let (X, d) be a complete metric space.*

1) *If $f : X \to X$ is a φ-function, then f has a unique fixed point α and, for every $x_0 \in X$ the sequence $(f^{[n]}(x_0))_n$ is convergent to α and moreover $d(f^{[n]}(x_0), \alpha) \le \varphi^{[n]}(d(x_0, \alpha)) \to 0$.*

2) *If $f : X \to X$ is a strong Meir-Keeler function, then f has a unique fixed point α and for every $x_0 \in X$ the sequence $(f^{[n]}(x_0))_n$ is convergent to α and moreover $d(f^{[n]}(x_0), \alpha) \le \varphi^{[n]}(d(x_0, \alpha)) \to 0$.*

Theorem 2.4 ([26])**.** *Let (X, d) be a complete ε-chainable metric space.*

1) *If $f : X \to X$ is an (ε, φ)-function, then f has a unique fixed point α and for every $x_0 \in X$ such that $d(x_0, \alpha) < \varepsilon$ the sequence $(f^{[n]}(x_0))_n$ is convergent to α and moreover $d(f^{[n]}(x_0), \alpha) \le \varphi^{[n]}(d(x_0, \alpha)) \to 0$.*

2) *If $f : X \to X$ is a strong ε-locally Meir-Keeler function, then f has a unique fixed point α and, for every $x_0 \in X$ such that $d(x_0, \alpha) < \varepsilon$*

the sequence $(f^{[n]}(x_0))_n$ *is convergent to* α *and moreover* $d(f^{[n]}(x_0), \alpha) \leq \varphi^{[n]}(d(x_0, \alpha)) \to 0$.

Now we can give the existence and uniqueness of the attractors of infinite iterated function systems containing some contraction type functions.

Theorem 2.5. *Let* (X, d) *be a complete metric space and* $S=(X, (f_i)_{i\in I})$ *an (IIFS), where the family of functions* $(f_i)_{i\in I}$ *is bounded and* $f_i : X \to X$ *are* φ*-functions for every* $i \in I$. *Then the function* $F_{\mathcal{S}} : B^*(X) \to B^*(X)$, $F_{\mathcal{S}}(Y) = \overline{\bigcup_{i\in I} f_i(Y)}$ *is an* φ*-function and has a unique fixed point.*

Theorem 2.6. *Let* (X, d) *be a complete metric space and* $S=(X, (f_i)_{i\in I})$ *an (IIFS), where the family of functions* $(f_i)_{i\in I}$ *is bounded and uniformly strong Meir–Keeler. Then the function* $F_{\mathcal{S}} : B^*(X) \to B^*(X)$, $F_{\mathcal{S}}(Y) = \overline{\bigcup_{i\in I} f_i(Y)}$ *is a strong Meir–Keeler function and has a unique fixed point.*

Theorem 2.7. *Let* (X, d) *be a complete uniformly* ε*-chainable metric space and* $S = (X, (f_i)_{i\in I})$ *an (IIFS), where the family* $(f_i)_{i\in I}$ *is bounded and* $f_i : X \to X$ *are* (ε_1, φ)*-functions for every* $i \in I$, *with* $\varepsilon_1 > \varepsilon > 0$. *Then the function* $F_{\mathcal{S}} : B^*(X) \to B^*(X)$, $F_{\mathcal{S}}(Y) = \overline{\bigcup_{i\in I} f_i(Y)}$ *is an* (ε_1, φ)*-function and has a unique fixed point.*

Theorem 2.8. *Let* (X, d) *be a complete uniformly* ε*-chainable metric space and* $S = (X, (f_i)_{i\in I})$ *an (IIFS), where the family of functions* $(f_i)_{i\in I}$ *is a bounded and uniformly strong* ε_1*-locally Meir-Keeler, with* $\varepsilon_1 > \varepsilon > 0$. *Then the function* $F_{\mathcal{S}} : B^*(X) \to B^*(X)$, $F_{\mathcal{S}}(Y) = \overline{\bigcup_{i\in I} f_i(Y)}$ *is a strong* ε_1*-locally Meir-Keeler function and has a unique fixed point.*

3. Connectedness of the attactors of (IIFS) formed by contraction type functions

In the next section we will prove sufficient conditions for attractors of (IIFS) formed by contraction type functions to be connected. Some topological properties of the attractors of IFSs were studied in [9], [13], [19] and [27]. Also, general aspects of topology can be found in [5]. Next, we will turn our attention towards the connecteness of the attractors of IIFSs. For the connectedness of the attractors of (IFS) we have the following important result:

Theorem 3.1 ([11]). *Let* (X, d) *be a complete metric space,* $S = (X, (f_k)_{k=\overline{1,n}})$ *be an (IFS) with* $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ *and* $A(S)$ *the attractor of* S. *The following are equivalent:*

1) *The family* $(A_i)_{i=\overline{1,n}}$ *is connected where* $A_i = f_i(A(S))$, *for all* $i \in I$.
2) $A(S)$ *is arcwise connected.*
3) $A(S)$ *is connected.*

The following results give a sufficient condition for the connectedness in the IIFSs' case.

Theorem 3.2. *Let* (X, d) *be a complete metric space and* $S = (X, (f_i)_{i\in I})$ *an (IIFS), where the family* $(f_i)_{i\in I}$ *is bounded and* $f_i : X \to X$ *are* φ*-functions for every* $i \in I$. *Let* $A(S)$ *be the attractor of* S *and* $I_j \subset I$, *for every* $j \in J$ *such that:*
1) $I = \bigcup_{j\in J} I_j$.
2) $\bigcup_{j\in J} B_j$ *is connected, where* $B_j = A(S_j)$ *is the attractor of* $S_j = (X, (f_i)_{i\in I_j})$, *for every* $j \in J$.
Then $A(S)$ *is a connected set.*

Proof. Let $C_0 = \bigcup_{j\in J} B_j$ and $(C_n)_n$ be a sequence of subsets of X defined by $C_n = F_{\mathcal{S}}(C_{n-1})$, for every $n \in \mathbb{N}^*$ where $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$, $F_S(B) = \overline{\bigcup_{i\in I} f_i(B)}$ for any $B \in \mathcal{B}^*(X)$. Then F_S is a φ-function.

We will prove that $C_n \subset C_{n+1}$, $\overline{\bigcup_{n\geq 1} C_n} = A(\mathcal{S})$ and by induction that C_n is connected for all n. From the basic properties of the connected sets will result that $A(\mathcal{S})$ is connected.

We prove by induction that $C_n \subset C_{n+1}$. Thus $B_j = F_{\mathcal{S}_j}(B_j) \subset F_{\mathcal{S}}(B_j)$, for every $j \in J$.

It results that $C_0 = \bigcup_{j\in J} B_j \subset \bigcup_{j\in J} F_{\mathcal{S}}(B_j) \subset F_{\mathcal{S}}(\bigcup_{j\in J} B_j) = F_{\mathcal{S}}(C_0) = C_1$. Thus $C \subset C_1 = F_{\mathcal{S}}(C_0)$.

By induction, suppose that $C_n \subset C_{n+1}$. It follows that $F_{\mathcal{S}}(C_n) \subset F_{\mathcal{S}}(C_{n+1})$. That is $C_{n+1} \subset C_{n+2}$.

Let $D = \bigcup_{n\geq 1} C_n$. Because $C_n = F_{\mathcal{S}}(C_{n-1})$, for every $n \in \mathbb{N}^*$ we have from Theorem 1.2 that $C_n \to A(\mathcal{S})$.

Let $n_0 \in N^*$ and $x \in C_{n_0}$. We consider the sequence $(x_n)_{n\geq n_0}$ defined by $x_n = x$, for every $n \geq n_0$. So, for every $n \geq n_0$ we have that $x_n \in C_n$, because $C_{n_0} \subset C_n$. Then from Proposition 1.1, point 4), $x = \lim x_n \in \lim C_n = A(\mathcal{S})$. Because x was arbitrarily chosen in C_{n_0} we have that $C_{n_0} \subset A(\mathcal{S})$. Then $D = \bigcup_{n\geq 1} C_n \subset A(\mathcal{S})$ and so $\overline{D} \subset \overline{A(\mathcal{S})} = A(\mathcal{S})$, because $A(\mathcal{S})$ is closed.

Now, let $a \in A(\mathcal{S})$. From Proposition 1.1, point 4), there exists $(x_n)_{n\geq 1}$ such that $x_n \in C_n \subset D = \bigcup_{n\geq 1} C_n$ for all $n \in \mathbb{N}^*$ and $x_n \to a$. Then $A(\mathcal{S}) \subset \overline{D}$ and it follows that $A(\mathcal{S}) = \overline{D}$.

We prove now that C_n is connected for all n. $C_0 = C$ is connected by hypothesis. Suppose now that C_n is connected for a fixed $n > 0$. We have $C_{n+1} = \overline{\bigcup_{i\in I} f_i(C_n)} = \overline{C_0 \cup (\bigcup_{i\in I} f_i(C_n))}$. To prove that C_{n+1} is connected it is sufficient to prove that $C_0 \cup (\bigcup_{i\in I} f_i(C_n))$ is connected, because the closure of every connected set is connected. We will prove that $C_0 \cap f_i(C_n) \neq \emptyset$, for all $i \in I$.

Let $i \in I = \bigcup_{j\in J} I_j$. It follows that there exists $l(I) \in J$ such that $i \in I_{l(i)}$. Because $B_{l(i)} = A(\mathcal{S}_{l(i)}) \subset C_0 \subset C_n$, for every $i \in I$, it results that $f_i(B_{l(i)}) \subset f_i(C_n)$. But $f_i(B_{l(i)}) \subset F_{\mathcal{S}_{I_{l(i)}}}(B_{l(i)}) = B_{l(i)} \subset C_0$. Then $\emptyset \neq f_i(B_{l(i)}) \subset C_0 \cap f_i(C_n)$, for every $i \in I$. Thus, we have proved that the set $C_0 \cap f_i(C_n)$ is nonempty.

Because C_n is connected it results that $f_i(C_n)$ is connected and then from Lemma 1.1, using the fact that $C_0 \cap f_i(C_n)$ is nonvoid, it follows that $C_0 \cup (\bigcup_{i\in I} f_i(C_n))$ is connected. With this we have proved that C_{n+1} is connected.

Hence, $\bigcap_{n\geq 0} C_n = C_0$ and C_n are connected for every $n \in \mathbb{N}$. This imply that $D = \bigcup_{n\geq 0} C_n$ is connected ant so $A(S) = \overline{D}$ is connected. □

Corollary 3.1. *Let (X, d) be a complete metric space and $S=(X, (f_i)_{i\in I})$ an (IIFS), where the family $(f_i)_{i\in I}$ is bounded and $f_i : X \to X$ are φ-functions for every $i \in I$. Let $A(S)$ be the attractor of S and $I_j \subset I$, for every $j \in J$ such that:*

1) $I = \bigcup_{j\in J} I_j$.

2) *B_j is connected, where $B_j := A(S_j)$ is the attractor of $S_j=(X, (f_i)_{i\in I_j})$, for all $j \in J$.*

3) *The family of sets $(B_j)_{j\in J}$ is connected.*

Then $A(S)$ is a connected set.

Proof. Since $(B_j)_{j\in J}$ is a connected family of connected sets, it follows from Lemma 1.1 that $\bigcup_{j\in J} B_j$ is connected. Therefore, by Theorem 3.2, $A(\mathcal{S})$ is connected. □

Corollary 3.2. *Let (X, d) be a complete metric space and $S=(X, (f_i)_{i\in I})$ an (IIFS), where the family $(f_i)_{i\in I}$ is bounded and $f_i : X \to X$ are φ-functions for every $i \in I$. Let $A(S)$ be the attractor of S and $I_j \subset I$, for every $j \in J$ such that:*

1) $I = \bigcup_{j\in J} I_j$.

2) *I_j are finite for all $j \in J$.*

3) *The families of sets* $(f_i(B_j))_{i\in I_j}$ *are connected, where* $B_j := A(S_j)$ *is the attractor of* $S_j = (X,(f_i)_{i\in I_j})$*, for all* $j \in J$*.*

4) *The family of sets* $(B_j)_{j\in J}$ *is connected.*

Then $A(S)$ *is a connected set.*

Proof. Since I_j is finite then $\mathcal{S}_j = (X,(f_i)_{i\in I_j})$ is an (IFS). Because the family $(f_i(B_j))_{i\in I_j}$ is connected where $B_j = A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j$,then by Theorem 3.1 we have that B_j is connected. This is true for all $j \in J$, because j was arbitrarily chosen. Thus $\bigcup_{j\in J} B_j$ is connected and by Theorem 3.2 $A(\mathcal{S})$ is connected. □

Theorem 3.3. *Let* (X, d) *be a complete uniformly* ε*-chainable metric space and* $S = (X,(f_i)_{i\in I})$ *an* (*IIFS*)*, where the family* $(f_i)_{i\in I}$ *is bounded and* $f_i : X \to X$ *are* (ε_1, φ)*-functions for every* $i \in I$*, with* $\varepsilon_1 > \varepsilon > 0$*. Let* $A(S)$ *be the attractor of* S *and* $I_j \subset I$*, for every* $j \in J$ *such that:*

1) $I = \bigcup_{j\in J} I_j$.

2) $\bigcup_{j\in J} B_j$ *is connected, where* $B_j = A(S_j)$ *is the attractor of* $S_j = (X,(f_i)_{i\in I_j})$*, for every* $j \in J$*.*

3) $h(B_j, A(S)) \le \varepsilon$*, for every* $j \in J$*.*

Then $A(S)$ *is a connected set.*

Proof. Let $C_0 = \bigcup_{j\in J} B_j$ and $(C_n)_n$ be a sequence of subsets of X defined by $C_n = F_{\mathcal{S}}(C_{n-1})$, for every $n \in \mathbb{N}^*$ where $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$, $F_S(B) = \overline{\bigcup_{i\in I} f_i(B)}$ for any $B \in \mathcal{B}^*(X)$. Then F_S is a (ε_1, φ)-function.

We have that $h(C_0, A(S)) = h(\bigcup_{j\in J} B_j, A(S)) \le \sup_{j\in J} h(B_j, A(S)) \le \varepsilon < \varepsilon_1$. Then $h(C_1, A(S)) = h(F_S(C_0), F_S(A(S))) \le \varphi(h(C_0, A(S))) < h(C_0, A(S)) < \varepsilon_1$.

Suppose now that $h(C_n,(A(S))) \le \varphi^{[n]}(h(C_0, A(S))) < \varepsilon_1$ for some $n \in \mathbb{N}$. Then $h(C_{n+1},(A(S))) \le h(F_S(C_n), F_S(A(S))) \le \varphi(\varphi^{[n]}(h(C_0, A(S)))) = \varphi^{[n+1]}(h(C_0, A(S))) < \varphi(\varepsilon_1) < \varepsilon_1$. Thus, $h(C_n,(A(S))) < \varepsilon_1$ for all $n \in \mathbb{N}$ and $h(C_n,(A(S))) \le \varphi^{[n]}(h(C_0, A(S))) \to 0$. Hence $C_n \to A(S)$. From now on, similar to Theorem 3.2, one can prove that $C_n \subset C_{n+1}$, $\overline{\bigcup_{n\ge1} C_n} = A(\mathcal{S})$ and C_n is connected for all $n \in \mathbb{N}$ and that completes the proof. □

Corollary 3.3. *Let* (X, d) *be a complete uniformly* ε_1*-chainable metric space and* $S = (X,(f_i)_{i\in I})$ *an* (*IIFS*)*, where the family* $(f_i)_{i\in I}$ *is bounded* $f_i : X \to X$ *are* (ε_1, φ)*-functions for every* $i \in I$*, with* $\varepsilon_1 > \varepsilon > 0$*. Let* $A(S)$ *be the attractor of* S *and* $I_j \subset I$*, for every* $j \in J$ *such that:*

1) $I = \bigcup_{j\in J} I_j$.

2) *B_j is connected, where B_j:=$A(S_j)$ is the attractor of S_j=$(X,(f_i)_{i\in I_j})$ for all $j \in J$.*

3) *The family of sets $(B_j)_{j\in J}$ is connected.*

4) *$h(B_j, A(S)) \leq \varepsilon$, for every $j \in J$.*

Then $A(S)$ is a connected set.

Proof. Because $(B_j)_{j\in J}$ is a connected family of connected sets, it follows from Lemma 1.1 that $\bigcup_{j\in J} B_j$ is connected. Therefore, by Theorem 3.3, $A(\mathcal{S})$ is connected. □

Corollary 3.4. *Let (X,d) be a complete uniformly ε_1-chainable metric space and $S = (X,(f_i)_{i\in I})$ an (IIFS), where the family $(f_i)_{i\in I}$ is bounded $f_i : X \to X$ are (ε_1, φ)-functions for every $i \in I$, with $\varepsilon_1 > \varepsilon > 0$. Let $A(S)$ be the attractor of S and $I_j \subset I$, for every $j \in J$ such that:*

1) *$I = \bigcup_{j\in J} I_j$.*

2) *I_j are finite, for all $j \in J$.*

3) *The families of sets $(f_i(B_j))_{i\in I_j}$ are connected, where $B_j := A(S_j)$ is the attractor of $S_j = (X,(f_i)_{i\in I_j})$, for all $j \in J$.*

4) *The family of sets $(B_j)_{j\in J}$ is connected.*

5) *$h(B_j, A(S)) \leq \varepsilon$, for every $j \in J$.*

Then $A(S)$ is a connected set.

Proof. Since I_j is finite then $\mathcal{S}_j = (X,(f_i)_{i\in I_j})$ is an (IFS). Because the family $(f_i(B_j))_{i\in I_j}$ is connected, where $B_j := A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j$, then by Theorem 3.1 we have that B_j is connected. This is true for all $j \in J$, because j was arbitrarily chosen, thus $\bigcup_{j\in J} B_j$ is connected and by Theorem 3.3, $A(\mathcal{S})$ is connected. □

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REFERENCES

1. Barnsley, M.F. – *Fractals Everywhere*, Second edition. Revised with the assistance of and with a foreword by Hawley Rising, III. Academic Press Professional, Boston, MA, 1993.

2. Chiţescu, I.; Miculescu, R. – *Approximation of fractals generated by Fredholm integral equations*, J. Comput. Anal. Appl., 11 (2009), 286–293.

3. Dumitru, D.; Mihail, A. – *The shift space of an iterated function system containing Meir-Keeler functions*, An. Univ. Bucureşti Mat., 57 (2008), 75–88.

4. Dumitru, D.; Mihail, A. – *A sufficient condition for the connectedness of the attractors of infinite iterated function systems*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 55 (2009), 87–94.

5. Engelking, R. – *General Topology*, Translated from the Polish by the author. Monografie Matematyczne, Tom 60, PWN-Polish Scientific Publishers, Warsaw, 1977.

6. Falconer, K.J. – *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics, 85, Cambridge University Press, Cambridge, 1986.

7. Falconer, K. – *Fractal Geometry*, Mathematical foundations and applications, John Wiley & Sons, Ltd., Chichester, 1990.

8. Gwóźdź-Łukawska, G.; Jachymski, J. – *The Hutchinson-Barnsley theory for infinite iterated function systems*, Bull. Austral. Math. Soc., 72 (2005), 441–454.

9. Hata, M. – *On the structure of self-similar sets*, Japan J. Appl. Math., 2 (1985), 381–414.

10. Hutchinson, J.E. – *Fractals and self-similarity*, Indiana Univ. Math. J., 30 (1981), 713–747.

11. Kigami, J. – *Analysis on Fractals*, Cambridge Tracts in Mathematics, 143, Cambridge University Press, Cambridge, 2001.

12. Kundu, S.; Jain, T. – *Atsuji spaces: equivalent conditions*, Spring Topology and Dynamical Systems Conference, Topology Proc., 30 (2006), 301–325.

13. Kwieciński, M. – *A locally connected continuum which is not an IFS attractor*, Bull. Polish Acad. Sci. Math., 47 (1999), 127–132.

14. Leśniak, K. – *Infinite iterated function systems: a multivalued approach*, Bull. Pol. Acad. Sci. Math., 52 (2004), 1–8.

15. Máté, L. – *The Hutchinson-Barnsley theory for certain noncontraction mappings*, Period. Math. Hungar., 27 (1993), 21–33.

16. Mauldin, R.D.; Urbański, M. – *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc., 73 (1996), 105–154.

17. Miculescu, R.; Mihail, A. – *Lipscomb's space ω^A is the attractor of an infinite IFS containing affine transformations of $l^2(A)$*, Proc. Amer. Math. Soc., 136 (2008), 587–592

18. Mihail, A.; Miculescu, R. – *The shift space for an infinite iterated function system*, Math. Rep. (Bucur.), 11 (2009), 21–32.

19. Mihail, A. – *On the connectivity of attractors of iterated function systems*, Rocky Mountain J. Math., 40 (2010), 1949–1964.

20. Mihail, A. – *A necessary and sufficient condition for the connectivity of the attractor of an infinite iterated function system*, Rev. Roumaine Math. Pures Appl., 55 (2010), 147–157.

21. Petruşel, A. – *Fixed point theory with applications to dynamical systems and fractals*, International Conference on Nonlinear Operators, Differential Equations and Applications (Cluj-Napoca, 2001), Semin. Fixed Point Theory Cluj-Napoca, 3 (2002), 305–315.

22. Rus, I.A.; Petruşel, A.; Petruşel, G. – *Fixed point theory: 1950-2000. Romanian contributions*, House of the Book of Science, Cluj-Napoca, 2002.

23. Secelean, N.-A. – *Countable iterated function systems*, Far East J. Dyn. Syst., 3 (2001), 149–167.

24. Secelean, N.-A. – *Măsură şi Fractali* (in romanian), Ed Univ. Sibiu, 2002.

25. Wicks, K.R. – *Fractals and Hyperspaces*, Lecture Notes in Mathematics, 1492, Springer-Verlag, Berlin, 1991.

26. Xu, H.K. – *ε-chainability and fixed points of set-valued mappings in metric spaces*, Math. Japon., 39 (1994), 353–356.

27. Yamaguti, M.; Hata, M.; Kigami, J. – *Mathematics of Fractals*, Translations of Mathematical Monographs, 167, American Mathematical Society, Providence, RI, 1997.

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