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HARMONIC AND HOLOMORPHIC VECTOR FIELDS ON AN f-MANIFOLD WITH PARALLELIZABLE KERNEL

BY

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Abstract. We consider a natural generalization of the metric almost contact manifolds that we call metric $f.pk$-manifolds. They are Riemannian manifolds with a compatible f-structure which admits a parallelizable kernel. With some additional conditions they are called $\mathcal{S}$-manifolds. We give some examples and study some properties of harmonic 1-forms on such manifolds. We also study harmonicity and holomorphicity of vector fields on them.

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1. Introduction

In the present paper we deal with a generalization of almost contact metric manifolds, that is we consider Riemannian manifolds of dimension $2n + s$ equipped with an f-structure φ of rank $2n$ which has parallelizable kernel, and is compatible with the Riemannian metric. We call them $f.pk$-structures. They are also known as *globally framed f-manifolds* (cf. [13, 14]). When certain conditions are satisfied we obtain more specific structures such as almost $\mathcal{S}$-structures and $\mathcal{S}$-structures that are natural generalizations of contact metric and Sasakian structures, respectively. There is a rich bibliography regarding these objects on manifolds (e.g. cf. [1, 5, 11]).

One of the reasons of the study of such structures is that there exist examples of even dimensional manifolds which are never Kähler but still admit an $\mathcal{S}$-structure. In fact, in [8] an $\mathcal{S}$-structure on the 4-dimensional manifold $U(2)$ is constructed (cf. Example 3.1 of the present paper).

In the next section we recall some definitions and properties which will be used later. The third section is dedicated to the study of the harmonic vector fields on a compact $\mathcal{S}$-manifold M^{2n+s} while in the fourth section we present some examples of harmonic 1-forms or vector fields on particular compact $\mathcal{S}$-manifolds, obtained using some results of [9]. In the fifth section from certain properties of the harmonic 1-forms we get that the Ricci curvature assumes strictly positive and strictly negative values. Finally, in the last section we study properties of $\mathcal{D}$-holomorphicity of vector fields on $f.pk$-manifolds as well as some pertinent examples, generalizing many results obtained in the contact case (cf. [4]).

All manifolds, maps, distributions considered here are smooth i.e. of the class C^∞; we denote by $\mathcal{F}(-)$ the algebra of differentiable functions over the corresponding manifold and by $\Gamma(-)$ the set of all sections of the corresponding bundle.

2. Preliminaries

Let M be a $(2n+s)$-dimensional manifold equipped with an f-structure φ, vector fields $\xi_1,\ldots,\xi_s$ and 1-forms $\eta^1,\ldots,\eta^s$ such that for all $i,j \in \{1,\ldots,s\}$, $\eta^i(\xi_j)=\delta^i_j$ and $\varphi^2=-\mathrm{Id}+\sum_{j=1}^s \eta^j\otimes\xi_j$, from which it follows that $\varphi(\xi_i)=0$, $\eta^i\circ\varphi=0$. The set (M,φ,ξ_i,η^i), $i\in\{1,\ldots,s\}$, is called an *f-manifold with parallelizable kernel* (shortly:*$f.pk$-manifold*). If g is a Riemannian metric compatible with the structure, that is it satisfies $g(\varphi X,\varphi Y)=g(X,Y)-\sum_{i=1}^s\eta^i(X)\eta^i(Y)$, for any $X,Y\in\Gamma(TM)$, the set $(M,\varphi,\xi_i,\eta^i,g)$ is called a metric *$f.pk$-manifold*. The distribution $\mathcal{D}:=\mathrm{Im}\,\varphi$ is clearly orthogonal to $\ker\varphi=\mathrm{span}\{\xi_1,\ldots,\xi_s\}$. With a metric $f.pk$-manifold there is naturally associated the Sasaki 2-form defined for each $X,Y\in\Gamma(TM)$ by $F(X,Y):=g(X,\varphi Y)$ and the tensor N of type $(1,2)$ such that $N:=[\varphi,\varphi]+2\sum_{i=1}^s d\eta^i\otimes\xi_i$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. When $N=0$ we say that M is *normal*. Furthermore, we can easily observe that using the Lie differentiation, the normality condition can be written as

$$(\mathcal{L}_{\varphi X}\varphi)Y-\varphi\left((\mathcal{L}_X\varphi)Y\right)=-2\sum_{i=1}^s d\eta^i(X,Y)\xi_i. \tag{2.1}$$

The normality of an $f.pk$-structure is equivalent to the integrability of the well known almost complex structure J on the manifold $\widetilde{M} := M \times \mathbb{R}^s$ defined, for each $\widetilde{X} = (X, \sum_{i=1}^{s} a^i \partial_i) \in \Gamma(T\widetilde{M})$, by:

$$J(X, \sum_{i=1}^{s} a^i \partial_i) := (\varphi X - \sum_{i=1}^{s} a^i \xi_i, \sum_{j=1}^{s} \eta^j(X) \partial_j), \tag{2.2}$$

where $x^1, \dots, x^s$ are natural coordinates on $\mathbb{R}^s$ and $\partial_i = \frac{\partial}{\partial x^i}$. Hence, if $X \in \Gamma(\mathcal{D})$, $i \in \{1, \dots, s\}$, then $J(X, 0) = (\varphi X, 0)$, $J(\xi_i, 0) = (0, \partial_i)$, $J(0, \partial_i) = (-\xi_i, 0)$.

Remark 2.1. It is well known (e.g. see [2]) that the normality condition on an almost contact manifold implies the annihilation of certain tensor fields $N^{(2)}$, $N^{(3)}$, $N^{(4)}$. In the more general case of $f.pk$-manifolds one can prove that if $(M, \varphi, \eta^i, \xi_i)$ is a normal $f.pk$-manifold, then the tensor fields defined by $N_i^{(2)}(X, Y) = (\mathcal{L}_{\varphi X} \eta^i) Y - (\mathcal{L}_{\varphi Y} \eta^i) X$, $N_i^{(3)} = \mathcal{L}_{\xi_i} \varphi$, $N_{i,j}^{(4)} = \mathcal{L}_{\xi_i} \eta^j$, $i, j \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$, vanish. Finally, we remark that one can also write:

$$N_i^{(2)}(X, Y) = 2d\eta^i(\varphi X, Y) - 2d\eta^i(\varphi Y, X), \quad N_{i,j}^{(4)}(X) = 2d\eta^j(\xi_i, X). \tag{2.3}$$

By definition, an almost $\mathcal{S}$-*manifold* is a metric $f.pk$-manifold such that $\eta^1 \wedge \cdots \wedge \eta^s \wedge F^n \neq 0$ (hence orientable) and, for each $i \in \{1, \dots, s\}$, $d\eta^i = F$; furthermore a normal almost $\mathcal{S}$-manifold is said to be an $\mathcal{S}$-manifold.

On an $\mathcal{S}$-manifold we have the following identities involving the Levi-Civita connection ∇ of g (cf. [5, 11])

$$\nabla \xi_i = -\varphi, \quad \nabla \bar{\xi} = -s\varphi, \quad [Z, \xi_i] \in \Gamma(\mathcal{D}) \tag{2.4}$$

$$(\nabla_X \varphi)(Y) = g(\varphi(X), \varphi(Y)) \bar{\xi} + \bar{\eta}(Y) \varphi^2(X), \tag{2.5}$$

where $i \in \{1, \dots, s\}$, $\bar{\xi} = \sum_{i=1}^{s} \xi_i$, $\bar{\eta} = \sum_{i=1}^{s} \eta^i$, $Z \in \Gamma(\mathcal{D})$, $X, Y \in \Gamma(TM)$.

We recall that if a is a positive real number, by a $\mathcal{D}$-homothetic deformation of constant a (cf. [6]) on a metric $f.pk$-manifold $(M, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, we mean a change of the structure tensors in the following way:

$$\widetilde{\varphi} = \varphi \quad \widetilde{\eta}^i = a\eta^i \quad \widetilde{\xi}_i = \frac{1}{a} \xi_i \quad \widetilde{g} = ag + a(a-1) \sum_{j=1}^{s} \eta^j \otimes \eta^j. \tag{2.6}$$

In [6], the following relationship between the Levi-Civita connections of g and $\tilde{g}$ has been proven on the almost $\mathcal{S}$-manifolds

$$a\tilde{\nabla}_X Y = a\nabla_X Y + (1-a)\big(\sum_{i=1}^{s} g(\varphi h_i X, Y)\xi_i + a\,(\bar{\eta}(Y)\varphi X + \bar{\eta}(X)\varphi Y)\big). \tag{2.7}$$

Here each $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$, $i \in \{1, \dots, s\}$, vanishes when ξ_i is Killing (cf. [5]). In particular when M is an $\mathcal{S}$-manifold all the h_i's are zero.

Looking at the Riemannian aspects of a metric $f.pk$-manifold, for the curvature we adopt the definition $R_{XY} := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ for each $X, Y \in \Gamma(TM)$. We denote by $\mathcal{R}$ the Ricci tensor field and by $\mathcal{R}^\sharp$ the Ricci operator defined by $g(\mathcal{R}^\sharp(X), Y) = \mathcal{R}(X, Y)$.

If T is a $(0, 2)$-tensor field on M and θ is a 1-form, we put $T(\theta, \theta) = T(Y, Y)$, where $Y = \theta^\sharp$. Let ϕ be a (1,1)-tensor. Since for any $X \in \Gamma(TM)$, $g((\theta \circ \phi)^\sharp, X) = g(\phi(X), \theta^\sharp) = -g(X, \phi(\theta^\sharp))$, then

$$(\theta \circ \phi)^\sharp = -\phi(\theta^\sharp) \tag{2.8}$$

and hence $g(\theta \circ \phi, \theta \circ \phi) = g(\phi(\theta^\sharp), \phi(\theta^\sharp))$.

For more details about Riemannian geometry and harmonic forms theory we refer to [15, 12].

3. Harmonic vector fields on compact $\mathcal{S}$-manifolds

We recall that a vector field X on a Riemannian manifold (M, g) is harmonic if and only if $X^\flat$ is a harmonic 1-form. Then we can adapt some results on harmonic 1-forms proved in [9] to harmonic vector fields on a compact $\mathcal{S}$-manifold $(M, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$. In particular, we recall that for a harmonic 1-form ω we have $\omega(\bar{\xi}) = 0$ and $\omega \circ \varphi$ is harmonic too. Furthermore, for each $i \in \{1, \dots, s\}$ we have $\mathcal{R}^\sharp(\xi_i) = 2n\bar{\xi}$. Again from [9], we know that the space $\mathcal{H}^1$ of the harmonic 1-forms on a compact $\mathcal{S}$-manifold M orthogonally decomposes as $\mathcal{H}^1_F \oplus \mathcal{H}^1_B$, where $\mathcal{H}^1_F := \{\sum_{i=1}^{s} a_i\eta^i \mid \sum_{i=1}^{s} a_i = 0\}$ is an $(s-1)$-dimensional vector subspace and $\mathcal{H}^1_B := \{\omega \in \mathcal{H}^1 \mid \omega(\xi_i) = 0, \text{ for all } i\}$ has even dimension since admits the almost complex structure $\omega \to \omega \circ \varphi$. It follows that the first Betti number of a compact $\mathcal{S}$-manifold has to be

$$b_1 = s - 1 + k, \tag{3.1}$$

where k is a nonnegative even integer. Then b_1 cannot be zero if $s \geq 2$.

Proposition 3.1. *For any harmonic vector field X one has $\bar{\eta}(X) = 0$ and $\mathcal{R}^\sharp(X) \in \Gamma(\mathcal{D})$.*

Proof. We get $X^\flat(\bar{\xi}) = 0$, that is $\bar{\eta}(X) = g(X, \bar{\xi}) = 0$. Furthermore, $g(\mathcal{R}^\sharp(X), \xi_i) = g(X, \mathcal{R}^\sharp(\xi_i)) = g(X, 2n\bar{\xi}) = 2n\bar{\eta}(X) = 0$. □

Proposition 3.2. *Let $a_1, \dots, a_s$ be real constants such that $\sum_{i=1}^s a_i = 0$. Then, one gets a harmonic vector field putting*

$$X = \sum_{i=1}^{s} a_i \xi_i. \tag{3.2}$$

Proof. In fact, the 1-form $\sum_{i=1}^s a_i \eta^i = X^\flat$ is harmonic. □

We call a vector field defined as in (3.2) a *foliate harmonic vector field*. On the other hand we call *basic harmonic vector field* a harmonic vector field X such that $\eta^i(X) = 0$ for each $i \in \{1, \dots, s\}$. These names are related to the foliation defined by $\ker \varphi$ (cf. [7]).

Remark 3.1. By a well-known result of de Rham, $\mathcal{R}^\sharp(X) = (\mathtt{trace}\nabla^2 X)^\flat$ if X is a harmonic vector field. Then, for any foliate harmonic vector field $X = \sum_{i=1}^s a_i \xi_i$ we get $\mathtt{trace}\nabla^2 X = \mathcal{R}^\sharp(X) = \mathcal{R}^\sharp(\sum_{i=1}^s a_i \xi_i) = (\sum_{i=1}^s a_i) 2n\bar{\xi} = 0$.

We denote by $Ha_F(M)$ the vector space of the foliate harmonic vector fields on M.

Proposition 3.3. *For any fixed $i \in \{1, \dots, s\}$, the set $\{(\xi_i - \xi_j) \mid i \neq j\}$ is a basis of $Ha_F(M)$. Moreover, $[X, X'] = 0$ for any $X, X' \in Ha_F(M)$.*

Proof. Clearly each $\xi_i - \xi_j$, $i \neq j$ is a foliate harmonic vector field and the linear independence of such $s-1$ vector fields is immediate. Finally, for $X = \sum_{i=1}^s a_i \xi_i$ and $X' = \sum_{j=1}^s b_j \xi_j$, being a_i and b_j constant we get $[X, X'] = \sum_{i,j=1}^s (a_i b_j [\xi_i, \xi_j] + a_i \xi_i(b_j)\xi_j - b_j \xi_j(a_i)\xi_i) = 0$. □

Proposition 3.4. *If X is a harmonic vector field, then φX is harmonic too.*

Proof. Since $X^\flat$ is harmonic then $X^\flat \circ \varphi$ is harmonic. Hence, using (2.8) we get that $\varphi X = \varphi((X^\flat)^\sharp) = -(X^\flat \circ \varphi)^\sharp$ is harmonic. □

The above results and the orthogonal decomposition $\mathcal{H}^1 = \mathcal{H}^1_F \oplus \mathcal{H}^1_B$ allows to obtain the following proposition.

Theorem 3.1. *Any harmonic vector field can be written in a unique way as sum of a foliate harmonic vector field and a basic harmonic vector field.*

Remark 3.2. If X is a harmonic vector field, then $\varphi(X)$ and $\varphi^2(X)$ are harmonic and, since $X = -\varphi^2(X) + \sum_{i=1}^{s} \eta^i(X)\xi_i$ is an orthogonal decomposition, we get that $-\varphi^2(X)$ is the basic component of X. Hence $\eta^i(X)$ is constant for any $i \in \{1, \dots, s\}$ and $\sum_{i=1}^{s} \eta^i(X) = 0$.

Since $\mathcal{D}$-homothetic deformations preserve the harmonicity of 1-forms (cf. [9]), then we get the following:

Proposition 3.5. *The harmonicity of vector fields is invariant under $\mathcal{D}$-homothetic deformations.*

We end this section describing harmonic 1-forms and harmonic vector fields on two examples of $\mathcal{S}$-manifolds.

Example 3.1 ([8]). We consider the $(4 = 2\cdot 1+2)$-dimensional manifold $U(2)$ and its Lie algebra $u(2)$ with basis $X = E_{12} - E_{21}$, $Y = \mathrm{i}(E_{12} + E_{21})$, $\xi_1 = \mathrm{i}E_{11}$ and $\xi_2 = -\mathrm{i}E_{22}$, where $\{E_{ij}\}$, $i, j \in \{1, 2\}$, is the canonical basis of $M_2(\mathbb{C})$. We consider the metric g such that the basis X, Y, ξ_1, ξ_2 is orthonormal, the 1-forms η^1, η^2 dual to ξ_1, ξ_2, and we define a (1,1)-tensor φ by putting $\varphi(X) = Y$, $\varphi(Y) = -X$, $\varphi(\xi_1) = \varphi(\xi_2) = 0$. Preserving the same symbols, one extends all these data to $U(2)$ by left-invariance and one proves that $\mathbf{U}(2) = (U(2), \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is an $\mathcal{S}$-manifold. It is known that $U(2)$ does not admit a Kähler or a symplectic structure; namely, (cf. [8]), its Betti numbers are

$$b_0 = 1,\ b_1 = 1,\ b_2 = 0,\ b_3 = 1,\ b_4 = 1. \tag{3.3}$$

We have the following:

Proposition 3.6. *The space of the harmonic 1-forms on the $\mathcal{S}$-manifold $\mathbf{U}(2)$ is spanned by the 1-form $\eta_1 - \eta_2$ and the harmonic vector fields are foliate and spanned by the vector field $\xi_1 - \xi_2$.*

Proof. From (3.1), (3.3) we get $1 = s - 1 + k$ and then $k = 0$. Hence any harmonic 1-form on $\mathbf{U}(2)$ belongs to $\mathcal{H}^1_F$ and is given by $a_1\eta^1 + a_2\eta^2$ with $a_1, a_2 \in \mathbb{R}$, $a_1 + a_2 = 0$. This proves the first assertion. Then, the second follows immediately. □

Example 3.2 ([1, 8])**.** The Hopf fibration $\pi_0 : S^{2n+1} \to \mathbb{C}P^n$. It is an $\mathbb{S}^1$-principal bundle and the projection π_0 is the Riemannian holomorphic fibration with respect to the canonical Sasakian structure on $\mathbb{S}^{2n+1}$ and the Fubini-Study structure on $\mathbb{C}P^n$. Let $\Delta : \mathbb{C}P^n \to (\mathbb{C}P^n)^s$, $s \geq 2$, be the canonical diagonal immersion and E^{2n+s} the induced pull-back bundle. Then the following diagram

$$\begin{array}{ccc} E^{2n+s} & \xrightarrow{\hat{\Delta}} & (\mathbb{S}^{2n+1})^s \\ \pi \big\downarrow & & (\pi_0)^s \big\downarrow \\ \mathbb{C}P^n & \xrightarrow{\Delta} & (\mathbb{C}P^n)^s \end{array} \tag{3.4}$$

commutes. The $(2n+s)$-dimensional manifold E^{2n+s} inherits a structure of $f.pk$-manifold from the toroidal bundle $(\pi_0)^s : (\mathbb{S}^{2n+1})^s \to (\mathbb{C}P^n)^s$ via the map $\hat{\Delta}$. It is proven in [1] that E^{2n+s} is an $\mathcal{S}$-manifold; moreover, it is compact as diffeomorphic to $\mathbb{S}^{2n+1} \times (\mathbb{S}^1)^{s-1}$. In [8], it is shown that if s is even, $s = 2t$, then b_1 is odd, which implies that E^{2n+2t} cannot carry a Kähler structure for any values $n, t \in \mathbb{N}^*$. In fact, the first Betti number of E^{2n+s} is $b_1 = s - 1$. In a similar way as in Example 3.1 we get the following

Proposition 3.7. *The harmonic 1-forms on E^{2n+s} are of the type $\sum_{i=1}^{s} a_i \eta^i$, $\sum_{i=1}^{s} a_i = 0$. Moreover, all the harmonic vector fields are foliate.*

4. Curvature and harmonic 1-forms

Bochner proved ([3]) that if the Ricci curvature of a compact Riemannian manifold is positive definite then there is no harmonic 1-form on the manifold and the first Betti number has to be zero. Furthermore, he proved that if the Ricci tensor of a compact oriented manifold is negative definite then every Killing vector field must be parallel. Then the Ricci curvature of a compact $\mathcal{S}$-manifold cannot be positive definite since $b_1 \neq 0$. Moreover, since the Killing vector fields $\xi_1, \ldots, \xi_s$ are not parallel, then the Ricci curvature cannot be negative definite. We will prove in this section that the Ricci curvature of a compact $\mathcal{S}$-manifold, $s \geq 2$, assumes strictly positive and strictly negative values. The following Proposition relates the curvature and the Ricci tensor fields of the metric g and the metric $\widetilde{g}$ of a $\mathcal{D}$-homothetic deformation of the given structure.

Proposition 4.1. *Let* $(M, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, *be an* $\mathcal{S}$*-manifold and let* $(\widetilde{\varphi}, \widetilde{\xi}_i, \widetilde{\eta}^i, \widetilde{g})$ *be an* $\mathcal{S}$*-structure on* M *obtained with a* $\mathcal{D}$*-homothetic deformation of constant* a. *Then, we have:*

$$
(4.1)\quad \begin{aligned}
\widetilde{R}_{XY}Z &= R_{XY}Z + (1-a)\{(\bar{\eta}(Y)g(\varphi X, \varphi Z) - \bar{\eta}(X)g(\varphi Y, \varphi Z))\bar{\xi} \\
&\quad + s(2F(X,Y)\varphi Z + F(X,Z)\varphi Y - F(Y,Z)\varphi X) \\
&\quad + (1+a)\bar{\eta}(Z)(\bar{\eta}(Y)\varphi^2 X - \bar{\eta}(X)\varphi^2 Y)\}
\end{aligned}
$$

$$
(4.2)\quad \widetilde{\mathcal{R}}(X,Y) = \mathcal{R}(X,Y) + 2s(1-a)g(\varphi X, \varphi Y) - 2n(1-a^2)\bar{\eta}(X)\bar{\eta}(Y).
$$

Proof. The following relationship between the Levi-Civita connections can be obtained using (2.7)

$$
(4.3)\quad \widetilde{\nabla}_X Y = \nabla_X Y + (1-a)(\bar{\eta}(Y)\varphi(X) + \bar{\eta}(X)\varphi(Y)).
$$

A long computation gives (4.1). Let now $\{E_1, \dots, E_{2n}, \xi_1, \dots, \xi_s\}$ be a φ-basis. Then $\left\{\tilde{E}_1 = \frac{1}{\sqrt{a}}E_1, \dots, \widetilde{E}_{2n} = \frac{1}{\sqrt{a}}E_{2n}, \widetilde{\xi}_1 = \frac{1}{a}\xi_1, \dots, \widetilde{\xi}_s = \frac{1}{a}\xi_s\right\}$ is a $\widetilde{g}$-orthonormal $\widetilde{\varphi}$-basis, and using such a $\widetilde{\varphi}$-basis and (4.1) we get (4.2). □

Remark 4.1. In the notation of Proposition 4.1, if we fix a local orthonormal basis with respect to g, then the local expression of $\widetilde{g}$ is given by

$$
\begin{pmatrix} aI_{2n} & 0 \\ 0 & a^2 I_s \end{pmatrix},
$$

where I_{2n} and I_s are the identity matrices of order $2n$ and s, respectively. Hence we have the relationship between the volume elements

$$
(4.4)\quad \nu_{\widetilde{g}} = \sqrt{det\widetilde{g}} = a^{n+s}\nu_g.
$$

The following result generalizes a result of Tanno ([20]).

Proposition 4.2. *Let* $(M, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, *be a compact* $\mathcal{S}$*-manifold. Then there is no harmonic 1-form* ω *such that*

$$
(4.5)\quad \mathcal{R}(\omega, \omega) + 2sg(\omega \circ \varphi, \omega \circ \varphi) \geq 0
$$

and the inequality holds at least at a point of M.

Proof. We suppose that there exists a harmonic 1-form ω satisfying (4.5) and such that the inequality holds at least at one point of M. Then

$$\int_M \mathcal{R}(\omega,\omega) + 2sg(\omega\circ\varphi,\omega\circ\varphi)\nu_g > 0. \tag{4.6}$$

Thus there is $\epsilon > 0$ such that $\int_M \mathcal{R}(\omega,\omega) + (2s-\epsilon)g(\omega\circ\varphi,\omega\circ\varphi)\nu_g > 0$. We choose $a \in \mathbb{R}$ such that $0 < a < \frac{\epsilon}{2s}$, that is $2s-\epsilon < 2s(1-a)$ and then

$$\int_M \mathcal{R}(\omega,\omega) + 2s(1-a)g(\omega\circ\varphi,\omega\circ\varphi)\nu_g > 0. \tag{4.7}$$

Let $(\widetilde{\varphi},\widetilde{\xi}_i,\widetilde{\eta}^i,\widetilde{g})$ be the $\mathcal{S}$-structure on M obtained with a $\mathcal{D}$-homothetic deformation of constant a. Since ω is harmonic with respect to g, then from (4.2) and $\omega(\bar{\xi}) = 0$ we get $\widetilde{\mathcal{R}}(\omega,\omega) = \mathcal{R}(\omega,\omega)+2s(1-a)g(\omega\circ\varphi,\omega\circ\varphi)$. Then by (4.4) and (4.7) we have that ω is a harmonic 1-form with respect to $\widetilde{g}$ such that $\int_M \widetilde{\mathcal{R}}(\omega,\omega)\nu_{\widetilde{g}} = a^{n+s}\int_M \widetilde{\mathcal{R}}(\omega,\omega)\nu_g > 0$, contradicting a well known result of Yano and Bochner (cf. [25]). □

Now we prove a result that is, in our context, stronger than the Bochner's.

Theorem 4.1. *Let $(M,\varphi,\xi_i,\eta^i,g)$, $i \in \{1,\dots,s\}$, be a compact $\mathcal{S}$-manifold of dimension $2n+s$, $s \geq 2$. Then there exist strictly positive and strictly negative Ricci curvatures.*

Proof. It easily follows from Proposition 4.2 that for each $t \in \mathbb{R}$, $0 \leq t \leq s$ there exists no harmonic form ω such that

$$\mathcal{R}(\omega,\omega) + 2tg(\omega\circ\varphi,\omega\circ\varphi) \geq 0 \tag{4.8}$$

and the inequality holds at least at a point of M. Moreover, for each $t \in \mathbb{R}$, $0 \leq t \leq s$, $\mathcal{R}'_t := \mathcal{R}+2tg(\varphi-,\varphi-)$ cannot be positive definite, otherwise the first Betti number should be zero, a contradiction. Thus, for any $t \in [0,s]$ there exist a point $x_t \in M$ and a vector $X_t \in \Gamma(\mathcal{D}_{x_t})$ with $\mathcal{R}'_t(X_t,X_t) \leq 0$. Hence, if we take $t \neq 0$, we have $\mathcal{R}(X_t,X_t) \leq -2tg(X_t,X_t) < 0$, proving that there are strictly negative Ricci curvatures. Finally, (cf. [9]), for any $i \in \{i,\dots,s\}$, $\mathcal{R}(\xi_i,\xi_i) = g(\mathcal{R}^\sharp(\xi_i),\xi_i) = 2n > 0$ and this ensures the existence of strictly positive Ricci curvatures. □

5. Holomorphicity and $f.pk$-manifolds

Definition 5.1. A vector field X on a $(2n+s)$-dimensional $f.pk$-manifold $(M, \varphi, \xi_i, \eta^i)$, $i \in \{1, \dots, s\}$, is said to be $\mathcal{D}$-*holomorphic* if for each $Y \in \Gamma(TM)$

$$(\mathcal{L}_X \varphi)Y \in \Gamma(\ker \varphi). \tag{5.1}$$

Moreover, we say that a distribution $\mathcal{V}$ is $\mathcal{D}$-*holomorphic* if around each point there is a local frame consisting of $\mathcal{D}$-holomorphic vector fields.

Remark 5.1. Definition 5.1 means that for each $Y \in \Gamma(TM)$ the component of $(\mathcal{L}_X \varphi)Y$ in $\mathcal{D}$ vanishes. Then the $\mathcal{D}$-holomorphicity of any vector field X is equivalent to $\varphi \circ (\mathcal{L}_X \varphi) = 0$. Denoting by $\alpha_i(X)$ the component of $(\mathcal{L}_X \varphi)Y$ in the direction of ξ_i and writing $(\mathcal{L}_X \varphi)Y = \sum_{j=1}^s \alpha_j(X)\xi_j$ we have $\alpha_k(X) = \eta^k((\mathcal{L}_X \varphi)Y) = \eta^k([X, \varphi Y])$. Thus condition (5.1) is equivalent to $(\mathcal{L}_X \varphi)Y = \sum_{i=1}^s \eta^i([X, \varphi Y])\xi_i$.

By Remark 2.1 we get immediately:

Lemma 5.1. *In a normal $f.pk$-manifold the vector fields $\xi_1, \dots, \xi_s$ are $\mathcal{D}$-holomorphic and* $\ker \varphi$ *is a $\mathcal{D}$-holomorphic distribution.*

Proposition 5.1. *If X is a $\mathcal{D}$-holomorphic vector field, then $[X, \xi_i] \in \Gamma(\ker \varphi)$. Moreover, $[X, \zeta] \in \Gamma(\ker \varphi)$ for any $\zeta \in \Gamma(\ker \varphi)$.*

Proof. In fact $\varphi([X, \xi_i]) = -(\mathcal{L}_X \varphi)\xi_i = 0$, as $\varphi \xi_i = 0$ and then, writing $\zeta = \sum_{i=1}^s f^i \xi_i$, we get $[X, \zeta] = \sum_{i=1}^s (f^i[X, \xi_i] + X(f^i)\xi_i) \in \Gamma(\ker \varphi)$. □

Proposition 5.2. *Let $(M, \varphi, \xi_i, \eta^i)$, $i \in \{1, \dots, s\}$, be an $f.pk$-manifold, $U_1, \dots, U_r$ be $\mathcal{D}$-holomorphic vector fields and $\lambda^1, \dots, \lambda^r \in \mathcal{F}(M)$. Then the vector field $X := \sum_{k=1}^r \lambda^k U_k$ is $\mathcal{D}$-holomorphic if and only if for each $Y \in \Gamma(TM)$ we have $\sum_{k=1}^r \big((\varphi Y)(\lambda^k)\varphi(U_k) - Y(\lambda^k)\varphi^2(U_k)\big) = 0$. Furthermore, if the structure is normal, then any $\mathcal{F}(M)$-linear combination of the ξ_i's is $\mathcal{D}$-holomorphic.*

Proof. We get the claimed equivalence by applying φ to both the sides of the identity $(\mathcal{L}_X \varphi)Y = \sum_{k=1}^s \big(\lambda^k(\mathcal{L}_{U_k}\varphi)(Y) - (\varphi Y)(\lambda^k)U_k + Y(\lambda^k)\varphi(U_k)\big)$. Furthermore, under the normality hypothesis, Lemma 5.1 ensures that any $\mathcal{F}(M)$-linear combination of the ξ_i's is $\mathcal{D}$-holomorphic. □

We denote by $holo_{\mathcal{D}}(M)$ the set of the $\mathcal{D}$-holomorphic vector fields on a normal $f.pk$-manifold M.

Proposition 5.3. *$holo_{\mathcal{D}}(M)$ is a Lie subalgebra of $\Gamma(TM)$ and $\Gamma(\ker\varphi)$ is an ideal of $holo_{\mathcal{D}}(M)$.*

Proof. With a direct computation, for each $Y \in \Gamma(TM)$, we get

$$\begin{aligned}(\mathcal{L}_{[X,X']}\varphi)(Y) &= [X,(\mathcal{L}_{X'}\varphi)(Y)] - (\mathcal{L}_{X'}\varphi)([X,Y]) \\ &\quad -[X',(\mathcal{L}_X\varphi)(Y)] + (\mathcal{L}_X\varphi)([X',Y]).\end{aligned}$$

X, X' being $\mathcal{D}$-holomorphic, Proposition 5.1 implies $(\mathcal{L}_{[X,X']}\varphi)(Y)\in\Gamma(\ker\varphi)$, that is $[X,X']$ is $\mathcal{D}$-holomorphic. Finally, for $X=\sum_{i=1}^{s} f^i\xi_i \in \Gamma(\ker\varphi)$ and $Y\in holo_{\mathcal{D}}(M)$, again Proposition 5.1 implies $[X,Y]\in\Gamma(\ker\varphi)$. □

Proposition 5.4. *Let (M,φ,ξ_i,η^i) be a normal f.pk-manifold. If X is a $\mathcal{D}$-holomorphic vector field then φX is also $\mathcal{D}$-holomorphic.*

Proof. The $\mathcal{D}$-holomorphicity of X and (2.1) yield

$$(\mathcal{L}_{\varphi X}\varphi)Y = -2\sum_{i=1}^{s} d\eta^i(X,Y)\xi_i, \tag{5.2}$$

for each $Y\in\Gamma(TM)$. Hence, φX is $\mathcal{D}$-holomorphic. We observe that, owing to the normality and using (2.3), we have $2d\eta^i(X,Y) = 2d\eta^i(\varphi X,\varphi Y) = -\eta^i([\varphi X,\varphi Y])$, according to Remark 5.1. □

The condition "$\mathcal{D}$-holomorphic" for vector fields can be expressed using an operator "∂" as it has been done for complex structures (e.g. [16]), and Sasakian structures (cf. [4]). We define an operator $\overline{\partial}$ as follows $\overline{\partial}X(Y) := \frac{1}{2}\varphi(\nabla_Y X + \varphi\nabla_{\varphi Y}X - \varphi(\nabla_X\varphi)Y)$. When M is normal, it is not difficult to show, using Remark 2.1, that a vector field X is $\mathcal{D}$-holomorphic if and only if X is annihilated by $\overline{\partial}$.

The vanishing of $\overline{\partial}X(Y)$ when $Y\in\Gamma(\ker\varphi)$, is equivalent to the fact that the vector field X is an infinitesimal automorphism of the foliation $\mathcal{F}$ defined by $\ker\varphi$. Therefore the main strength of the condition is on the $\mathcal{D}$-level.

Let us consider the splitting of the tangent bundle $TM=\ker\varphi\oplus\mathcal{D}$ with the natural projection $\pi\colon TM\to D$. In the subbundle $\mathcal{D}$ we can define a Bott (adapted) connection $\nabla^{\mathcal{D}}$ which, for sections of $\mathcal{D}$, is the restriction of the Levi-Civita connection (e.g. cf. [21], p. 21)

$$\nabla^D_X Z = \begin{cases}\pi[X,Z], & \text{for } X\in\Gamma(\ker\ \varphi),\ \ Z\in\Gamma(\mathcal{D}),\\ \pi(\nabla_X Z), & \text{for } X\in\Gamma(\mathcal{D}),\quad Z\in\Gamma(\mathcal{D}).\end{cases}$$

Using the same formula we define the operator $\partial^{\mathcal{D}}$ for the connection ∇^D. Then for any $X, Y \in \Gamma(\mathcal{D})$ we have $\partial^{\mathcal{D}} X(Y) = \overline{\partial} X(Y)$. From the foliation point of view normal $f.pk$-manifolds are manifolds equipped with a transversely Kähler foliation defined by a locally free action of an abelian Lie group (cf. [7, 10] and for the Sasakian case see [18, 23, 24]), i.e. the foliation $\mathcal{F}$ determined by ker φ is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$, where

1. $\{U_i\}$ is an open covering of M,
2. $f_i \colon U_i \longrightarrow N_0$ are submersions with connected fibres,
3. g_{ij} are local diffeomorphisms of N_0 such that $g_{ij} f_j = f_i$ on $U_i \cap U_j$.

such that on the transverse manifold $N = \amalg_i N_i$, where $N_i = f_i(U_i)$, there are a Riemannian metric $\bar{g}$ and a complex structure J such that $(N, \bar{g}, J)$ is a Kähler manifold and g_{ij} its holomorphic isometries. Moreover, the submersions f_i are Kählerian. Let $\overline{\nabla}$ be the Levi-Civita connection of $(N, \bar{g}, J)$.

Let $\bar{X}$ and $\bar{Y}$ be two vector fields on some N_i, and $X^{\mathcal{D}}$ and $Y^{\mathcal{D}}$ their $\mathcal{D}$-lifts, respectively. These vector fields are infinitesimal automorphisms of the foliation and $\partial^{\mathcal{D}} X^{\mathcal{D}}(Y^{\mathcal{D}}) = \overline{\partial}\bar{X}(\bar{Y})^{\mathcal{D}}$. Therefore we have proved the following proposition:

Proposition 5.5. *Let $(M, \varphi, \xi_i, \eta^i)$, $i \in \{1, \ldots, s\}$, be a normal f.pk manifold. Then a vector field X is $\mathcal{D}$-holomorphic if and only it is an infinitesimal automorphism of the foliation and its transverse part projects to a holomorphic vector field on the transverse manifold, i.e. it is a transversely holomorphic vector field, cf.* [17].

The following result regarding the manifold $\widetilde{M} = M \times \mathbb{R}^s$, equipped with the almost complex structure J described in (2.2), relates holomorphicity in the $\mathcal{D}$- and in the classical sense.

Proposition 5.6. *If $\widetilde{X} = (X, \sum_{i=1}^s a^i \partial_i)$ is a holomorphic vector field on $\widetilde{M} = M \times \mathbb{R}^s$, and M is a normal f.pk-manifold, then X is $\mathcal{D}$-holomorphic.*

Proof. One can easily check that for each $i \in \{1, \ldots, s\}$ and $Y \in \Gamma(TM)$ the following identities hold:

$$(5.3) \qquad (\mathcal{L}_{\widetilde{X}} J)(0, \partial_i) = \big(-[X, \xi_i] - \sum_{j=1}^{s} \partial_i a^j \xi_j, \sum_{j=1}^{s} \xi_i(a^j) \partial_j\big).$$

$$(\mathcal{L}_{\widetilde{X}}J)(Y,0) = \big((\mathcal{L}_X\varphi)Y - \sum_{i=1}^{s} Y(a^i)\xi_i, \sum_{j=1}^{s} b^j\partial_j\big), \tag{5.4}$$

where $b^j := X(\eta^j(Y)) - (\varphi Y)(a^j) - \sum_{i=1}^s \eta^i(Y)\frac{\partial a^j}{\partial x^i} - \eta^j([X,Y])$. Since the holomorphicity of $\widetilde{X}$ means $\mathcal{L}_{\widetilde{X}}J = 0$, by (5.4) we have $(\mathcal{L}_X\varphi)Y = \sum_{i=1}^s Y(a^i)\xi_i$. Hence X is $\mathcal{D}$-holomorphic. □

Theorem 5.1. *Let $\widetilde{X} = (X, \sum_{i=1}^s a^i\partial_i)$ be a vector field on $\widetilde{M}$, with M a normal f.pk-manifold. Then $\widetilde{X}$ is holomorphic if and only if the following properties are verified for each $i,j \in \{1,\dots,s\}$, $Y \in \Gamma(TM)$*

(*a*) $(\mathcal{L}_X\varphi)Y = \sum_{j=1}^s Y(a^j)\xi_j$;

(*b*) $[X,\xi_i] + \sum_{j=1}^s \frac{\partial a^j}{\partial x^i}\xi_j = 0$.

Proof. If $\widetilde{X}$ is holomorphic then (5.3), (5.4) imply (a) and (b). Vice versa, we put ξ_i in place of Y in (a) getting $-\varphi([X,\xi_i]) = \sum_{j=1}^s \xi_i(a^j)\xi_j = 0$ and then $\xi_i(a^j) = 0$, which together with (b) implies $(\mathcal{L}_{\widetilde{X}}J)(0,\partial_i) = 0$ in (5.3). Finally, putting φY in place of Y in (a), we have $(\mathcal{L}_X\varphi)\varphi Y \in \Gamma(\ker\varphi)$ so $(\mathcal{L}_X\varphi)\varphi Y = \sum_{i=1}^s \eta^i([X,\varphi^2 Y])\xi_i$. Using also (b) we get $(\varphi Y)(a^i) = -\eta^i([X,Y]) - \sum_{j=1}^s \eta^j(Y)\frac{\partial a^i}{\partial x^j} + X(\eta^i(Y))$ that is $b_j = 0$, which together with (a) implies $(\mathcal{L}_{\widetilde{X}}J)(Y,0) = 0$ in (5.4). Hence $\widetilde{X}$ is holomorphic. □

Proposition 5.7. *Let M be a normal f.pk-manifold, $X \in \Gamma(\ker\varphi)$. Then $(X,0)$ is holomorphic on $\widetilde{M}$ if and only if $\eta^i(X)$ is constant, for any $i \in \{1,\dots,s\}$. In particular, each $(\xi_i,0)$ is holomorphic and each $(0,\frac{\partial}{\partial x^i})$ is holomorphic too.*

Proof. Consider $X \in \Gamma(\ker\varphi)$. Then $X = \sum_{i=1}^s f^i\xi_i$ is a $\mathcal{D}$-holomorphic vector field on M. We rewrite (a) and (b) of the above theorem for the vector field $(X,0)$ on $\widetilde{M}$, obtaining:
(*a*) $\Leftrightarrow \forall Y \in \Gamma(TM), \forall i \in \{1,\dots,s\}\ \varphi(Y)(f^i) = 0$;
(*b*) $\Leftrightarrow \forall i,j \in \{1,\dots,s\}\ \xi_j(f^i) = 0$.
Then we apply the above theorem, observing that $f^i = \eta^i(X)$. The last assertion follows immediately since $J(\xi_i,0) = (0,\frac{\partial}{\partial x^i})$. □

6. Holomorphic vector fields on $\mathcal{S}$-manifolds

Proposition 6.1. *Suppose that M is an almost $\mathcal{S}$-manifold and X is a vector field on M. Then any two of the following conditions imply the remaining*

(i) $(\mathcal{L}_X g)(Y, Z) = 0$, $\forall Y, Z \in \Gamma(\mathcal{D})$;

(ii) $i_X(d\eta^i)$ *is closed for each* $i \in \{1, \dots, s\}$;

(iii) X *is $\mathcal{D}$-holomorphic.*

Proof. By the Cartan formula, since $F = d\eta^1 = \cdots = d\eta^s$, we have

$$\mathcal{L}_X F = i_X(dF) + d(i_X F) = d(i_X F). \tag{6.1}$$

Moreover, since for each $X, Y, Z \in \Gamma(TM)$ we have

$$(\mathcal{L}_X g)(Y, \varphi Z) = (\mathcal{L}_X F)(Y, Z) - g(Y, (\mathcal{L}_X \varphi) Z), \tag{6.2}$$

then by (6.1) we easily obtain the claim. □

Proposition 6.2. *Let M be an almost $\mathcal{S}$-manifold. If $X \in \Gamma(\mathcal{D})$ is a $\mathcal{D}$-holomorphic vector field, then $[X, \xi_i] = 0$ for any $i \in \{1, \dots, s\}$.*

Proof. By Proposition 5.1 we know that $[X, \xi_i] \in \Gamma(\ker \varphi)$. If M is an almost $\mathcal{S}$ manifold then $\nabla \xi_i = -\varphi - \varphi \circ h_i$ (cf. [11]) so that $\nabla_{\xi_i} \xi_k = 0$. Then we easily get that $g([X, \xi_i], \xi_k) = \eta^k([X, \xi_i]) = 0$. □

From now on we consider $\mathcal{S}$-manifolds. Since the ξ_i's are $\mathcal{D}$-holomorphic, any foliate harmonic vector field is $\mathcal{D}$-holomorphic too. Moreover, by Proposition 3.3 we obtain that $Ha_F(M)$ is an abelian Lie subalgebra of $holo_{\mathcal{D}}(M)$.

Example 6.1. We describe an $\mathcal{S}$-structure on $\mathbb{R}^{2n+s}$ that generalizes the classical Sasakian structure on $\mathbb{R}^{2n+1}$ given by Sasaki (cf. [19]). We put for each $i \in \{1, \dots, s\}$

$$\eta^i := \frac{1}{2}\left(dz^i - \sum_{\alpha=1}^{n} y^\alpha dx^\alpha\right), \quad \xi_i := 2\frac{\partial}{\partial z^i},$$

where $(x^1, \dots, x^n, y^1, \dots, y^n, z^1, \dots, z^s)$ are the natural coordinates of $\mathbb{R}^{2n+s}$. We have $d\eta^1 = \cdots = d\eta^s = \sum_{\alpha=1}^{n} dx^\alpha \wedge dy^\alpha$, $\eta^1 \wedge \cdots \wedge \eta^s \wedge (d\eta^i)^n \neq 0$ and $d\eta^i(\xi_j, X) = 0$, for each $i, j \in \{1, \dots, s\}$, $X \in \Gamma(T\mathbb{R}^{2n+s})$. We put

$$g := \sum_{i=1}^{s} \eta^i \otimes \eta^i + \frac{1}{4}\sum_{\alpha=1}^{n} (dx^\alpha)^2 + (dy^\alpha)^2.$$

We define the metric f–structure φ by giving its matrix with respect to the canonical basis of vector fields of $T\mathbb{R}^{2n+s}$:

$$\begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & B^t & 0 \end{pmatrix},$$

where the (n,s)-matrix B is given by $B_{\alpha i} = y^\alpha$, $\alpha \in \{1,\dots,n\}, i \in \{1,\dots,s\}$. Then $E_1 = 2\frac{\partial}{\partial y^1},\dots,E_n = 2\frac{\partial}{\partial y^n}$, $E_{n+1} = 2\frac{\partial}{\partial x^1} + y^1\overline{\xi},\dots,E_{2n} = 2\frac{\partial}{\partial x^n} + y^n\overline{\xi}$ span $\mathcal{D}$, and $\{E_1,\dots,E_n,\varphi E_1 = E_{n+1},\dots,\varphi E_n = E_{2n},\xi_1,\dots,\xi_s\}$ is a φ-basis. We are going to give a characterization of $\mathcal{D}$-holomorphic vector fields on this structure.

Let us write any vector field X on $\mathbb{R}^{2n+s}$ as

$$X = \sum_{i=1}^{s} \alpha^i \xi_i + \sum_{\sigma=1}^{n} \{\beta^\sigma\ E_\sigma + \gamma^\sigma \varphi E_\sigma\}. \tag{6.3}$$

Theorem 6.1. *A vector field X on $\mathbb{R}^{2n+s}$ is $\mathcal{D}$-holomorphic if and only if for any $\rho,\sigma \in \{1,\dots,n\}$, $i \in \{1,\dots,s\}$*

$$\frac{\partial \beta^\sigma}{\partial z^i} = \frac{\partial \gamma^\sigma}{\partial z^i} = 0, \quad \frac{\partial \beta^\sigma}{\partial x^\rho} + \frac{\partial \gamma^\sigma}{\partial y^\rho} = 0, \quad \frac{\partial \beta^\sigma}{\partial y^\rho} - \frac{\partial \gamma^\sigma}{\partial x^\rho} = 0. \tag{6.4}$$

Proof. With a direct calculation we get that $\frac{\partial}{\partial x^i}$ is $\mathcal{D}$-holomorphic and hence by Proposition 5.4 also $E_i = 2\varphi\frac{\partial}{\partial x^i} = -2\frac{\partial}{\partial y^i}$, φE_i are all $\mathcal{D}$-holomorphic. Moreover, the vector field in (6.3) is $\mathcal{D}$-holomorphic if and only if the vector field $\sum_{\sigma=1}^{n}\big(\beta^\sigma\ E_\sigma + \gamma^\sigma \varphi E_\sigma\big)$ is $\mathcal{D}$-holomorphic. We apply Proposition 5.2 to the vector fields $E_1,\dots,E_n,\varphi E_1,\dots,\varphi E_n$ and get for each vector field Y

$$\sum_{\sigma=1}^{n} \big\{(\varphi Y)(\beta^\sigma) - Y(\gamma^\sigma)\varphi E_\sigma + \big(Y(\beta^\sigma) - (\varphi Y)(\gamma^\sigma)\big)E_\sigma\big\} = 0.$$

Hence we obtain (6.4) taking first $Y = \xi_i$, and then $Y = \frac{\partial}{\partial x^\rho}$. $\square$

One can observe that the last two equations in (6.4) are the Cauchy-Riemann equations referred to the basis $\{X_i = \varphi E_i, X_{n+i} = -E_i,\}$ of $\mathcal{D}$.

Proposition 6.3. *Let $(M,\varphi,\xi_i,\eta^i,g)$, $i \in \{1,\dots,s\}$, be an $\mathcal{S}$-manifold. A vector field X is $\mathcal{D}$-holomorphic if and only if the following conditions hold:*

(i) $[\nabla X, \varphi](Y) \in \Gamma(\ker \varphi)$, *for each* $Y \in \Gamma(\mathcal{D})$,

(ii) $X^{\mathcal{D}} = \varphi(\nabla_{\xi_i} X)$, *for any* $i \in \{1, \dots, s\}$.

Proof. Directly by (2.4) we have that for any $Y \in \Gamma(TM)$

$$(\mathcal{L}_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X - [\nabla X, \varphi](Y). \tag{6.5}$$

If X is $\mathcal{D}$-holomorphic, for $Y \in \Gamma(\mathcal{D})$, (6.5) immediately implies (i). Moreover, by Proposition 6.2 we have $[X, \xi_i] = 0$ and then $\nabla_{\xi_i} X = \nabla_X \xi_i = -\varphi X$. Applying φ we obtain (ii), as $X^{\mathcal{D}} = -\varphi^2 X$. Vice versa, (i) and (6.5) imply $\varphi((\mathcal{L}_X \varphi)(Y)) = 0$ for any $Y \in \Gamma(\mathcal{D})$. Finally, (ii) implies $(\mathcal{L}_X \varphi)(\xi_i) = -\varphi(\nabla_X \xi_i - \nabla_{\xi_i} X) = \varphi^2 X + X^{\mathcal{D}} = 0$. Hence, X is $\mathcal{D}$-holomorphic. □

By a direct calculation we get the following:

Lemma 6.1. *Let $\mathcal{V}$ be an invariant distribution on an $\mathcal{S}$-manifold M^{2n+s} and $\mathcal{H}$ the distribution orthogonal to $\mathcal{V}$. Then for each $X \in \Gamma(\mathcal{H})$, $V \in \Gamma(\mathcal{V})$, $Z \in \Gamma(TM)$ we have*

$$g(X, (\mathcal{L}_V \varphi - \nabla_V \varphi)Z) = g(\nabla_{\varphi Z} X + \nabla_Z \varphi X, V). \tag{6.6}$$

If $\mathcal{V}$ is a distribution on a Riemannian manifold, we denote by $B^{\mathcal{V}}$ and $I^{\mathcal{V}}$, respectively, the second fundamental form and the integrability tensor field of the distribution $\mathcal{V}$, i.e. for each $V, W \in \Gamma(\mathcal{V})$, $I^{\mathcal{V}}(V, W) = [V, W]^{\mathcal{H}}$, $2B^{\mathcal{V}}(V, W) = (\nabla_V W + \nabla_W V)^{\mathcal{H}}$, where $\mathcal{H}$ is the distribution orthogonal to $\mathcal{V}$.

Proposition 6.4. *Let $\mathcal{V}$ be an invariant distribution of an $\mathcal{S}$-manifold M such that* $\ker \varphi \subset \mathcal{V}$ *and $\mathcal{H}$ be the distribution orthogonal to $\mathcal{V}$. Then we have the following identities, for any $U, V \in \Gamma(\mathcal{V})$, $i \in \{1, \dots, s\}$*

$$2\left(B^{\mathcal{V}}(U, \varphi V) - \varphi B^{\mathcal{V}}(U, V)\right) = \varphi I^{\mathcal{V}}(U, V) - I^{\mathcal{V}}(U, \varphi V) \tag{6.7}$$

$$2B^{\mathcal{V}}(U, \xi_i) = -I^{\mathcal{V}}(U, \xi_i) \tag{6.8}$$

$$B^{\mathcal{V}}(\varphi U, \xi_i) = \varphi B^{\mathcal{V}}(U, \xi_i) \tag{6.9}$$

Proof. We observe that under the hypotheses on $\mathcal{V}$, we have $\dim \mathcal{V} = 2p + s$, as the restriction of φ to $\mathcal{D} \cap \mathcal{V}$ is an almost complex structure. Furthermore, the invariance of $\mathcal{V}$ implies the invariance of $\mathcal{H}$; then denoted by $h : TM \to \mathcal{H}$ the natural projection we have $h \circ \varphi = \varphi \circ h$. Hence by (2.5) and a straightforward computation we get

$$2\left(B^{\mathcal{V}}(U, \varphi U) - \varphi B^{\mathcal{V}}(U, V)\right) = h\left(\nabla_{\varphi V} U - \varphi \nabla_V U\right) = \varphi I^{\mathcal{V}}(U, V) - I^{\mathcal{V}}(U, \varphi V).$$

For proving (6.8) we apply (6.7) to ξ_i and obtain $-2\varphi B^{\mathcal{V}}(U,\xi_i) = \varphi I^{\mathcal{V}}(U,\xi_i)$ and then $2B^{\mathcal{V}}(U,\xi_i) + I^{\mathcal{V}}(U,\xi_i) \in \Gamma(\ker\varphi \ \cap \ \mathcal{H}) = \{0\}$. Finally, since $U \in \Gamma(\mathcal{V})$ we can substitute φU in place of U in (6.8) obtaining

$$2B^{\mathcal{V}}(\varphi U,\xi_i) = -I^{\mathcal{V}}(\varphi U,\xi_i). \tag{6.10}$$

On the other hand applying φ to (6.8) and summing with (6.10) we obtain $B^{\mathcal{V}}(\varphi U,\xi_i) - \varphi B^{\mathcal{V}}(U,\xi_i) = \frac{1}{2}h\left([\xi_i,\varphi U] - \varphi[\xi_i,U]\right) = h\left((\mathcal{L}_{\xi_i}\varphi)U\right) = 0.$ □

Proposition 6.5. *Let $\mathcal{V}$ be an invariant $\mathcal{D}$-holomorphic distribution of a $(2n+s)$-dimensional $\mathcal{S}$-manifold M such that $\ker\varphi \subset \mathcal{V}$ and $\mathcal{H}$ be the orthogonal distribution. We denote by $\mathrm{v}: TM \to \mathcal{V}$ the natural projection. Then for any $X,Y \in \Gamma(\mathcal{H})$, $Z \in \Gamma(TM)$ we have*

$$\mathrm{v}(\nabla_{\varphi Z}X + \nabla_Z\varphi X) = 0 \tag{6.11}$$

$$\varphi B^{\mathcal{H}}(X,Y) + g(X,Y)\bar{\xi} = \frac{1}{2}I^{\mathcal{H}}(X,\varphi Y) \tag{6.12}$$

$$|B^{\mathcal{H}}|^2 + 2(n-p)s = \frac{1}{4}|I^{\mathcal{H}}|^2. \tag{6.13}$$

Moreover, $\mathcal{V}$ is minimal.

Proof. For any vector field $V \in \Gamma(\mathcal{V})$, being $\mathcal{V}$ invariant, by (2.5) we get

$$g(X,(\nabla_V\varphi)Z) = 0. \tag{6.14}$$

For any $V \in \Gamma(\mathcal{V})$ we locally write $V = \sum_{i=1}^s f^iU_i$ with U_i $\mathcal{D}$-holomorphic. Hence $(\mathcal{L}_V\varphi)(Z) = \sum_{i=1}^s\big(f^i(\mathcal{L}_{U_i}\varphi)(Z) - (\varphi Z)(f^i)U_i + Z(f^i)\varphi U_i\big)$ and, since $X \in \Gamma(\mathcal{H})$, we get $g\left((\mathcal{L}_V\varphi)(Z),X\right) = 0$ which, together with (6.14) and (6.6), implies $g(\nabla_{\varphi Z}X + \nabla_Z\varphi X, V) = 0$. Hence (6.11) follows.

Using (6.14), (2.5) once again and $\mathcal{H} \subset \mathcal{D}$, we straightforwardly obtain $\frac{1}{2}g(I^{\mathcal{H}}(X,\varphi Y),V) = g(\varphi B^{\mathcal{H}}(X,Y) + g(X,Y)\bar{\xi},V)$ and hence (6.12). We notice that for each $i \in \{1,\dots,s\}$, $\eta^i(B^{\mathcal{H}}(X,Y)) = 0$, as $\nabla\xi_i = -\varphi$; hence $B^{\mathcal{H}}(X,Y) \in \Gamma(\mathcal{D})$ and then $|\varphi B^{\mathcal{H}}(X,Y)| = |B^{\mathcal{H}}(X,Y)|$. (6.13) follows, since $|\bar{\xi}|^2 = s$.

By (6.7) we have $2\left(B^{\mathcal{V}}(U,\varphi V) - \varphi B^{\mathcal{V}}(U,V)\right) = \mathrm{v}(\varphi[U,V] - [U,\varphi V]) = \mathrm{v}((\mathcal{L}_U\varphi)(V)) = 0$, where $U \in \Gamma(\mathcal{V})$ is $\mathcal{D}$-holomorphic. Then we get $B^{\mathcal{V}}(\varphi U,\varphi V) = -B^{\mathcal{V}}(U,V)$, for each $U,V \in \Gamma(\mathcal{V})$, because $B^{\mathcal{V}}(U,\xi_i) = 0$ and $B^{\mathcal{V}}$ is a symmetric tensor field. Hence using a local $\mathcal{D}$-holomorphic φ-basis of $\mathcal{V}$ we get $\mathtt{trace}(B^{\mathcal{V}}) = 0$, that is $\mathcal{V}$ is minimal. □

We recall the Walczak formula in the case of a Riemannian manifold with two orthogonal distributions $\mathcal{V}$ and $\mathcal{H}$, cf. [22]:

$$div^{\mathcal{V}}(\mathtt{trace}B^{\mathcal{H}})+div^{\mathcal{H}}(\mathtt{trace}B^{\mathcal{V}})+\frac{1}{4}|I^{\mathcal{V}}|^2+\frac{1}{4}|I^{\mathcal{H}}|^2=s_{mix}+|B^{\mathcal{V}}|^2+|B^{\mathcal{H}}|^2,$$

where $s_{mix}=s_{mix}(\mathcal{V},\mathcal{H})=\sum_{j,\alpha}K(e_j\wedge f_\alpha)$, $\{e_j\}$, $j\in\{1,\dots,dim(\mathcal{V})\}$ and $\{f_\alpha\}$, $\alpha\in\{1,\dots,dim(\mathcal{H})\}$ are local basis of $\mathcal{V}$ and $\mathcal{H}$ respectively. In the case of the $\mathcal{S}$-manifolds we have:

Proposition 6.6. *Let $\mathcal{V}$ be an invariant $\mathcal{D}$-holomorphic distribution of a $(2n+s)$-dimensional $\mathcal{S}$-manifold such that $\ker\varphi\subset\mathcal{V}$ and $\mathcal{H}$ be the orthogonal distribution. Then the Walczak formula becomes:*

$$div^{\mathcal{V}}(\mathtt{trace}B^{\mathcal{H}})+2(n-p)s+\frac{1}{4}|I^{\mathcal{V}}|^2=s_{mix}+|B^{\mathcal{V}}|^2. \tag{6.15}$$

Proof. The identity follows from (6.13) and the minimality of $\mathcal{V}$. □

Corollary 6.1. *Let $\mathcal{V}$ be an invariant $\mathcal{D}$-holomorphic distribution of a $(2n+s)$-dimensional $\mathcal{S}$-manifold such that $\ker\varphi\subset\mathcal{V}$ and $\mathcal{H}$ be the orthogonal distribution. If $\mathcal{V}$ is integrable and $\overline{M}$ is a compact leaf then:*

$$\int_{\overline{M}} s_{mix}-2s(n-p)+|B^{\mathcal{V}}|^2=0. \tag{6.16}$$

If, moreover, $s_{mix}\geq 2s(n-p)$ each compact leaf is totally geodesic. Finally, if $s_{mix}>2s(n-p)$, then there are no compact leaves.

Proof. Identity (6.16) follows by (6.15) and $I^{\mathcal{V}}=0$. If $s_{mix}\geq 2s(n-p)$ then $s_{mix}=2s(n-p)$, $|B^{\mathcal{V}}|^2=0$ and $\overline{M}$ is totally geodesic. The last assertion is obvious. □

Corollary 6.2. *In a $(2n+s)$-dimensional $\mathcal{S}$-manifold we have*

$$s_{mix}(\ker\varphi,\mathcal{D})=2ns.$$

Proof. Since $\ker\varphi$ is integrable and totally geodesic with $\mathcal{D}$ as orthogonal distribution, the Walczak formula becomes:

$$div^{\ker\varphi}(\mathtt{trace}B^{\mathcal{D}})+2ns=s_{mix}(\ker\varphi,\mathcal{D}). \tag{6.17}$$

On the other hand $\mathtt{trace}B^{\mathcal{D}}=0$ and this completes the proof. □

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