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# ALEXANDROFF THEOREM IN HAUSDORFF TOPOLOGY FOR NULL-NULL-ADDITIVE SET MULTIFUNCTIONS

BY

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**Abstract.** In this paper we further a previous study concerning abstract regularity for monotone set multifunctions, with has immediate applications in well-known situations such as the Borel $\sigma$-algebra of a Hausdorff space and/or the Borel (Baire, respectively) $\delta$-ring or $\sigma$-ring of a locally compact Hausdorff space. We also study relationships among abstract regularities and other properties of continuity. Especially, a set-valued Alexandroff type theorem is obtained.

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## 1. Introduction and preliminary

Regularity is a very important continuity property. It connects measure theory and topology, approximating general Borel sets by more tractable sets, such as compact and/or open sets.

In the last years, many authors studied different problems and applications of continuity properties, especially regularity, in order to obtain Alexandroff and Lusin type theorems, in convergence problems for Choquet integrals (see Dinculeanu [3] for real normed space-valued measures, Belley and Morales [2], Asahina, Uchino and Murofushi [1], Jiang and Suzuki [14], Pap [23], Ha and Wang [12], Li and Yasuda [19], Li, Yasuda and Song [20], Narukawa [21], Narukawa, Murofushi and Sugeno [22], Song and Li [28], Wu and Ha [29], Wu and Wu [30] for fuzzy measures, Kawabe [15-18] for Riesz space-valued measures etc.).

In the set-valued case, different results were obtained by Guo and Zhang [11], Gavriluţ [4-9], Precupanu [24, 25], Precupanu, Gavriluţ and Croitoru [26], Precupanu and Gavriluţ [27], Zhang and Guo [31], Zhang and Wang [32] and many others).

In this paper, we continue the study [10] concerning different abstract regularities in Hausdorff topology for fuzzy (i.e., monotone) set multifunctions. The relationships among these types of regularity and other properties of continuity (such as, increasing/decreasing convergence, (S)-fuzziness, order continuity) are presented and a set-valued Alexandroff type theorem is obtained.

## 2. Terminology and notations

Let $T$ be an abstract set, $\mathcal{C}$ a ring of subsets of $T$, $X$ a real normed space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of $X, \mathcal{P}_f(X)$ the family of closed, nonvoid sets of $X$, $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of $X$ and $h$ the Hausdorff pseudometric on $\mathcal{P}_f(X)$.

By [13], $h(M,N) = \max\{e(M,N), e(N,M)\}$, for every $M, N \in \mathcal{P}_f(X)$, where $e(M,N) = \sup_{x\in M} d(x,N)$ is *the excess* of $M$ over $N$. On $\mathcal{P}_{bf}(X)$, $h$ becomes a metric.

We denote $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$.

If $X$ is complete, then the same is $\mathcal{P}_f(X)$. We observe that $e(N,M) = h(M,N)$, for every $M,N \in \mathcal{P}_f(X)$, with $M \subseteq N$. Also, $e(M,N) \le e(M,P)$, for every $M,N,P \in \mathcal{P}_f(X)$, with $P \subseteq N$ and $e(M,P) \le e(N,P)$, for every $M,N,P \in \mathcal{P}_f(X)$, with $M \subseteq N$. We denote by $\mathbb{N}$ the set of all naturals, by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+$ the set $[0,\infty)$. Also, by $cA$ we usually mean $T\backslash A$, where $A \subset T$.

Let us first recall from [10] the following notions:

**Definition 2.1.** A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be:

I) *increasing convergent* (with respect to $h$) if $\lim_{n\to\infty} h(\mu(A_n),\mu(A))=0$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}}\subset\mathcal{C}$, with $A_n\nearrow A \in \mathcal{C}$.

II) *decreasing convergent* (with respect to $h$) if $\lim_{n\to\infty} h(\mu(A_n),\mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}}\subset\mathcal{C}$, with $A_n\searrow A \in \mathcal{C}$.

III) i) *fuzzy* (or, *monotone*) if $\mu(A)\subseteq\mu(B)$, for every $A, B\in\mathcal{C}$, with $A\subseteq B$.

ii) *fuzzy in the sense of Sugeno* (briefly, $(S)$-*fuzzy*) if it is fuzzy, increasing convergent, decreasing convergent and $\mu(\emptyset) = \{0\}$.

IV) *exhaustive* (with respect to $h$) if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$.

V) *order continuous* (with respect to $h$) if $\lim_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

VI) *autocontinuous from above* if for every $A \in \mathcal{C}$ and every $(B_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $\lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} h(\mu(A \cup B_n), \mu(A)) = 0$.

VII) *uniformly autocontinuous* if for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have $h(\mu(A \cup B), \mu(A)) < \varepsilon$.

VIII) i) *null-additive* if $\mu(A \cup B){=}\mu(A)$, for every $A, B{\in}\mathcal{C}$, with $\mu(B){=}\{0\}$.

ii) *null-null-additive* if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$.

IX) *single asymptotic null-additive* if for every $A \in \mathcal{C}$ with $\mu(A) = \{0\}$ and every sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $\lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} |\mu(A \cup B_n)| = 0$.

X) *asymptotic null-additive* if for every sequences $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{C}$, with $\lim_{n\to\infty} |\mu(A_n)| = \lim_{n\to\infty} |\mu(B_n)| = 0$, we have $\lim_{n\to\infty} |\mu(A_n \cup B_n)| = 0$.

All over the paper, unless stated otherwise, $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is supposed to be a *fuzzy* (*i.e., monotone*) *set multifunction, with* $\mu(\emptyset) = \{0\}$.

We shall need the following notions and results from [10]:

**Definition 2.2.** We say that $\mu$ has the *pseudometric generating property* (briefly, *PGP*) if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(A)| < \delta$ and $|\mu(B)| < \delta$, we have $|\mu(A \cup B)| < \varepsilon$.

**Definition 2.3.** i) A double sequence $\{A_{m,n}\}_{(m,n)\in\mathbb{N}^2} \subset \mathcal{C}$ is called a *$\mu$-regulator* if:

$(R_1)$ $A_{m,n} \supset A_{m,n'}$, whenever $m, n, n' \in \mathbb{N}$ and $n \le n'$;

$(R_2')$ $\mu(\bigcap_{n=1}^{\infty} A_{m,n}) = \{0\}$, for any $m \in \mathbb{N}$.

ii) $\mu$ is said to fulfil *condition* $(E')$ if for any $\varepsilon > 0$ and any $\mu$-regulator $\{A_{m,n}\}_{(m,n)\in\mathbb{N}^2} \subset \mathcal{C}$, there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of naturals such that $|\mu(\bigcup_{i=1}^{\infty} A_{i,n_i})| < \varepsilon$.

**Remark 2.4.** i) If $\mu$ is asymptotic null-additive, then it is single asymptotic null-additive. If $\mu$ is single asymptotic null-additive, then it is null-null-additive.

ii) If $\mu$ is autocontinuous from above, then $\mu$ is asymptotic null-additive.

iii) If $\mathcal{C}$ is a $\sigma$-ring and $\mu$ is fuzzy, increasing convergent and autocontinuous from above, then $\mu$ has PGP.

iv) a) Any autocontinuous from above order continuous set multifunction is increasing convergent and decreasing convergent.

b) Any decreasing convergent set multifunction is order continuous. Consequently, if $\mu$ is fuzzy and autocontinuous from above, then $\mu$ is (S)-fuzzy if and only if it is order continuous.

v) Let $\mu$ be (S)-fuzzy. Then $\mu$ fulfils condition (E') if and only if it is null-null-additive.

Let $\mathcal{M}, \mathcal{N} \subset \mathcal{P}(T)$ be two arbitrary nonvoid families of subsets of $T$.

As we saw in [10], for the consistency of the following notions, one may place itself in one of the following situations:

(i) $T$ is a Hausdorff space, $\mathcal{C}$ is the Borel $\sigma$-algebra $\widetilde{\mathcal{B}}$ generated by the open sets of $T$, $\mathcal{M} = \mathcal{F}$, the family of closed subsets of $T$ and $\mathcal{N} = \mathcal{D}$, the family of open subsets of $T$ or $\mathcal{M} = \mathcal{K}$, the family of compact subsets of $T$ and $\mathcal{N} = \mathcal{D}$;

(ii) $T$ is, particularly, a locally compact Hausdorff space, $\mathcal{C}$ is $\mathcal{B}_0$ (respectively, $\mathcal{B}'_0$) - the Baire $\delta$-ring (respectively, $\sigma$-ring) generated by compact sets, which are $G_\delta$ (i.e., countable intersections of open sets) or $\mathcal{C}$ is $\mathcal{B}$ (respectively, $\mathcal{B}'$) - the Borel $\delta$-ring (respectively, $\sigma$-ring) generated by the compact sets of $T$, $\mathcal{M} = \mathcal{K}$ and $\mathcal{N} = \mathcal{D}$. Note that $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{B}_0 \subset \mathcal{B}'_0$, $\mathcal{B}'_0 \subset \mathcal{B}'$ and $\mathcal{B} \subset \mathcal{B}'$.

According to the usage of $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{F}$ and $\mathcal{D}$ or $\mathcal{K}$ and $\mathcal{D}$, it will be understood that we place ourselves, respectively, in the general situation, situation (i)/(ii) or situation (ii) (we remark that in situation (i) we may have $\mathcal{F}$ and $\mathcal{D}$ or $\mathcal{K}$ and $\mathcal{D}$ and in situation (ii) we may have $\mathcal{K}$ and $\mathcal{D}$).

We shall particularly study situation (i) when $T$ is a locally compact Hausdorff space.

**Definition 2.5.** I) A set $A \in \mathcal{C}$ is said to be:

(i) $R_{\mathcal{M},\mathcal{N}}$*-regular* if for every $\varepsilon > 0$, there are $M \in \mathcal{M} \cap \mathcal{C}, M \subset A$ and $N \in \mathcal{N} \cap \mathcal{C}, N \supset A$ so that $e(\mu(N), \mu(M)) < \varepsilon$.

(ii) $R_{\mathcal{M}}$*-regular* if for every $\varepsilon > 0$, there exists $M \in \mathcal{M} \cap \mathcal{C}, M \subset A$ so that $e(\mu(A), \mu(M)) < \varepsilon$.

(iii) $R_{\mathcal{N}}$*-regular* if for every $\varepsilon > 0$, there exists $N \in \mathcal{N} \cap \mathcal{C}, N \supset A$ such that $e(\mu(N), \mu(A)) < \varepsilon$.

(iv) $R'_{\mathcal{M},\mathcal{N}}$*-regular* if for every $\varepsilon > 0$, there are $M \in \mathcal{M} \cap \mathcal{C}, M \subset A$ and $N \in \mathcal{N} \cap \mathcal{C}, A \subset N$ so that $|\mu(N \backslash M)| < \varepsilon$.

(v) $R'_{\mathcal{M}}$*-regular* if for every $\varepsilon > 0$, there is $M \in \mathcal{M} \cap \mathcal{C}, M \subset A$ so that $|\mu(A \backslash M)| < \varepsilon$.

(vi) $R'_{\mathcal{N}}$*-regular* if for every $\varepsilon > 0$, there is $N \in \mathcal{N} \cap \mathcal{C}, A \subset N$ such that $|\mu(N \backslash A)| < \varepsilon$.

II) $\mu$ is said to be:

i) $R_{\mathcal{M},\mathcal{N}}$*-regular* (respectively, $R_{\mathcal{M}}$*-regular*, $R_{\mathcal{N}}$*-regular*) if every set $A \in \mathcal{C}$ is $R_{\mathcal{M},\mathcal{N}}$-regular (respectively, $R_{\mathcal{M}}$-regular, $R_{\mathcal{N}}$-regular).

ii) $R'_{\mathcal{M},\mathcal{N}}$*-regular* (respectively, $R'_{\mathcal{M}}$*-regular*, $R'_{\mathcal{N}}$*-regular*) if every set $A \in \mathcal{C}$ is $R'_{\mathcal{M},\mathcal{N}}$-regular (respectively, $R'_{\mathcal{M}}$-regular, $R'_{\mathcal{N}}$-regular).

The reader is refered to [10] for various properties and remarks concerning abstract types of regularity.

## 3. Set-valued Alexandroff type theorem

In this section, we establish different relationships among regularities and other properties of continuity:

**Proposition 3.1.** *Suppose $T$ is a Hausdorff space.*

*i) If $\mathcal{C}$ is a $\sigma$-ring and $\mu$ is $R_{\mathcal{K}}$-regular, then $\mu$ is increasing convergent on $\mathcal{D} \cap \mathcal{C}$.*

*ii) If $\mathcal{C}$ is a $\delta$-ring and $\mu$ is $R_{\mathcal{D}}$-regular, then $\mu$ is decreasing convergent on $\mathcal{K} \cap \mathcal{C}$.*

**Proof.** i) Let $(D_n)_n, D \subset \mathcal{D} \cap \mathcal{C}$ be so that $D_n \nearrow D$. Obviously, $e(\mu(D_n), \mu(D)) = 0$, for every $n \in \mathbb{N}$. Because $\mu$ is $R_{\mathcal{K}}$-regular, for every $\varepsilon > 0$, there is $K \in \mathcal{K} \cap \mathcal{C}, K \subset D$ so that $e(\mu(D), \mu(K)) < \varepsilon$. Since $K \subset D = \bigcup_{n=1}^{\infty} D_n$, there is $n_0 \in \mathbb{N}$ so that $K \subset D_{n_0}$. Consequently, for every $n \geq n_0$,

$$\begin{aligned} e(\mu(D), \mu(D_n)) &\leq e(\mu(D), \mu(K)) + e(\mu(K), \mu(D_{n_0})) + e(\mu(D_{n_0}), \mu(D_n)) \\ &= e(\mu(D), \mu(K)) < \varepsilon, \end{aligned}$$

so $h(\mu(D), \mu(D_n)) < \varepsilon$, for every $n \geq n_0$, which means that $\mu$ is increasing convergent on $\mathcal{D} \cap \mathcal{C}$.

ii) Let $(K_n)_n, K \subset \mathcal{K} \cap \mathcal{C}$ be so that $K_n \searrow K$. Obviously, $e(\mu(K), \mu(K_n)) = 0$, for every $n \in \mathbb{N}$. Because $\mu$ is $R_{\mathcal{D}}$-regular, for every $\varepsilon > 0$, there is $D \in \mathcal{D} \cap \mathcal{C}, K \subset D$ so that $e(\mu(D), \mu(K)) < \varepsilon$.

We demonstrate that there exists $n_0 \in \mathbb{N}$ so that $K \subset K_{n_0} \subset D$. For this, we suppose that $K_n \cap cD \neq \emptyset$, for every $n \in \mathbb{N}$. Since $K_n \cap cD$ is compact, for every $n \in \mathbb{N}$ and $\bigcap_{i=1}^{p} K_i \cap cD = K_p \cap cD \neq \emptyset$, for every $p \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} K_i \cap cD \neq \emptyset$. Consequently, $\emptyset \neq \bigcap_{i=1}^{\infty} K_i \cap cD = K \cap cD = \emptyset$, which is a contradiction. Then, for every $n \geq n_0$,

$$\begin{aligned} e(\mu(K_n), \mu(K)) &\leq e(\mu(K_n), \mu(K_{n_0})) + e(\mu(K_{n_0}), \mu(D)) + e(\mu(D), \mu(K)) \\ &= e(\mu(D), \mu(K)) < \varepsilon, \end{aligned}$$

so, finally, $h(\mu(K_n), \mu(K)) < \varepsilon$, for every $n \geq n_0$, which says that $\mu$ is decreasing convergent on $\mathcal{K} \cap \mathcal{C}$. □

**Remark 3.2.** In the above theorem, $\mathcal{C}$ can be, for instance, the $\sigma$-algebra $\widetilde{\mathcal{B}}$ or, if $T$ is, moreover, locally compact, in i) $\mathcal{C}$ can be the $\sigma$-ring $\mathcal{B}'$ or $\mathcal{B}'_0$ and in ii) $\mathcal{C}$ can be the $\delta$-ring $\mathcal{B}$ or $\mathcal{B}_0$.

If $\mathcal{C}$ is $\widetilde{\mathcal{B}}$ or $\mathcal{B}'$ ($\mathcal{B}'_0$, respectively) - in case when $T$ is locally compact and $\mu$ is $R_{\mathcal{K},\mathcal{D}}$-regular, then $\mu$ is increasing convergent on $\mathcal{D} \cap \mathcal{C}$ and decreasing convergent on $\mathcal{K} \cap \mathcal{C}$.

In the following Alexandroff type result, we prove that, if, moreover, $\mu$ is autocontinuous from above and $R'_{\mathcal{K}}$-regular, then $\mu$ is, moreover, order continuous, hence increasing convergent and decreasing convergent on $\mathcal{C}$, not only increasing convergent on $\mathcal{D} \cap \mathcal{C}$/decreasing convergent on $\mathcal{K} \cap \mathcal{C}$.

**Theorem 3.3** (Alexandroff type theorem)**.** *If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is autocontinuous from above and $R'_{\mathcal{K}}$-regular, then $\mu$ is order-continuous.*

**Proof.** Let $(A_n)_n \subset \mathcal{C}$ be such that $A_n \searrow \emptyset$.

Since $\mu$ is $R'_{\mathcal{K}}$-regular, for every $n \in \mathbb{N}$, there is an increasing sequence $(K_{n,m})_m \subset \mathcal{K} \cap \mathcal{C}$ so that for every $m \in \mathbb{N}$, $K_{n,m} \subset A_n$ and $\lim_{m\to\infty} |\mu(A_n \backslash K_{n,m})| = 0$.

Since $\lim_{m\to\infty} |\mu(A_1\backslash K_{1,m})|$=0, there is $m_1 \in \mathbb{N}$ so that $|\mu(A_1\backslash K_{1,m_1})|< \frac{\varepsilon}{2^1}$. Because $\lim_{m\to\infty} |\mu(A_2\backslash K_{2,m})| = 0$ and $\mu$ is autocontinuous from above, $\lim_{m\to\infty} h(\mu((A_1\backslash K_{1,m_1})\cup(A_2\backslash K_{2,m})), \mu(A_1\backslash K_{1,m_1})) = 0$, so, there is $m_2 \in \mathbb{N}$ such that $|\mu((A_1\backslash K_{1,m_1})\cup(A_2\backslash K_{2,m_2}))| \le h(\mu((A_1\backslash K_{1,m_1})\cup(A_2\backslash K_{2,m_2}))$, $\mu(A_1\backslash K_{1,m_1}))+|\mu(A_1\backslash K_{1,m_1})| < \frac{\varepsilon}{2^1}+\frac{\varepsilon}{2^2}$. Recurrently, there is an increasing sequence of naturals $(m_k)_k$ so that for every $p \in \mathbb{N}$, $|\mu(\bigcup_{i=1}^{p}(A_i\backslash K_{i,m_i}))| < \sum_{i=1}^{p}\frac{\varepsilon}{2^i}$.

Since $\bigcap_{i=1}^{\infty} A_i = \emptyset$, $\bigcap_{i=1}^{\infty} K_{i,m_i} = \emptyset$, so there is $i_0 \in \mathbb{N}$ such that $\bigcap_{i=1}^{i_0} K_{i,m_i} = \emptyset$. Consequently, for every $i \ge i_0, |\mu(A_i)| \le |\mu(A_{i_0})| \le |\mu(\bigcup_{i=1}^{i_0}(A_i\backslash K_{i,m_i}))| < \sum_{i=1}^{i_0}\frac{\varepsilon}{2^i} < \varepsilon$, so $\lim_{i\to\infty} |\mu(A_i)| = 0$ and this means $\mu$ is order continuous. □

By Theorem 3.32 and Remark 2.4 iv), we get:

**Corollary 3.4.** *If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is autocontinuous from above and $R'_{\mathcal{K}}$-regular, then $\mu$ is $(S)$-fuzzy.*

**Remark 3.5.** In Theorem 3.3 and Corollary 3.4, $\mathcal{C}$ can be, for instance, $\widetilde{\mathcal{B}}, \mathcal{B}, \mathcal{B}_0, \mathcal{B}'$ or $\mathcal{B}'_0$ etc.

**Theorem 3.6.** *If $\mathcal{C}$ is a $\sigma$-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is decreasing convergent, $R_{\mathcal{K}}$-regular and has PGP, then $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular.*

**Proof.** We observe that, since $\mu$ is decreasing convergent, then, by Remark 2.5 iv), $\mu$ is exhaustive.

We suppose that, on the contrary, there exists $A_0 \in \mathcal{C}$ and $\varepsilon_0 > 0$ so that, for every $K \in \mathcal{K}\cap\mathcal{C}$ and every $D \in \mathcal{D}\cap\mathcal{C}$, with $K \subset A_0 \subset D$, we have $|\mu(D\backslash K)| \ge \varepsilon_0$.

We fix such arbitrary sets $K_0$ and $D_0$.

Since $\mu$ has PGP, then for $\varepsilon_0 > 0$, there is $\delta_0(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(A)| < \delta_0$ and $|\mu(B)| < \delta_0$, we have $|\mu(A \cup B)| < \varepsilon_0$. Because $\mu$ is $R_{\mathcal{K}}-$regular, for $D_0\backslash A_0$ there is $C'_1 \in \mathcal{K}\cap\mathcal{C}$ so that $C'_1 \subset D_0\backslash A_0$ and $e(\mu(D_0\backslash A_0), \mu(C'_1)) < \frac{\delta_0}{3}$.

Analogously, for $A_0\backslash K_0$ there is $C_1'' \in \mathcal{K} \cap \mathcal{C}$ so that $C_1'' \subset A_0\backslash K_0$ and $e(\mu(A_0\backslash K_0), \mu(C_1'')) < \frac{\delta_0}{3}$.

We denote $D_1 = D_0\backslash C_1'$ and $K_1 = K_0 \cup C_1''$. Then $D_1 \in \mathcal{D}\cap\mathcal{C}$, $K_1 \in \mathcal{K}\cap\mathcal{C}$ and $K_1 \subset A_0 \subset D_1$. Therefore, $|\mu(D_1\backslash K_1)| \geq \varepsilon_0$. By induction we obtain the existence of two sequences $(C_n')_{n\in\mathbb{N}}$, $(C_n'')_{n\in\mathbb{N}} \subset \mathcal{K}\cap\mathcal{C}$ of pairwise disjoint sets such that for every $n \in \mathbb{N}, C_{n+1}' \subset D_n\backslash A_0$, $C_{n+1}'' \subset A_0\backslash K_n$ and

$$e(\mu(D_n\backslash A_0), \mu(C_{n+1}')) < \frac{\delta_0}{3}, e(\mu(A_0\backslash K_n), \mu(C_{n+1}'')) < \frac{\delta_0}{3}.$$

Also, for every $n \in \mathbb{N}, D_{n+1} = D_n\backslash C_{n+1}'$, $K_{n+1} = K_n \cup C_{n+1}''$ and $K_n \subset A_0 \subset D_n$, so, $|\mu(D_n\backslash K_n)| \geq \varepsilon_0$.

Since $\mu$ is exhaustive, then $\lim_{n\to\infty} |\mu(C_n')| = \lim_{n\to\infty} |\mu(C_n'')| = 0$, so there is $n_0(\varepsilon_0)$ such that $|\mu(C_n')| < \frac{\delta_0}{3}$ and $|\mu(C_n'')| < \frac{\delta_0}{3}$, for every $n \geq n_0(\varepsilon_0)$. Consequently,

$$|\mu(D_{n_0}\backslash A_0)| \leq e(\mu(D_{n_0}\backslash A_0), \mu(C_{n_0+1}')) + |\mu(C_{n_0}')| < \frac{2\delta_0}{3} < \delta_0,$$
$$|\mu(A_0\backslash K_{n_0})| \leq e(\mu(A_0\backslash K_{n_0}), \mu(C_{n_0+1}'')) + |\mu(C_{n_0}'')| < \frac{2\delta_0}{3} < \delta_0,$$

whence $|\mu(D_{n_0}\backslash K_{n_0})| < \varepsilon_0$, which is a contradiction. □

By Theorem 3.6, Remark 2.4 iii) and by Theorems 3.3 ii) and 3.5 i) from [10], we get the following:

**Corollary 3.7.** *If $\mathcal{C}$ is a $\sigma$-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and autocontinuous from above, then i) $\mu$ is $R_{\mathcal{K}}$-regular $\Leftrightarrow$ ii) $\mu$ is $R'_{\mathcal{K}}$-regular $\Leftrightarrow$ iii) $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular.*

**Remark 3.8.** In Theorem 3.6 and Corollary 3.7, $\mathcal{C}$ can be, for instance, $\widetilde{\mathcal{B}}, \mathcal{B}'$ or $\mathcal{B}_0'$ etc.

In what follows, we present two situations under which the entire space $T$ is $R'_{\mathcal{K}}$-regular (and, consequently, also, $R'_{\mathcal{K},\mathcal{D}}$-regular).

**Theorem 3.9.** *If $\mathcal{C}$ is a $\sigma$-ring, $(T, d)$ is a complete separable metric space and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ fulfils $(E')$, then $T$ is $R'_{\mathcal{K}}$-regular.*

**Proof.** Let $\{t_k\}_{k\in\mathbb{N}}$ be a countable dense subset of $T$. For every $n, k \in \mathbb{N}$, we consider $T_n(t_k) = \{t \in T; d(t, t_k) \leq \frac{1}{n}\}$ and for every $n, m \in \mathbb{N}$, we denote $A_{n,m} = T\backslash(\bigcup_{k=1}^{m} T_n(t_k))$.

We observe that, for every $n \in \mathbb{N}$, $A_{n,m} \underset{m\to\infty}{\searrow} \emptyset$, so, since $\mu$ fulfils (E'), there is an increasing sequence $(j_{n_i})_i \subset \mathbb{N}$ so that $\lim_{i\to\infty} |\mu(\bigcup_{n=1}^{\infty} A_{n,j_{n_i}})| = 0$. For every $i \in \mathbb{N}$, we denote $B_i = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{j_{n_i}} T_n(t_k)$. Obviously, for every $i \in \mathbb{N}$, $B_i$ is closed and totally bounded in the complete metric space $T$, so, it is compact.

It remains to establish that $\lim_{i\to\infty} |\mu(T\backslash B_i)| = 0$. Indeed, since for every $i \in \mathbb{N}$, $T\backslash B_i = \bigcup_{n=1}^{\infty}(T\backslash(\bigcup_{k=1}^{j_{n_i}} T_n(t_k))) = \bigcup_{n=1}^{\infty} A_{n,j_{n_i}}$, the conclusion follows. □

By Remark 2.4 v) and Theorem 3.9, we obtain the following:

**Corollary 3.10.** *If $\mathcal{C}$ is a $\sigma$-ring, $(T, d)$ is a complete separable metric space and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and null-null-additive, then $T$ is $R'_{\mathcal{K}}$-regular.*

**Theorem 3.11.** *If $\mathcal{C}$ is a $\sigma$-ring, $(T, d)$ is a locally compact separable metric space and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is $R'_{\mathcal{F}}$-regular, decreasing convergent and fulfils condition $(E')$, then $T$ is $R'_{\mathcal{K}}$-regular.*

**Proof.** Since $(T, d)$ is a locally compact separable metric space, then $T = \bigcup_{n=1}^{\infty} B_n$, where for every $n \in \mathbb{N}$, $B_n$ is a relatively compact, open set. Since $\mu$ is $R'_{\mathcal{F}}$-regular, then for every $n \in \mathbb{N}$, there exists an increasing sequence $(F_{n,m})_m \subset \mathcal{F} \cap \mathcal{C}$ so that for every $m \in \mathbb{N}$, $F_{n,m} \subset B_n$ and $\lim_{m\to\infty} |\mu(B_n\backslash F_{n,m})| = 0$. Because for every $n \in \mathbb{N}, B_n\backslash F_{n,m} \underset{m\to\infty}{\searrow} \bigcap_{m=1}^{\infty}(B_n\backslash F_{n,m})$ and $\mu$ is decreasing convergent, then $|\mu(\bigcap_{m=1}^{\infty}(B_n\backslash F_{n,m}))| = 0$, so, for every $n \in \mathbb{N}, \mu(\bigcap_{m=1}^{\infty}(B_n\backslash F_{n,m})) = \{0\}$.

Since $\mu$ fulfils (E'), for every $n \in \mathbb{N}$, there exists an increasing sequence of naturals $(m_{n_l})_l$ such that $\lim_{l\to\infty} |\mu(\bigcup_{n=1}^{\infty}(B_n\backslash F_{n,m_{n_l}}))| = 0$. For every $l \in \mathbb{N}$, we denote $C_l = T\backslash(\bigcup_{n=1}^{\infty} F_{n,m_{n_l}})$. Then $C_l \subseteq \bigcup_{n=1}^{\infty}(B_n\backslash F_{n,m_{n_l}})$, so $\lim_{l\to\infty} |\mu(C_l)| = 0$. Now, for every $l, s{\in}\mathbb{N}$ we denote $C_{l,s}{=}T\backslash(\bigcup_{n=1}^{s} F_{n,m_{n_l}})$. Since for every $l \in \mathbb{N}, C_{l,s} \underset{s\to\infty}{\searrow} C_l$ and $\mu$ is decreasing convergent, then for every $\varepsilon > 0$ and every $l \in \mathbb{N}$, there is $s_0(l){\in}\mathbb{N}$ so that $h(\mu(C_{l,s_0(l)}), \mu(C_l)){<}\frac{\varepsilon}{2^l}$.

Since $\lim_{l\to\infty} |\mu(C_l)| = 0$, then

$$\lim_{l\to\infty} |\mu(T\backslash(\bigcup_{n=1}^{s_0(l)} F_{n,m_{n_l}}))| = \lim_{l\to\infty} |\mu(C_{l,s_0(l)})| = 0.$$

For every $l \in \mathbb{N}$, we denote $R_l = \bigcup_{n=1}^{s_0(l)} F_{n,m_{n_l}}$. Evidently, $\lim_{l\to\infty} |\mu(T\backslash R_l)| = 0$. Since $\lim_{l\to\infty} |\mu(T\backslash \overline{R}_l)| \le \lim_{l\to\infty} |\mu(T\backslash R_l)|$, then $\lim_{l\to\infty} |\mu(T\backslash \overline{R}_l)| = 0$, where $\overline{R}_l$ denotes the closure of $R_l$. Also, for every $l \in \mathbb{N}$, we have $\overline{R}_l \subseteq \overline{\bigcup_{n=1}^{s_0(l)} B_n} = \bigcup_{n=1}^{s_0(l)} \overline{B}_n$, so $\overline{R}_l$ is compact. If $(\overline{R}_l)_l$ is increasing, the proof finishes. If $(\overline{R}_l)_l$ is not increasing, for every $p \in \mathbb{N}$, we denote $R'_p = \bigcup_{l=1}^{p} \overline{R}_l$. Then for every $p \in \mathbb{N}, (R'_p)_p$ is an increasing sequence of compact sets and $\lim_{p\to\infty} |\mu(T\backslash R'_p)| \le \lim_{p\to\infty} |\mu(T\backslash \overline{R}_p)| = 0$. □

**Remark 3.12.** In Theorem 3.9, Corollary 3.10 and Theorem 3.11, $\mathcal{C}$ can be, for instance, $\widetilde{\mathcal{B}}$, and in Theorem 3.9 and Corollary 3.10, $\mathcal{C}$ can also be $\mathcal{B}'$ or $\mathcal{B}'_0$.

By Remark 2.4 i), Theorem 3.9 and by Proposition 4.7 from [10], we get:

**Corollary 3.13.** *Suppose $(T, d)$ is a complete separable metric space and $\mu : \widetilde{\mathcal{B}} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and asymptotic null-additive. Then $\mu$ is $R'_{\mathcal{F},\mathcal{D}}$-regular if and only if $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular.*

Also, by Remark 2.4 i), Theorem 3.11 and by Theorem 3.3 ii) and Proposition 4.7 from [10], we have:

**Corollary 3.14.** *Suppose $(T, d)$ is a locally compact separable metric space and $\mu : \widetilde{\mathcal{B}} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and asymptotic null-additive. Then $\mu$ is $R'_{\mathcal{F},\mathcal{D}}$-regular if and only if $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular.*

By Remark 2.4 ii), Corollary 3.13, Corollary 3.14, Corollary 3.7 and Remark 3.8 and also by Theorems 3.3 i) and 3.5 iii) from [10], we have:

**Corollary 3.15.** *Suppose $(T, d)$ is a complete separable metric space or a locally compact separable metric space and $\mu : \widetilde{\mathcal{B}} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and autocontinuous from above. Then i) $\mu$ is $R'_{\mathcal{F},\mathcal{D}}$-regular $\Leftrightarrow$ ii) $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular $\Leftrightarrow$ iii) $\mu$ is $R_{\mathcal{K},\mathcal{D}}$-regular $\Leftrightarrow$ iv) $\mu$ is $R'_{\mathcal{K}}$-regular $\Leftrightarrow$ v) $\mu$ is $R_{\mathcal{K}}$-regular.*

By Corollary 3.15, Remark 2.4 ii) and by Corollary 4.1 from [10], we immediately have:

**Corollary 3.16.** *In the conditions of Corollary 3.15, i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) $\Leftrightarrow$ v) $\Leftrightarrow$ vi) $\mu$ is $R'_{\mathcal{F}}$-regular $\Leftrightarrow$ vii) $\mu$ is $R'_{\mathcal{D}}$-regular.*

**Theorem 3.17.** *Suppose $\mathcal{C}$ is the $\sigma$-ring generated by a family $\mathcal{S}$ of subsets of $T$ and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is null-null-additive and $(S)$-fuzzy.*

*If $\mathcal{M} = \mathcal{F}$ ($\mathcal{M} = \mathcal{K}$, respectively), $\mathcal{S} \subset \mathcal{M}$ and every set $A \in \mathcal{S}$ is $R'_{\mathcal{D}}$-regular, then $\mu$ is $R'_{\mathcal{M},\mathcal{D}}$-regular.*

**Proof.** Let $\mathcal{A} = \{A \in \mathcal{C}; A$ is $R'_{\mathcal{M},\mathcal{D}}$-regular$\}$. We prove that $\mathcal{A} = \mathcal{C}$.

We observe that, since $\mathcal{S} \subset \mathcal{M}$ and every set $A \in \mathcal{S}$ is $R'_{\mathcal{D}}$-regular, then $A$ is also $R'_{\mathcal{M},\mathcal{D}}$-regular, so $\mathcal{S} \subset \mathcal{A}$.

We demonstrate that $\mathcal{A}$ is a $\sigma-$ring.

If $A_1, A_2 \in \mathcal{A}$, there are two sequences $(M_n^1), (M_n^2) \subset \mathcal{M} \cap \mathcal{C}$ and two sequences $(D_n^1), (D_n^2) \subset \mathcal{D} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n^1 \subset A_1 \subset D_n^1$, $M_n^2 \subset A_2 \subset D_n^2$, $\lim_{n\to\infty} \lim|\mu(D_n^1 \backslash M_n^1)| = 0$ and $\lim_{n\to\infty} |\mu(D_n^2 \backslash M_n^2)| = 0$. By Corollary 2.4 from [10] $\mu$ is, equivalently, asymptotic null-additive, so we also have $\lim_{n\to\infty} |\mu((D_n^1 \backslash M_n^1) \cup \mu(D_n^2 \backslash M_n^2))| = 0$. Because for every $n \in \mathbb{N}$, $D_n^1 \backslash M_n^2 \in \mathcal{D} \cap \mathcal{C}$, $M_n^1 \backslash D_n^2 \in \mathcal{M} \cap \mathcal{C}$, $M_n^1 \backslash D_n^2 \subset A_1 \backslash A_2 \subset D_n^1 \backslash M_n^2$ and

$$|\mu((D_n^1 \backslash M_n^2) \backslash (M_n^1 \backslash D_n^2))| \le |\mu((D_n^1 \backslash M_n^1) \cup \mu(D_n^2 \backslash M_n^2))|,$$

then $A_1 \backslash A_2 \in \mathcal{A}$.

Now, let $(A_n)_n \subset \mathcal{A}$ be so that $A_n \nearrow A$. Since $(A_n)_n \subset \mathcal{A}$, then for every $n \in \mathbb{N}$, there is an increasing sequence $(M_{n,m})_m \subset \mathcal{M} \cap \mathcal{C}$ and a decreasing sequence $(D_{n,m})_m \subset \mathcal{D} \cap \mathcal{C}$ so that for every $m \in \mathbb{N}$, $M_{n,m} \subset A_n \subset D_{n,m}$ and $\lim_{m\to\infty} |\mu(D_{n,m} \backslash M_{n,m})| = 0$. For every $n \in \mathbb{N}$, we denote $N_n = \bigcap_{m=1}^{\infty} (D_{n,m} \backslash M_{n,m})$. Because for any $n \in \mathbb{N}, (D_{n,m})_m$ is decreasing and $(M_{n,m})_m$ is increasing, then $D_{n,m} \backslash M_{n,m} \underset{m\to\infty}{\searrow} N_n$. Since $\mu$ is decreasing convergent, we have $\lim_{m\to\infty} h(\mu(D_{n,m} \backslash M_{n,m}), \mu(N_n)) = 0$. Because for every $n \in \mathbb{N}$, $\lim_{m\to\infty} |\mu(D_{n,m} \backslash M_{n,m})| = 0$ and for any $m, n \in \mathbb{N}$,

$$|\mu(N_n)| \le h(\mu(D_{n,m} \backslash M_{n,m}), \mu(N_n)) + |\mu(D_{n,m} \backslash M_{n,m})|,$$

by passing $m$ to the limit, we have that for any $n \in \mathbb{N}$, $|\mu(N_n)| = 0$, so $\mu(N_n) = \{0\}$.

Consequently, because $\mu$ is null-null-additive and (S)-fuzzy, then $\mu$ fulfils condition (E'), so, for each $n \in \mathbb{N}$, there is a sequence of naturals $(k_{n_l})_l$ such that $\lim_{l\to\infty} |\mu(\bigcup_{n=1}^{\infty} (D_{n,k_{n_l}} \backslash M_{n,k_{n_l}}))| = 0$.

Since for every $l \in \mathbb{N}$,

$$\{0\} \subseteq \mu((\bigcup_{n=1}^{\infty} (D_{n,k_{n_l}}) \backslash (\bigcup_{n=1}^{\infty} M_{n,k_{n_l}}))) \subseteq \mu(\bigcup_{n=1}^{\infty} (D_{n,k_{n_l}} \backslash M_{n,k_{n_l}})),$$

we get that $\lim_{l\to\infty}|\mu((\bigcup_{n=1}^{\infty} D_{n,k_{n_l}})\backslash(\bigcup_{n=1}^{\infty} M_{n,k_{n_l}}))| = 0$. For every $l \in \mathbb{N}$, we denote $R_l = (\bigcup_{n=1}^{\infty} D_{n,k_{n_l}})\backslash(\bigcup_{n=1}^{\infty} M_{n,k_{n_l}})$ and for every $p \in \mathbb{N}$, we denote $S_{l,p} = (\bigcup_{n=1}^{\infty} D_{n,k_{n_l}})\backslash(\bigcup_{n=1}^{p} M_{n,k_{n_l}})$. So, we have $\lim_{l\to\infty}|\mu(R_l)| = 0$ and for every $l \in \mathbb{N}$, $S_{l,p} \underset{p\to\infty}{\searrow} R_l$, hence, by the decreasing convergence of $\mu$, $\lim_{p\to\infty} h(\mu(S_{l,p}), \mu(R_l)) = 0$.

Consequently, for every $l \in \mathbb{N}$, there is $p_l \in \mathbb{N}$ so that $h(\mu(S_{l,p_l}), \mu(R_l)) < \frac{\varepsilon}{2}$. Because $\lim_{l\to\infty}|\mu(R_l)| = 0$, then $\lim_{l\to\infty}|S_{l,p_l})| = 0$. Finally, since for every $l \in \mathbb{N}$, $\bigcup_{n=1}^{p_l} M_{n,k_{n_l}} \subset \bigcup_{n=1}^{\infty} A_n = A \subset \bigcup_{n=1}^{\infty} D_{n,k_{n_l}}, \bigcup_{n=1}^{\infty} D_{n,k_{n_l}} \in \mathcal{D}\cap\, \mathcal{C}$ and $\bigcup_{n=1}^{p_l} M_{n,k_{n_l}} \in \mathcal{M}\cap\, \mathcal{C}$, then $A \in \mathcal{A}$.

Consequently, $\mathcal{A}$ is a $\sigma$-ring, so, the conclusion immediately follows. □

**Remark 3.18.** With the notations from the above theorem, if:

(i) $T$ is a metric space, $\mathcal{C} = \widetilde{\mathcal{B}}$ and $\mathcal{S} = \mathcal{F} = \mathcal{M}$ or

(ii) $T$ is a locally compact Hausdorff space, $\mathcal{C} = \mathcal{B}_0', \mathcal{S}$ is the family of all compact, $G_\delta$-sets of $T$ and $\mathcal{M} = \mathcal{K}$ and if $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is decreasing convergent, then every set $A \in \mathcal{S}$ is $R'_{\mathcal{D}}$-regular.

Indeed, in both situations (i) or (ii), if $A \in \mathcal{S}$ is arbitrary, there is a decreasing sequence $(D_n)_n \subset \mathcal{D}\cap\, \mathcal{C}$ so that $D_n \searrow A$. Because $\mu$ is decreasing convergent, by Example 3.12 ii) from [10], we get that $A$ is $R'_{\mathcal{D}}$-regular.

By Theorem 3.17 and Remark 3.18, we have:

**Corollary 3.19.** *i) If $T$ is a metric space and $\mu : \widetilde{\mathcal{B}} \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and null-null-additive, then $\mu$ is $R'_{\mathcal{F},\mathcal{D}}$-regular.*

*ii) If $T$ is a locally compact Hausdorff space and $\mu : \mathcal{B}_0' \to \mathcal{P}_f(X)$ is $(S)$-fuzzy and null-null-additive, then $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular.*

By Remark 2.4 ii) and iv), Corollary 3.19 i), Corollary 3.15 and Theorem 3.3, we get:

**Corollary 3.20.** *Suppose $(T, d)$ is a complete separable metric space or a locally compact separable metric space and $\mu : \widetilde{\mathcal{B}} \to \mathcal{P}_f(X)$ is autocontinuous from above. Then i) $\mu$ is $R'_{\mathcal{F},\mathcal{D}}$-regular $\Leftrightarrow$ ii) $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular $\Leftrightarrow$ iii) $\mu$ is $R_{\mathcal{K},\mathcal{D}}$-regular $\Leftrightarrow$ iv) $\mu$ is $R'_{\mathcal{K}}$-regular $\Leftrightarrow$ v) $\mu$ is $R_{\mathcal{K}}$-regular $\Leftrightarrow$ vi) $\mu$ is $(S)$-fuzzy $\Leftrightarrow$ vii) $\mu$ is order continuous.*

By Theorem 3.3, Remark 2.4 iv) and Corollary 3.19 ii) and also by Theorem ii) from [10], we have:

**Corollary 3.21.** *If $T$ is a locally compact Hausdorff space and $\mu : \mathcal{B}_0' \to \mathcal{P}_f(X)$ is autocontinuous from above, then $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular if and only if it is order continuous (if and only if it is $(S)$-fuzzy).*

By Corollary 3.21, Corollary 3.7 and Remark 3.8, we get:

**Corollary 3.22.** *Suppose $T$ is a locally compact Hausdorff space and $\mu : \mathcal{B}_0' \to \mathcal{P}_f(X)$ is autocontinuous from above. Then i) $\mu$ is $(S)$-fuzzy $\Leftrightarrow$ ii) $\mu$ is order continuous $\Leftrightarrow$ iii) $\mu$ is $R'_{\mathcal{K},\mathcal{D}}$-regular $\Leftrightarrow$ iv) $\mu$ is $R'_{\mathcal{K}}$-regular $\Leftrightarrow$ v) $\mu$ is $R_{\mathcal{K}}$-regular.*

## 4. Concluding remarks

In this paper, the study of abstract regularity in the fuzzy set-valued case is furthered in order to obtain concrete applications in problems concerning continuity properties. Especially, a set-valued Alexandroff type theorem is obtained.

We shall apply this study concerning abstract regularity in further researches, for instance, in order to obtain an abstract set-valued Lusin type theorem under classical measurability.

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