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SOLUTIONS AND STABILITY OF GENERALIZED MIXED TYPE QCA-FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES*

BY

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Abstract. In this paper, we investigate the general solution and the generalized stability for the quartic, cubic and additive functional equation (briefly, QCA–functional equation)

$$\begin{aligned} f(x+ky)+f(x-ky) &= k^2 f(x+y)+k^2 f(x-y) \\ &\quad + (k^2-1)(k^2 f(y)+k^2 f(-y)-2f(x)), \end{aligned}$$

for any $k \in \mathbb{Z} - \{0, \pm 1\}$ in Menger probabilistic normed spaces.

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1. Introduction and preliminaries

In 1942, Menger [39] introduced the notion of a probabilistic metric space. Since then, the theory of probabilistic metric spaces has developed by many authors in many directions (see [4], [48]). The idea of Menger was to use the distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic

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generalization of metric spaces appears to be interested in the investigation of physical quantities, physiological thresholds and some other fields. It is also of fundamental importance in probabilistic functional analysis.

On the other hand, in 1962, Serstnev [49] introduced the concept of a probabilistic normed space introduced by means of a definition that was closely modelled on the theory of (classical) normed spaces and used to study the problem of best approximation in statistics.

In the sequel, we adopt the usual terminology, notation and conventions of the theory of probabilistic normed spaces used in [1, 2, 4, 18, 19, 48].

Throughout this paper, let Δ^+ is the space of distribution functions, that is,

$$\Delta^+ : \ = \{F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1] \ : \ F \ \text{ is left-continuous, } \\ \text{non-decreasing on } \ \mathbb{R}, \ F(0) = 0 \text{ and } \ F(+\infty) = 1\}$$

and a subset D^+ of Δ^+ is defined by $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element of Δ^+ with order $\leq$ is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1 ([48])**.** A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$, for all $a \in [0,1]$;
(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$.

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1, & \text{if } n = 1, \\ T(T_{i=1}^{n-1} x_i, x_n), & \text{if } n \geq 2 \end{cases}$$

and $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$ (see [32, 33]) .

Definition 1.2. A Menger probabilistic normed spaces (briefly, Menger PN-space) is a triple (X, Λ, T), where X is a vector space, T is a continuous t-norm and Λ is a mapping from X into D^+ satisfying the following conditions hold:

(PN1) $\Lambda_x(0) = 0$, for all $x \in X$;

(PN2) $\Lambda_x(t) = \varepsilon_0(t)$, for all $t > 0$ if and only if $x = 0$;

(PN3) $\Lambda_{\alpha x}(t) = \Lambda_x(\frac{t}{|\alpha|})$, for all $x \in X$, $\alpha \neq 0$ and $t > 0$;

(PN4) $\Lambda_{x+y}(t + s) \geq T(\Lambda_x(t), \Lambda_y(s))$, for all $x, y \in X$ and $t, s \geq 0$.

Clearly, every Menger PN-space is probabilistic metric space having a metrizable uniformity on X if $\sup_{a<1} T(a, a) = 1$.

Definition 1.3. Let (X, Λ, T) be a Menger PN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\Lambda_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\Lambda_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A Menger PN-space (X, Λ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. *If (X, Λ, T) is a Menger PN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$.*

A basic question in the theory of functional equations is as follows:

"When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?"

If the problem has a solution, we say that the equation is stable. In 1940, the first stability problem concerning group homomorphisms was raised by Ulam [50] and, in 1941, the stability problem affirmatively solved by Hyers [34]. Since then, the result of Hyers was generalized by Aoki [3] for approximate additive function in 1950 and by Rassias [44] for approximate linear functions by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\varepsilon(\|x\|^p+\|y\|^p)$ in 1978. Because of a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon proved by Rassias is called the Hyers-Ulam-Rassias stability (see also [5, 22, 35, 37, 41, 42, 43, 45, 46]). In 1994, a generalization of Rassias theorem was obtained by Găvruta [21], who replaced $\varepsilon(\|x\|^p + \|y\|^p)$ by the general control function $\varphi(x, y)$.

In 2002, Jun and Kim [36] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and established the general solution and the generalized Hyers-Ulam-Rassias stability for functional equation (1.1). They proved that a function f between two real vector spaces X and Y is a solution of the equation (1.1) if and only if there exists a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$ and, moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. In fact, the function C is given by

$$C(x,y,z)=\frac{1}{24}(f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)),$$

for all $x,y,z \in X$. Obviously, the function $f(x) = cx^3$ satisfies the functional equation (1.1) and so it is natural to call the equation (1.1) the cubic functional equation. Every solution of the cubic functional equation is called a cubic function. In 2005, Lee et al. [38] considered the following functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of the equation (1.2) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x,x)$ for all $x \in X$. In fact, the bi-quadratic function B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y)),$$

for all $x,y \in X$. It is easy to show that the function $f(x) = dx^4$ satisfies the functional equation (1.2), which is called the quartic functional equation.

In 2008, Najati and Zamani [40] obtained the generalized Hyers-Ulam-Rassias stability for a mixed type of cubic and additive functional equation. In addition, in 2009, Eshaghi Gordji and Khodaei [20] established the general solution and investigated the Hyers-Ulam-Rassias stability for a mixed type of cubic, quadratic and additive functional equation (briefly, AQC–functional equation) with $f(0) = 0$,

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \tag{1.3}$$

in quasi-Banach spaces, where k is nonzero integer numbers with $k \neq \pm 1$. Obviously, the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.3). For other mixed type functional equations, see [6]-[20] and [23]-[31].

In 2009, Shakeri et al. [47] proved the stability of cubic functional equation in Menger PN-spaces.

In this paper, we deal with the following functional equation derived from additive, cubic and quartic functions

$$\begin{aligned} f(x+ky) + f(x-ky) &= k^2 f(x+y) + k^2 f(x-y) \\ &\quad + (k^2-1)(k^2 f(y) + k^2 f(-y) - 2f(x)), \end{aligned} \tag{1.4}$$

for fixed integers k with $k \neq 0, \pm 1$. It is easy to see that the function $f(x) = ax + bx^3 + cx^4$ is a solution of the functional equation (1.4). The main purpose of this paper is to establish the general solution of the equation (1.4) and to investigate the generalized stability for the equation (1.4) in Menger probabilistic normed spaces.

2. Generalized mixed type quartic, cubic and additive functional equation

In this section, we establish the general solution of the equation (1.4).

Theorem 2.1. *Let X and Y be vector spaces. A function $f : X \to Y$ whit $f(0) = 0$ satisfies the equation (1.4) for all $x, y \in X$ if and only if there exist a unique symmetric bi-quadratic function $B : X \times X \longrightarrow Y$, a unique function $C : X \times X \times X \longrightarrow Y$ and a unique additive function $A : X \to Y$ such that*

$$f(x) = B(x, x) + C(x, x, x) + A(x),$$

for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

Proof. Let f satisfies the equation (1.4). We decompose f into the even part and odd part by putting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \ \forall x \in X.$$

It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the functions f_e and f_o satisfy the equation (1.4).

Now, we show that the function $f_e : X \to Y$ is quartic. In fact, it follows from the equation (1.4) that

$$\begin{aligned} f_e(x+ky) + f_e(x-ky) = k^2 f_e(x+y) + k^2 f_e(x-y) \\ + 2(1-k^2)f_e(x) + 2k^2(k^2-1)f_e(y), \ \forall x, y \in X. \end{aligned} \tag{2.1}$$

Letting $x = y = 0$ in (2.1), we have $f(0) = 0$. Putting $x = 0$ in (2.1), we get

$$f_e(ky) = k^4 f_e(y), \quad \forall y \in X. \tag{2.2}$$

Replacing x by $2x$ in (2.1), we get

$$\begin{aligned} f_e(2x+ky) + f_e(2x-ky) = k^2 f_e(2x+y) + k^2 f_e(2x-y) \\ + 2(1-k^2)f_e(2x) + 2k^2(k^2-1)f_e(y), \ \forall x, y \in X. \end{aligned} \tag{2.3}$$

If we put $y = x+y$ in the equation (2.1) and then $y = x-y$ in the equation (2.1) again, then it follows from the evenness of f_e that

$$\begin{aligned} f_e(k(x+y)+x) + f_e(k(x+y)-x) = k^2 f_e(2x+y) + k^2 f_e(y) \\ + 2(1-k^2)f_e(x) + 2k^2(k^2-1)f_e(x+y) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} f_e(k(x-y)+x) + f_e(k(x-y)-x) = k^2 f_e(2x-y) + k^2 f_e(y) \\ + 2(1-k^2)f_e(x) + 2k^2(k^2-1)f_e(x-y), \ \forall x, y \in X. \end{aligned} \tag{2.5}$$

Adding the equations (2.4) and (2.5), we have

$$\begin{aligned} & f_e(k(x+y)+x) + f_e(k(x+y)-x) \\ & + f_e(k(x-y)+x) + f_e(k(x-y)-x) \\ & = k^2 f_e(2x+y) + k^2 f_e(2x-y) + 2k^2 f_e(y) + 4(1-k^2)f_e(x) \\ & + 2k^2(k^2-1)(f_e(x+y) + f_e(x-y)), \ \forall x, y \in X. \end{aligned} \tag{2.6}$$

Interchanging x with y in (2.1) and using the evenness of f_e, we obtain

$$\begin{aligned} & f_e(kx+y) + f_e(kx-y) = k^2 f_e(x+y) + k^2 f_e(x-y) \\ & + 2k^2(k^2-1)f_e(x) + 2(1-k^2)f_e(y), \ \forall x, y \in X. \end{aligned} \tag{2.7}$$

We substitute $y = x + ky$ in (2.7) and then $y = x - ky$ in (2.7) and use (2.2), we obtain

$$\begin{aligned} &f_e(k(x+y)+x) + f_e(k(x-y)-x) \\ &= k^2 f_e(2x+ky) + k^6 f_e(y) + 2k^2(k^2-1) f_e(x) + 2(1-k^2) f_e(x+ky) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &f_e(k(x-y)+x) + f_e(k(x+y)-x) \\ &= k^2 f_e(2x-ky) + k^6 f_e(y) + 2k^2(k^2-1) f_e(x) \\ &+ 2(1-k^2) f_e(x-ky), \ \forall x, y \in X. \end{aligned} \tag{2.9}$$

Adding the equations (2.8) and (2.9), we have

$$\begin{aligned} &f_e(k(x+y)+x) + f_e(k(x+y)-x) \\ &+ f_e(k(x-y)+x) + f_e(k(x-y)-x) \\ &= k^2 f_e(2x+ky) + k^2 f_e(2x-ky) + 2k^6 f_e(y) + 4k^2(k^2-1) f_e(x) \\ &+ 2(1-k^2)(f_e(x+ky) + f_e(x-ky)), \ \ \forall x, y \in X. \end{aligned} \tag{2.10}$$

It follows from the equations (2.1), (2.3), (2.6) and (2.10) that

$$\begin{aligned} &f_e(2x+y) + f_e(2x-y) \\ &= 4(f_e(x+y) + f_e(x-y)) + 2f_e(2x) - 8f_e(x) - 6f_e(y), \ \forall x, y \in X. \end{aligned} \tag{2.11}$$

Letting $y = x$ in (2.11), we have $f_e(3x) = 6f_e(2x) - 15f_e(x)$ and letting $y = 2x$ in (2.11), we have $f_e(4x) = 20f_e(2x) - 64f_e(x)$. Thus, by induction, we get

$$f_e(mx) = \frac{m(m^2-1)}{12} f_e(2x) + \frac{m^2(4-m^2)}{3} f_e(x), \tag{2.12}$$

for each fixed integer $m \neq 0, \pm 1, \pm 2$ and $x \in X$. But, $k \neq 0, \pm 1$ and, also if $k = \pm 2$, then it follows from the equation (2.7) that f_e is quartic. Otherwise, if we use the equation (2.12) for $m = k$ and the equation (2.2), then we obtain $f_e(2x) = 16 f_e(x)$ and so it follows from the equation (2.11) that

$$f_e(2x+y) + f_e(2x-y) = 4f_e(x+y) + 4f_e(x-y) + 24f_e(x) - 6f_e(y), \forall x, y \in X.$$

This shows that f_e is quartic and so there exists a unique symmetric bi-quadratic function $B : X \times X \longrightarrow Y$ such that

$$f_e(x) = B(x, x), \quad \forall x \in X. \tag{2.13}$$

On the other hand, we show that the function $f_o : X \to Y$ is cubic-additive. In fact, it follows from the equation (1.4) that

$$f_o(x + ky) + f_o(x - ky) = k^2 f_o(x + y) + k^2 f_o(x - y) + 2(1 - k^2)f_o(x)$$

for all $x, y \in X$. By the same method as in Lemma 2.2 of [20], we can show that f_o is cubic-additive. Therefore, it follows that

$$f_o(x) = C(x, x, x) + A(x), \ \forall x \in X, \tag{2.14}$$

C is symmetric for each fixed one variable and is additive for fixed two variables and A is additive. Hence, from the equations (2.13) and (2.14), it follows that

$$f(x) = f_e(x) + f_o(x) = B(x, x) + C(x, x, x) + A(x), \ \forall x \in X.$$

Conversely, let $f(x) = B(x, x) + C(x, x, x) + A(x)$ for all $x \in X$, where the function B is symmetric bi-quadratic, C is symmetric for each fixed one variable and is additive for fixed two variables and A is additive. By a simple computation, we can show that the functions $x \mapsto B(x, x)$, $x \mapsto C(x, x, x)$ and $x \mapsto A(x)$ satisfy the functional equation (1.4). Therefore, the function f satisfies the equation (1.4). This completes the proof. □

3. Generalized stability in Menger probabilistic normed spaces

In this section, we investigate the stability problem of the functional equation (1.4).

Let X be a real linear space and (Y, Λ, T) be a complete Menger PN-space. Now, we define a difference operator $\Delta f : X \times X \to Y$ by

$$\begin{aligned}\Delta f(x, y) : &= f(x + ky) + f(x - ky) - k^2 f(x + y) - k^2 f(x - y) \\ &\quad - (k^2 - 1)(k^2 f(y) + k^2 f(-y) - 2f(x)), \ \ \forall x, y \in X,\end{aligned}$$

where $f : X \to Y$ is a mapping.

Theorem 3.1. *Let $\xi : X^2 \to D^+$ ($\xi(x,y)$ is denoted by $\xi_{x,y}$) be a function such that*

$$\lim_{m\to\infty} \xi_{k^m x, k^m y}(k^{4m}t) = 1 \tag{3.1}$$

and

$$\lim_{m\to\infty} T_{\ell=1}^{\infty}\Big(\xi_{0,k^{m+\ell-1}x}\Big(\frac{k^{4(m+\ell)}}{2^{\ell-1}}t\Big)\Big) = 1, \quad \forall x \in X, t > 0. \tag{3.2}$$

Suppose that an even function $f : X \to Y$ whit $f(0) = 0$ satisfies the inequality

$$\Lambda_{\Delta f(x,y)}(t) \geq \xi_{x,y}(t), \ \forall x, y \in X, t > 0. \tag{3.3}$$

Then there exists a unique quartic function $Q : X \to Y$ such that

$$\Lambda_{f(x)-Q(x)}(t) \geq T_{\ell=1}^{\infty}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell-1}}t\Big)\Big), \quad \forall x, y \in X, t > 0. \tag{3.4}$$

Proof. Setting $x = 0$ in (3.3) and using $f(0) = 0$, the evenness of f, we obtain

$$\Lambda_{2f(ky)-2k^4 f(y)}(t) \geq \xi_{0,y}(t), \quad \forall y \in X, t > 0. \tag{3.5}$$

Replacing y by x in (3.5), we have

$$\Lambda_{\frac{f(kx)}{k^4}-f(x)}(t) \geq \xi_{0,x}(2k^4 t) \geq \xi_{0,x}(k^4 t), \quad \forall x \in X, t > 0. \tag{3.6}$$

If we replace x by $k^\ell x$ in (3.6), we have

$$\Lambda_{\frac{f(k^{\ell+1}x)}{k^{4(\ell+1)}}-\frac{f(k^\ell x)}{k^{4\ell}}}(t) \geq \xi_{0,k^\ell x}(2k^{4(\ell+1)}t) \tag{3.7}$$

for all $x \in X$, $t > 0$ and $\ell \in \mathbb{N}$. Thus it follows from (3.7) and (PN_4) that

$$\begin{aligned}
\Lambda_{\frac{f(k^2x)}{k^8}-f(x)}(t) &\geq T\Big(\Lambda_{\frac{f(k^2x)}{k^8}-\frac{f(kx)}{k^4}}\Big(\frac{t}{2}\Big), \Lambda_{\frac{f(kx)}{k^4}-f(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T(\xi_{0,kx}(k^8 t), \xi_{0,x}(k^4 t))\\
&\geq T\Big(\xi_{0,kx}\Big(\frac{k^8}{2}t\Big), \xi_{0,x}(k^4 t)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{\frac{f(k^3x)}{k^{12}}-f(x)}(t) &\geq T\Big(\Lambda_{\frac{f(k^3x)}{k^{12}}-\frac{f(kx)}{k^4}}\Big(\frac{t}{2}\Big),\Lambda_{\frac{f(kx)}{k^4}-f(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T\Big(T\Big(\Lambda_{\frac{f(k^3x)}{k^{12}}-\frac{f(k^2x)}{k^8}}\Big(\frac{t}{4}\Big),\Lambda_{\frac{f(k^2x)}{k^8}-\frac{f(kx)}{k^4}}\Big(\frac{t}{2})\Big),\Lambda_{\frac{f(kx)}{k^4}-f(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T\Big(T\Big(\xi_{0,k^2x}\Big(\frac{k^{12}}{2}t\Big),\xi_{0,kx}\Big(\frac{k^8}{2}t\Big),\xi_{0,x}(k^4t)\Big)\\
&\geq T\Big(T\Big(\xi_{0,k^2x}\Big(\frac{k^{12}}{4}t\Big),\xi_{0,kx}\Big(\frac{k^8}{2}t)\Big),\xi_{0,x}(k^4t)\Big)\\
&= T(\xi_{0,x}(k^4t),T(\xi_{0,k^2x}(\frac{k^{12}}{4}t),\xi_{0,kx}(\frac{k^8}{2}t)))\\
&= T\Big(\xi_{0,x}(k^4t),T(\xi_{0,kx}\Big(\frac{k^8}{2}t\Big),\xi_{0,k^2x}\Big(\frac{k^{12}}{4}t\Big)\Big)\\
&= T\Big(T\Big(\xi_{0,x}(k^4t),\xi_{0,kx}\Big(\frac{k^8}{2}t)\Big),\xi_{0,k^2x}\Big(\frac{k^{12}}{4}t\Big)\Big),
\end{aligned}$$

for all $x \in X$ and $t > 0$ and so

$$\Lambda_{\frac{f(k^mx)}{k^{4m}}-f(x)}(t) \geq T_{\ell=1}^{m}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell-1}}t\Big)\Big), \quad \forall x \in X, t > 0. \tag{3.8}$$

In order to prove the convergence of the sequence $\{\frac{f(k^mx)}{k^{4m}}\}$, if we replace x with $k^{m'}x$ in (3.8), then we get

$$\Lambda_{\frac{f(k^{m+m'}x)}{k^{4(m+m')}}-\frac{f(k^{m'}x)}{k^{4m'}}}(t) \geq T_{\ell=1}^{m}\Big(\xi_{0,k^{m'+\ell-1}x}\Big(\frac{k^{4(m'+\ell)}}{2^{\ell-1}}t\Big)\Big), \quad \forall x \in X, t > 0.$$

Since the right hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{\frac{f(k^mx)}{k^{4m}}\}$ is a Cauchy sequence. Therefore, one can define the function $Q : X \to Y$ by $Q(x) := \lim_{m\to\infty} \frac{1}{k^{4m}}f(k^mx)$ for all $x \in X$. Now, if we replace x, y with k^mx, k^my in (3.3), respectively, it follows that

$$\Lambda_{\frac{\Delta f(k^mx,k^my)}{k^{4m}}}(t) \geq \xi_{k^mx,k^my}(k^{4m}t), \ \forall x, y \in x, t > 0. \tag{3.9}$$

By letting $m \to \infty$ in (3.9), we find that $\Lambda_{\Delta Q(x,y)}(t) = 1$ for all $t > 0$, which implies $\Delta Q(x, y) = 0$ and so Q satisfies the functional equation (1.4). Hence, by Theorem 2.1, the function $Q : X \to Y$ is quartic. To prove (3.4), if we take the limit as $m \to \infty$ in (3.8), then we can get (3.4).

Finally, to prove the uniqueness of the quartic function Q subject to (3.4), let us assume that there exists a quartic function Q' which satisfies (3.4). Since $Q(k^m x) = k^{4m}Q(x)$ and $Q'(k^m x) = k^{4m}Q'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, it follows from (3.4) that

$$\begin{aligned}
&\Lambda_{Q(x)-Q'(x)}(t) = \Lambda_{Q(k^m x)-Q'(k^m x)}(k^{4m}t)\\
&\geq T\Big(\Lambda_{Q(k^m x)-f(k^m x)}\Big(\frac{k^{4m}}{2}t\Big), \Lambda_{f(k^m x)-Q'(k^m x)}\Big(\frac{k^{4m}}{2}t\Big)\Big)\\
&\geq T\Big(T_{\ell=1}^{\infty}\Big(\xi_{0,k^{m+\ell-1}x}\Big(\frac{k^{4(m+\ell)}}{2^{\ell-1}}t\Big)\Big), T_{\ell=1}^{\infty}\Big(\xi_{0,k^{m+\ell-1}x}\Big(\frac{k^{4(m+\ell)}}{2^{\ell-1}}t\Big)\Big)\Big),
\end{aligned} \tag{3.10}$$

for all $x \in X$ and $t > 0$. By letting $m \to \infty$ in (3.10), we find that $Q = Q'$. This completes the proof. □

Theorem 3.2. *Let $\xi : X^2 \to D^+$ be a function such that*

$$\lim_{m\to\infty} T(\xi_{2^m x,2^m y}(2^m t), \xi_{2^m x,2^m y}(2^{m-4}t)) = 1 \tag{3.11}$$

and

$$\lim_{m\to\infty} T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell+m-1}x}(2^{m-1}t)) = 1, \quad \forall x \in X, t > 0, \tag{3.12}$$

where

$$\begin{aligned}
\tilde{T}_x(t) = T\Big(&T\Big(T\Big(T\Big(\xi_{x,2x}\Big(\frac{k^2}{2^5}t\Big), \xi_{(2k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big),\\
&T\Big(\xi_{(2k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big), \xi_{x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big)\Big),\\
&T\Big(\xi_{2x,2x}\Big(\frac{k^2-1}{2^2}t\Big), \xi_{x,3x}\Big(\frac{k^2(k^2-1)}{2^2}t\Big)\Big)\Big),\\
&T\Big(T\Big(T\Big(\xi_{x,x}\Big(\frac{k^2}{2^6}t\Big), \xi_{(k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big),\\
&T\Big(\xi_{(k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big), \xi_{x,2x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big)\Big), \xi_{2x,x}\Big(\frac{k^2-1}{2^3}t\Big)\Big)\Big),
\end{aligned}$$

for all $x \in X$ and $t > 0$. Suppose that an odd function $f : X \to Y$ satisfies (3.3) *for all $x, y \in X$ and $t > 0$. Then the limit*

$$A(x) = \lim_{m\to\infty} \frac{1}{2^m}\left(f(2^{m+1}x) - 8f(2^m x)\right)$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\Lambda_{f(2x)-8f(x)-A(x)}(t) \geq T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(t)), \ \forall x \in X, t > 0. \tag{3.13}$$

Proof. It follows from (3.3) and the oddness of f that

$$\Lambda_{f(ky+x)-f(ky-x)-k^2f(x+y)-k^2f(x-y)+2(k^2-1)f(x)}(t) \geq \xi_{x,y}(t), \ \forall x,y \in X, t>0. \tag{3.14}$$

Putting $y = x$ in (3.14), we have

$$\Lambda_{f((k+1)x)-f((k-1)x)-k^2f(2x)+2(k^2-1)f(x)}(t) \geq \xi_{x,x}(t), \quad \forall x \in X, t>0. \tag{3.15}$$

It follows from (3.15) that

$$\Lambda_{f(2(k+1)x)-f(2(k-1)x)-k^2f(4x)+2(k^2-1)f(2x)}(t) \geq \xi_{2x,2x}(t), \quad \forall x \in X, t>0. \tag{3.16}$$

Replacing x and y by $2x$ and x in (3.14), respectively, we get

$$\Lambda_{f((k+2)x)-f((k-2)x)-k^2f(3x)-k^2f(x)+2(k^2-1)f(2x)}(t) \geq \xi_{2x,x}(t), \quad \forall x \in X, t>0. \tag{3.17}$$

Setting $y = 2x$ in (3.14) gives

$$\Lambda_{f((2k+1)x)-f((2k-1)x)-k^2f(3x)-k^2f(-x)+2(k^2-1)f(2x)}(t) \geq \xi_{x,2x}(t), \quad \forall x \in X, t>0. \tag{3.18}$$

Putting $y = 3x$ in (3.14), we obtain

$$\Lambda_{f((3k+1)x)-f((3k-1)x)-k^2f(4x)-k^2f(-2x)+2(k^2-1)f(x)}(t) \geq \xi_{x,3x}(t), \quad \forall x \in X, t>0. \tag{3.19}$$

Replacing x and y by $(k+1)x$ and x in (3.14), respectively, we get

$$\Lambda_{f((2k+1)x)-f((-x)-k^2f((k+2)x)-k^2f(kx)+2(k^2-1)f((k+1)x)}(t) \geq \xi_{(k+1)x,x}(t), \quad \forall x \in X, t>0. \tag{3.20}$$

Replacing x and y by $(k-1)x$ and x in (3.14), respectively, one gets

$$\Lambda_{f((2k-1)x)-f(x)-k^2f((k-2)x)-k^2f(kx)+2(k^2-1)f((k-1)x)}(t) \geq \xi_{(k-1)x,x}(t), \quad \forall x \in X, t>0. \tag{3.21}$$

Replacing x and y by $(2k+1)x$ and x in (3.14), respectively, we obtain

$$\begin{aligned}\Lambda_{f((3k+1)x)-f(-(k+1)x)-k^2f(2(k+1)x)+2(k^2-1)f((2k+1)x)-k^2f(2kx)}(t)\\ \geq \xi_{(2k+1)x,x}(t), \quad \forall x\in X, t>0.\end{aligned} \tag{3.22}$$

Replacing x and y by $(2k-1)x$ and x in (3.14), respectively, we have

$$\begin{aligned}\Lambda_{f((3k-1)x)-f(-(k-1)x)-k^2f(2(k-1)x)+2(k^2-1)f((2k-1)x)-k^2f(2kx)}(t)\\ \geq \xi_{(2k-1)x,x}(t), \quad \forall x\in X, t>0.\end{aligned} \tag{3.23}$$

Thus it follows from (3.15), (3.17), (3.18), (3.20) and (3.21) that

$$\begin{aligned}&\Lambda_{f(3x)-4f(2x)+5f(x)}(t)\\ &\geq T\Big(T\Big(T\Big(\xi_{x,x}\Big(\frac{k^2}{2^4}t\Big),\xi_{(k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big)\Big),\\ &\qquad T\Big(\xi_{(k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big),\xi_{x,2x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big)\Big)\Big),\\ &\qquad \xi_{2x,x}\Big(\frac{k^2-1}{2}t\Big)\Big), \quad \forall x\in X, t>0.\end{aligned} \tag{3.24}$$

Also, from (3.15), (3.16), (3.18), (3.19), (3.22) and (3.23), we have

$$\begin{aligned}&\Lambda_{f(4x)-2f(3x)-2f(2x)+6f(x)}(t)\\ &\geq T\Big(T\Big(T\Big(\xi_{x,2x}\Big(\frac{k^2}{2^4}t\Big),\xi_{(2k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big)\Big),\\ &\qquad T\Big(\xi_{(2k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big),\xi_{x,x}\Big(\frac{k^2(k^2-1)}{2^3}t\Big)\Big)\Big),\\ &\qquad T\Big(\xi_{(2x,2x}\Big(\frac{k^2-1}{2}t\Big),\xi_{x,3x}\Big(\frac{k^2(k^2-1)}{2}t\Big)\Big), \quad \forall x\in X, t>0.\end{aligned} \tag{3.25}$$

Finally, by using (3.24) and (3.25), we obtain

$$\begin{aligned}&\Lambda_{f(4x)-10f(2x)+16f(x)}(t)\\ &\geq T\Big(T\Big(T\Big(T\Big(\xi_{x,2x}\Big(\frac{k^2}{2^5}t\Big),\xi_{(2k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big),\\ &\qquad T\Big(\xi_{(2k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big),\xi_{x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big)\Big),\\ &\qquad T\Big(\xi_{2x,2x}\Big(\frac{k^2-1}{2^2}t\Big),\xi_{x,3x}\Big(\frac{k^2(k^2-1)}{2^2}t)\Big)\Big),\\ &\qquad T\Big(T\Big(T\Big(\xi_{x,x}\Big(\frac{k^2}{2^6}t\Big),\xi_{(k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big),\end{aligned} \tag{3.26}$$

$$T\Big(\xi_{(k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big),\xi_{x,2x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big)\Big),$$
$$\xi_{2x,x}\Big(\frac{k^2-1}{2^3}t\Big)\Big)\Big), \quad \forall x\in X, t>0.$$

Let

$$\begin{aligned}\tilde{T}_x(t)=T\Big(&T\Big(T\Big(T\Big(\xi_{x,2x}\Big(\frac{k^2}{2^5}t\Big),\xi_{(2k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big),\\ &T\Big(\xi_{(2k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big),\xi_{x,x}\Big(\frac{k^2(k^2-1)}{2^4}t\Big)\Big)\Big),\\ &T\Big(\xi_{2x,2x}\Big(\frac{k^2-1}{2^2}t\Big),\xi_{x,3x}\Big(\frac{k^2(k^2-1)}{2^2}t\Big)\Big)\Big),\\ &T\Big(T\Big(T\Big(\xi_{x,x}\Big(\frac{k^2}{2^6}t\Big),\xi_{(k-1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big),\\ &T\Big(\xi_{(k+1)x,x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big),\xi_{x,2x}\Big(\frac{k^2(k^2-1)}{2^5}t\Big)\Big)\Big),\\ &\xi_{2x,x}\Big(\frac{k^2-1}{2^3}t\Big)\Big)\Big), \quad \forall x\in X, t>0.\end{aligned} \tag{3.27}$$

Thus (3.26) means that

$$\Lambda_{f(4x)-10f(2x)+16f(x)}(t)\geq \tilde{T}_x(t), \quad \forall x\in X, t>0. \tag{3.28}$$

Let $g: X\to Y$ be a function defined by $g(x):=f(2x)-8f(x)$ for all $x\in X$. From (3.28), we conclude that

$$\Lambda_{\frac{g(2x)}{2}-g(x)}(t)\geq \tilde{T}_x(2t)\geq \tilde{T}_x(t), \quad \forall x\in X, t>0,$$

which implies that

$$\Lambda_{\frac{g(2^{\ell+1}x)}{2^{\ell+1}}-\frac{g(2^\ell x)}{2^\ell}}(t)\geq \tilde{T}_{2^\ell x}(2^{\ell+1}t), \tag{3.29}$$

for all $x\in X$, $t>0$ and $\ell\in\mathbb{N}$. Thus it follows from (3.29) and (PN4) that

$$\begin{aligned}\Lambda_{\frac{g(2^2x)}{2^2}-g(x)}(t)&\geq T\Big(\Lambda_{\frac{g(2^2x)}{2^2}-\frac{g(2x)}{2}}\Big(\frac{t}{2}\Big),\Lambda_{\frac{g(2x)}{2}-g(x)}\Big(\frac{t}{2}\Big)\Big)\\ &\geq T(\tilde{T}_{2x}(2t),\tilde{T}_x(t))\geq T(\tilde{T}_{2x}(t),\tilde{T}_x(t))\end{aligned}$$

and

$$\begin{aligned}
&\Lambda_{\frac{g(2^3x)}{2^3}-g(x)}(t) \geq T\Big(\Lambda_{\frac{g(2^3x)}{2^3}-\frac{g(2x)}{2}}\Big(\frac{t}{2}\Big), \Lambda_{\frac{g(2x)}{2}-g(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T\Big(T\Big(\Lambda_{\frac{g(2^3x)}{2^3}-\frac{g(2^2x)}{2^2}}\Big(\frac{t}{4}\Big), \Lambda_{\frac{g(2^2x)}{2^2}-\frac{g(2x)}{2}}\Big(\frac{t}{4}\Big)\Big), \Lambda_{\frac{g(2x)}{2}-g(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T(T(\tilde{T}_{2^2x}(2t), \tilde{T}_{2x}(t)), \tilde{T}_x(t))\\
&\geq T(T(\tilde{T}_{2^2x}(t), \tilde{T}_{2x}(t)), \tilde{T}_x(t)) = T(T(\tilde{T}_x(t), \tilde{T}_{2x}(t)), \tilde{T}_{2^2x}(t)), \forall x \in X, t > 0.
\end{aligned}$$

Thus

$$\Lambda_{\frac{g(2^m x)}{2^m}-g(x)}(t) \geq T_{\ell=1}^m(\tilde{T}_{2^{\ell-1}x}(t)), \quad \forall x \in X, t > 0. \tag{3.30}$$

In order to prove the convergence of the sequence $\{\frac{g(2^m x)}{2^m}\}$, if we replace x with $2^{m'}x$ in (3.30), then we have

$$\Lambda_{\frac{g(2^{m+m'}x)}{2^{m+m'}}-\frac{g(2^{m'}x)}{2^{m'}}}(t) \geq T_{\ell=1}^m(\tilde{T}_{2^{m'+\ell-1}x}(2^{m'}t)), \quad \forall x \in X, t > 0.$$

Since the right hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{\frac{g(2^m x)}{2^m}\}$ is a Cauchy sequence. Therefore, one can define the function $A : X \to Y$ by $A(x) := \lim_{m\to\infty} \frac{1}{2^m} g(2^m x)$ for all $x \in X$.

Now, if we replace x, y with $2^m x, 2^m y$ in (3.3), respectively, then it follows that

$$\begin{aligned}
\Lambda_{\frac{\Delta g(2^m x, 2^m y)}{2^m}}(t) &= \Lambda_{\frac{\Delta f(2^{m+1}x, 2^{m+1}y)}{2^m}-8\frac{\Delta f(2^m x, 2^m y)}{2^m}}(t)\\
&\geq T\Big(\Lambda_{\frac{\Delta f(2^{m+1}x,2^{m+1}y)}{2^m}}\Big(\frac{t}{2}\Big), \Lambda_{\frac{\Delta f(2^m x, 2^m y)}{2^{m-3}}}\Big(\frac{t}{2}\Big)\Big)\\
&\geq T(\xi_{2^{m+1}x, 2^{m+1}y}(2^{m-1}t), \xi_{2^m x, 2^m y}(2^{m-4}t)),
\end{aligned} \tag{3.31}$$

for all $x, y \in X$ and $t > 0$. By letting $m \to \infty$ in (3.31), we have $\Lambda_{\Delta A(x,y)}(t) = 1$ for all $t > 0$ and so $\Delta A(x, y) = 0$. Therefore, A satisfies (1.4). Hence, by Theorem 2.1 (see Lemma 2.2 in [20]), the function $A : X \to Y$ is additive.

To prove (3.13), if we take the limit as $m \to \infty$ in (3.30), then we can get (3.13).

Finally, to prove the uniqueness of the additive function A subject to (3.13), assume that there exists a additive function A' which satisfies (3.13).

Since $A(2^m x) = 2^m A(x)$ and $A'(2^m x) = 2^m A'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, it follows from (3.13) that

$$\begin{aligned} &\Lambda_{A(x)-A'(x)}(t) = \Lambda_{A(2^m x)-A'(2^m x)}(2^m t)\\ (3.32)\quad &\geq T(\Lambda_{A(2^m x)-g(2^m x)}(2^{m-1}t), \Lambda_{g(2^m x)-A'(2^m x)}(2^{m-1}t))\\ &\geq T(T_{\ell=1}^{\infty}(\tilde{T}_{2^{m+\ell-1}x}(2^{m-1}t)), T_{\ell=1}^{\infty}(\tilde{T}_{2^{m+\ell-1}x}(2^{m-1}t))), \ \ \forall x{\in}X, t{>}0. \end{aligned}$$

By letting $m \to \infty$ in (3.32), we get $A = A'$. This completes the proof. □

Theorem 3.3. *Let $\xi : X^2 \to D^+$ be a function such that*

$$\lim_{m\to\infty} T(\xi_{2^m x, 2^m y}(2^{3m}t), \xi_{2^m x, 2^m y}(2^{3m-2}t)) = 1 \tag{3.33}$$

and

$$\lim_{m\to\infty} T_{\ell=1}^{\infty}(\tilde{T}_{2^{m+\ell-1}x}(2^{3m+2\ell-1}t) = 1, \ \ \forall x \in X, t > 0. \tag{3.34}$$

Suppose that an odd function $f : X \to Y$ satisfies (3.3) *for all $x, y \in X$ and $t > 0$. Then the limit*

$$C(x) = \lim_{m\to\infty} \frac{1}{2^{3m}} \left(f(2^{m+1}x) - 2f(2^m x)\right)$$

exists for all $x \in X$ and $C : X \to Y$ is a unique cubic function satisfying

$$\Lambda_{f(2x)-2f(x)-C(x)}(t) \geq T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(2^{2\ell}t)), \ \ \forall x \in X, t > 0, \tag{3.35}$$

where $\tilde{T}_x(t)$ is defined as in Theorem 3.2.

Proof. By the similar method as in the proof of Theorem 3.2, we can obtain

$$\Lambda_{f(4x)-10f(2x)+16f(x)}(t) \geq \tilde{T}_x(t), \ \ \forall x \in X, t > 0.$$

Let $h : X \to Y$ be a function defined by $h(x) := f(2x) - 2f(x)$, for all $x \in X$. Thus (3.28) implies that

$$\Lambda_{\frac{h(2x)}{2^3}-h(x)}(t) \geq \tilde{T}_x(2^3 t) \geq \tilde{T}_x(2^2 t), \ \ \forall x \in X, t > 0,$$

which implies that

$$\Lambda_{\frac{h(2^{\ell+1}x)}{2^{3(\ell+1)}}-\frac{h(2^\ell x)}{2^{3\ell}}}(t) \geq \tilde{T}_{2^\ell x}(2^{3(\ell+1)}t) \tag{3.36}$$

for all $x \in X$, $t > 0$ and $\ell \in \mathbb{N}$. Thus it follows from (3.36) and (PN_4) that

$$\Lambda_{\frac{h(2^m x)}{2^{3m}} - h(x)}(t) \geq T_{\ell=1}^{m}(\tilde{T}_{2^{\ell-1}x}(2^{2\ell}t)), \quad \forall x \in X, t > 0. \tag{3.37}$$

In order to prove the convergence of the sequence $\{\frac{h(2^m x)}{2^{3m}}\}$, if we replace x with $2^{m'}x$ in (3.37), then we get

$$\Lambda_{\frac{h(2^{m+m'}x)}{2^{3(m+m')}} - \frac{h(2^{m'}x)}{2^{3m'}}}(t) \geq T_{\ell=1}^{m}(\tilde{T}_{2^{m'+\ell-1}x}(2^{3m'+2\ell}t)), \quad \forall x \in X, t > 0.$$

Since the right hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{\frac{h(2^m x)}{2^{3m}}\}$ is a Cauchy sequence. Therefore, one can define the function $C : X \to Y$ by $C(x) := \lim_{m\to\infty} \frac{1}{2^{3m}} h(2^m x)$ for all $x \in X$.

Now, if we replace x, y with $2^m x, 2^m y$ in (3.3), respectively, then it follows that

$$\begin{aligned} &\Lambda_{\frac{\Delta h(2^m x, 2^m y)}{2^{3m}}}(t) = \Lambda_{\frac{\Delta f(2^{m+1}x, 2^{m+1}y)}{2^{3m}} - 2\frac{\Delta f(2^m x, 2^m y)}{2^{3m}}}(t) \\ &\geq T\left(\Lambda_{\frac{\Delta f(2^{m+1}x, 2^{m+1}y)}{2^{3m}}}\left(\frac{t}{2}\right), \Lambda_{\frac{\Delta f(2^m x, 2^m y)}{2^{3m-1}}}\left(\frac{t}{2}\right)\right) \\ &\geq T(\xi_{2^{m+1}x, 2^{m+1}y}(2^{3m-1}t), \xi_{2^m x, 2^m y}(2^{3m-2}t)), \quad \forall x, y \in X, t > 0. \end{aligned} \tag{3.38}$$

By letting $m \to \infty$ in (3.38), we find that $\Lambda_{\Delta C(x,y)}(t) = 1$ for all $t > 0$, which implies $\Delta C(x, y) = 0$ and so C satisfies (1.4). Therefore, by Theorem 2.1 (see Lemma 2.2 in [20]), the function $C : X \to Y$ is cubic. The rest of the proof is similar to the proof of Theorem 3.2. This completes the proof. □

Theorem 3.4. *Let $\xi : X^2 \to D^+$ be a function satisfies* (3.11) *for all $x, y \in X$ and $t > 0$. Suppose that* (3.12) *for all $x \in X$ and $t > 0$ holds and an odd function $f : X \to Y$ satisfies* (3.3) *for all $x, y \in X$ and $t > 0$. Then there exists an additive function $A : X \to Y$ and a cubic function $C : X \to Y$ such that*

$$\Lambda_{f(x)-A(x)-C(x)}(t) \geq T(T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3t)), T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3(2^{2\ell})t))), \tag{3.39}$$

for all $x \in X$ and $t > 0$, where $\tilde{T}_x(t)$ is defined as in Theorem 3.2.

Proof. By Theorems 3.2 and 3.3, there exist a unique additive function $A_o : X \to Y$ and a unique cubic function $C_o : X \to Y$ such that

$$\Lambda_{f(2x)-8f(x)-A_0(x)}(t) \geq T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(t)) \tag{3.40}$$

and

$$\Lambda_{f(2x)-2f(x)-C_0(x)}(t) \geq T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(2^{2\ell}t)), \quad \forall x \in X, t > 0. \tag{3.41}$$

Thus it follows from (3.40) and (3.41) that

$$\Lambda_{f(x)+\frac{1}{6}A_0(x)-\frac{1}{6}C_0(x)}(t) \geq T(\Lambda_{f(2x)-8f(x)-A_0(x)}(3t), \Lambda_{f(2x)-2f(x)-C_0(x)}(3t)),$$

for all $x \in X$ and $t > 0$. Thus we obtain (3.39) by letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ for all $x \in X$. This completes the proof. □

Finally, we are ready to prove the main theorem concerning the stability problem for the functional equation (1.4) in Menger probabilistic normed spaces.

Theorem 3.5. *Let $\xi : X^2 \to D^+$ be a function satisfies* (3.1) *and* (3.11) *for all $x, y \in X$ and $t > 0$. Suppose that* (3.2) *and* (3.12) *for all $x \in X$ and $t > 0$ hold and a function $f : X \to Y$ satisfies* (3.3) *for all $x, y \in X$ and $t > 0$. Furthermore, assume that $f(0) = 0$ in* (3.3), *where f is even. Then there exists an additive function $A : X \to Y$, a cubic function $C : X \to Y$ and a unique quartic function $Q : X \to Y$ satisfying* (1.4) *and*

$$\begin{aligned}
&\Lambda_{f(x)-A(x)-C(x)-Q(x)}(t) \\
&\geq T\Big(T\Big(T_{\ell=1}^{\infty}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell}}t\Big)\Big), T_{\ell=1}^{\infty}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell}}t\Big)\Big)\Big), \\
&\quad T\Big(T\Big(T_{\ell=1}^{\infty}\Big(\tilde{T}_{2^{\ell-1}x}\Big(\frac{3}{2}t\Big)\Big), T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3(2^{2\ell-1})t)\Big)\Big), \\
&\quad T\Big(T_{\ell=1}^{\infty}\Big(\tilde{T}_{2^{\ell-1}x}\Big(\frac{3}{2}t\Big)\Big), T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3(2^{2\ell-1})t))\Big)\Big)\Big),
\end{aligned} \tag{3.42}$$

for all $x \in X, t > 0$, where $\tilde{T}_x(t)$ is defined as in Theorem 3.2.

Proof. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\begin{aligned}
\Lambda_{\Delta f_e(x,y)}(t) &= \Lambda_{\frac{\Delta f(x,y)+\Delta f(-x,-y)}{2}}(t) \geq T(\Lambda_{\Delta f(x,y)}(t), \Lambda_{\Delta f(-x,-y)}(t)) \\
&\geq T(\xi_{x,y}(t), \xi_{-x,-y}(t)) = T(\xi_{x,y}(t), \xi_{x,y}(t)), \quad \forall x, y \in X, t > 0.
\end{aligned}$$

Hence, in view of Theorem 3.1, there exist a unique quartic function $Q : X \to Y$ such that

$$\begin{aligned}
\Lambda_{f_e(x)-Q(x)}(t) &\geq T\Big(T_{\ell=1}^{\infty}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell-1}}t\Big)\Big), \\
&\quad T_{\ell=1}^{\infty}\Big(\xi_{0,k^{\ell-1}x}\Big(\frac{k^{4\ell}}{2^{\ell-1}}t\Big)\Big)\Big), \quad \forall x \in X, t > 0.
\end{aligned} \tag{3.43}$$

On the other hand, let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and, by using the above method, it follows from Theorem 3.4 that there exist an additive function $A : X \to Y$ and a cubic function $C : X \to Y$ such that

$$\begin{aligned} &\Lambda_{f_o(x)-A(x)-C(x)}(t) \geq T(T(T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3t)), T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3(2^{2\ell})t))), \\ &T(T_{\ell=1}^{\infty}(U\tilde{T}_{2^{\ell-1}x}(3t)), T_{\ell=1}^{\infty}(\tilde{T}_{2^{\ell-1}x}(3(2^{2\ell})t)))), \quad \forall x \in X, t > 0. \end{aligned} \tag{3.44}$$

Hence (3.42) follows from (3.43) and (3.44). This completes the proof. □

REFERENCES

1. Alsina, C.; Schweizer, B.; Sklar, A. – *Continuity properties of probabilistic norms*, J. Math. Anal. Appl., 208 (1997), 446–452.
2. Alsina, C.; Schweizer, B.; Sklar, A. – *On the definition of a probabilistic normed space*, Aequationes Math., 46 (1993), 91–98.
3. Aoki, T. – *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64–66.
4. Chang, S.S.; Cho, Y.J.; Kang, S.M. – *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers, Inc., New York, 2001.
5. Bavand, Savadkouhi M.; Gordji, M. Eshaghi; Rassias, J. M.; Ghobadipour, N. – *Approximate ternary Jordan derivations on Banach ternary algebras*, J. Math. Phys., 50 (2009), 042303, 9 pp.
6. Ebadian, A.; Najati, A.; Eshaghi Gordji, M. – *On approximate additive-quartic and quadratic-cubic functional equations in two variables on abelian groups*, Results Math., 58 (2010), 39–53.
7. Eshaghi Gordji, M. – *Stability of a functional equation deriving from quartic and additive functions*, Bull. Korean Math. Soc., 47 (2010), 491–502.
8. Eshaghi Gordji, M. – *Stability of an additive-quadratic functional equation of two variables in F-spaces*, J. Nonlinear Sci. Appl., 2 (2009), 251–259.
9. Eshaghi Gordji, E.; Savadkouhi, M.B. – *Stability of cubic and quartic functional equations in non-Archimedean spaces*, Acta Appl. Math., 110 (2010), 1321–1329.
10. Eshaghi Gordji, M.; Savadkouhi, M.B. – *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces*, Appl. Math. Lett., 23 (2010), 1198–1202.
11. Eshaghi Gordji, M.; Khodaei, H. – *On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations*, Abstr. Appl. Anal., 2009, Art. ID 923476, 11 pp.

12. Eshaghi Gordji, M.; Ebadian, A.; Zolfaghari, S. – *Stability of a functional equation deriving from cubic and quartic functions*, Abstr. Appl. Anal., 2008, Art. ID 801904, 17 pp.

13. Eshaghi Gordji, M.; Ghaemi, M.B.; Kaboli Gharetapeh, S.; Shams, S.; Ebadian, A. – *On the stability of J^*-derivations*, J. Geom. Phys., 60 (2010), 454–459.

14. Eshaghi Gordji, M.; Najati, A. – *Approximately J^*-homomorphisms: a fixed point approach*, J. Geom. Phys., 60 (2010), 809–814.

15. Eshaghi Gordji, M.; Khodaei, H.; Khodabakhsh, R. – *General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 72 (2010), 69–84.

16. Eshaghi Gordji, M.; Khodabakhsh, R.; Jung, S.-M.; Khodaei, H. – *AQCQ-functional equation in non-Archimedean normed spaces*, Abstr. Appl. Anal., 2010, Art. ID 741942, 22 pp.

17. Eshaghi Gordji, M.; Khodaei, H. – *The fixed point method for fuzzy approximation of a functional equation associated with inner product spaces*, Discrete Dyn. Nat. Soc., 2010, Art. ID 140767, 15 pp.

18. Eshaghi Gordji, M.; Ghaemi, M.B.; Majani, H. – *Generalized Hyers-Ulam-Rassias theorem in Menger probabilistic normed spaces*, Discrete Dyn. Nat. Soc., 2010, Art. ID 162371, 11 pp.

19. Eshaghi Gordji, M.; Ghaemi, M.B.; Majani, H.; Park, C. – *Generalized Ulam-Hyers Stability of Jensen Functional Equation in Serstnev PN-Spaces*, J. Inequal. Appl. 2010, Article ID 868193, 14 pages.

20. Eshaghi Gordji, M.; Khodaei, H. – *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, Nonlinear Anal., 71 (2009), 5629–5643.

21. Găvruta, P. – *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431–436.

22. Găvruta, P.; Găvruta, L. – *A new method for the generalized Hyers-Ulam-Rassias stability*, Internat. J. Nonlinear Anal. Appl., 1 (2010), 11–18.

23. Eshaghi Gordji, M.; Zolfaghari, S.; Rassias, J.M.; Savadkouhi, M.B. – *Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces*, Abstr. Appl. Anal., 2009, Art. ID 417473, 14 pp.

24. Eshaghi-Gordji, M.; Kaboli-Gharetapeh, S.; Park, Choonkil; Zolfaghari, S. – *Stability of an additive-cubic-quartic functional equation*, Adv. Difference Equ., 2009, Art. ID 395693, 20 pp.

25. Eshaghi Gordji, M.; Kaboli Gharetapeh, S.; Rassias, J.M.; Zolfaghari, S. – *Solution and stability of a mixed type additive, quadratic, and cubic functional equation*, Adv. Difference Equ., 2009, Art. ID 826130, 17 pp.

26. Eshaghi Gordji, M.; Zolfaghari, S.; Rassias, J.M.; Savadkouhi, M.B. – *Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces*, Abstr. Appl. Anal., 2009, Art. ID 417473, 14 pp.

27. Eshaghi Gordji, M.; Savadkouhi, M.B. – *Stability of mixed type cubic and quartic functional equations in random normed spaces*, J. Inequal. Appl., 2009, Art. ID 527462, 9 pp.

28. Eshaghi Gordji, M.; Abbaszadeh, S.; Park, C. – *On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces*, J. Inequal. Appl., 2009, Art. ID 153084, 26 pp.

29. Eshaghi Gordji, M.; Savadkouhi, M.B.; Park, C. – *Quadratic-quartic functional equations in RN-spaces*, J. Inequal. Appl., 2009, Art. ID 868423, 14 pp.

30. Eshaghi Gordji, M.; Savadkouhi, M.B. – *Approximation of generalized homomorphisms in quasi-Banach algebras*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., 17 (2009), 203–213.

31. Eshaghi Gordji, M.; Savadkouhi, M.B.; Bidkham, M. – *Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces*, J. Comput. Anal. Appl., 12 (2010), 454–462.

32. Hadžić, O.; Pap, E. – *Fixed Point Theory in Probabilistic Metric Spaces*, Mathematics and its Applications, 536, Kluwer Academic Publishers, Dordrecht, 2001.

33. Hadžić, O.; Pap, E.; Budinčević, M. – *Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces*, Uncertainty modelling, 2001 (Bratislava), Kybernetika (Prague), 38 (2002), 363–382.

34. Hyers, D.H. – *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., 27 (1941), 222–224.

35. Hyers, D.H.; Isac, G.; Rassias, T.M. – *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser Boston, Inc., Boston, MA, 1998.

36. Jun, K.-W.; Kim, H.-M. – *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl., 274 (2002), 267–278.

37. Khodaei, H.; Rassias, Th.M. – *Approximately generalized additive functions in several variables*, Internat. J. Nonlinear Anal. Appl., 1 (2010), 22–41.

38. Lee, S.H.; Im, S.M.; Hwang, I.S. – *Quartic functional equations*, J. Math. Anal. Appl., 307 (2005), 387–394.

39. Menger, K. – *Statistical metric*, Proc. Nat. Acad. Sci. USA, 28 (1942), 535–537.

40. Najati, A.; Eskandani, G.Z. – *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., 342 (2008), 1318–1331.

41. Najati, A.; Rassias, T.M. – *Pexider type operators and their norms in X_λ spaces*, Czechoslovak Math. J., 59 (2009), 1087–1094.

42. Park, C.; Najati, A. – *Generalized additive functional inequalities in Banach algebras*, Interant. J. Nonlinear Anal. Appl., 1 (2010), 54–62.

43. Park, C.; Rassias, T.M. – *Isomorphisms in unital C^*-algebras*, Interant. J. Nonlinear Anal. Appl., 1 (2010), 1–10.

44. Rassias, T.M. – *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978), 297–300.

45. Rassias, T.M. – *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., 62 (2000), 23–130.

46. Rassias, T.M.; Šemrl, P. – *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl., 173 (1993), 325–338.

47. Shakeri, S.; Saadati, R.; Sadeghi, Gh.; Vaezpour, S.M. – *Stability of cubic equation in Menger probabilistic normed spaces*, J. Appl. Sci., 9 (2009), 1795–1797.

48. Schweizer, B.; Sklar, A. – *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

49. Sherstnev, A.N. – *On the notion of a random normed space*, Dokl. Akad. Nauk SSSR, 149 (1963), 280–283.

50. Ulam, S.M. – *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

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