

CHARACTERIZATION OF SUPER QUASI-EINSTEIN MANIFOLD

BY

K. HALDER, B. PAL^{*}, A. BHATTACHARYYA and T. DE

Abstract. In this paper we give characterizations of super quasi-Einstein manifold, mixed super quasi-Einstein manifold and mixed generalized quasi-Einstein manifold for both even and odd dimensions.

Mathematics Subject Classification 2010: 53C25.

Key words: Einstein manifold, quasi-Einstein manifold, mixed generalized quasi-Einstein manifold, super quasi-Einstein manifold, mixed super quasi-Einstein manifold.

1. Introduction

A Riemannian manifold (M, g) with dimension $(n \geq 2)$, is said to be an Einstein manifold if the following condition $S(X, Y) = \frac{r}{n}g(X, Y)$, holds on M , where S and r denote the Ricci tensor and the scalar curvature of (M, g) respectively. The notion of quasi-Einstein manifold was defined in [6]. A non-flat Riemannian manifold (M, g) , $(n \geq 2)$ is defined to be a quasi-Einstein manifold if the condition $S(X, Y) = \alpha g(X, Y) + \beta \rho(X)\rho(Y)$, is fulfilled on M , where α and β are scalars of which $\beta \neq 0$ and ρ is non-zero 1-form such that $g(X, \xi) = \rho(X)$ for all vector field X .

We call ρ the associated 1-form and ξ is called the generator of the manifold. In this paper we consider ξ as a unit vector field.

In [4], CHAKI introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super

^{*}Second author is supported by DST PURSE Project grant P-1/RS/33/10 of India.

quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$(1.1) \quad \begin{aligned} S(X, Y) = & \alpha g(X, Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] \\ & + \delta D(X, Y), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are scalars such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0$, $\forall X \in \chi(M)$.

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

In [2], BHATTACHARYYA, TARAFDAR and DEBNATH introduced the notion of mixed super quasi-Einstein manifold, denoted by $MS(QE)_n$ with an example. A non-flat Riemannian manifold (M^n, g) , $(n \geq 3)$ is called mixed super quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(1.2) \quad \begin{aligned} S(X, Y) = & \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) \\ & + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y), \end{aligned}$$

where $\alpha, \beta, \varrho, \gamma, \delta$ are scalars such that $\beta, \varrho, \gamma, \delta$ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0$, $\forall X \in \chi(M)$.

Here $\alpha, \beta, \varrho, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

In [3], BHATTACHARYYA and DE introduced the notion of mixed generalized quasi-Einstein manifold with an example. A non-flat Riemannian manifold is called mixed generalized quasi-Einstein manifold if its Ricci tensor is non-zero and satisfies the condition

$$(1.3) \quad \begin{aligned} S(X, Y) = & \alpha g(X, Y) + \beta \omega(X)\omega(Y) + \gamma \mu(X)\mu(Y) \\ & + \delta[\omega(X)\mu(Y) + \omega(Y)\mu(X)], \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are non-zero scalars, $g(X, \xi_1) = \omega(X)$, $g(X, \xi_2) = \mu(X)$ and $g(\xi_1, \xi_2) = 0$.

ω, μ are two non-zero 1-forms, ξ_1, ξ_2 are unit vector fields corresponding to the 1-forms ω and μ respectively.

Let M be an m -dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(U \wedge V)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M$, where $\{U, V\}$ is an orthonormal basis of π . For any n -dimensional subspace $L \subseteq T_p M$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, \dots, e_n\}$ is any orthonormal basis of L ([8]). When $L = T_p M$, the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p .

The well-known characterization of 4-dimensional Einstein spaces was given by SINGER and THORPE in [8]. Later we have seen that the generalization of the characterization of 4-dimensional Einstein spaces was given by CHEN in [5]. On the other hand, in [7] DUMITRU obtained the result for odd dimensional Einstein spaces. Also these results (both even and odd dimensions) were generalized to quasi Einstein manifold by BEJAN in [1]. From the above studies, as generalizations of quasi-Einstein manifold, we give characterization of super quasi-Einstein manifold, mixed super quasi-Einstein manifold and mixed generalized quasi-Einstein manifold for both even and odd dimensions.

2. Characterization of super quasi-Einstein manifold

Theorem 2.1. A Riemannian manifold of dimension $(2n+1)$ with $n \geq 2$ is super quasi-Einstein if and only if the Ricci operator Q has eigen vector fields ξ_1 and ξ_2 such that at any $p \in M$, there exist three real numbers a, b and c satisfying

$$\begin{aligned}\tau(P) + a &= \tau(P^\perp); & \xi_1, \xi_2 &\in T_p P^\perp, \\ \tau(N) + b &= \tau(N^\perp); & \xi_1 &\in T_p N, \xi_2 \in T_p N^\perp \\ \tau(R) + c &= \tau(R^\perp); & \xi_1 &\in T_p R, \xi_2 \in T_p R^\perp,\end{aligned}$$

for any n -plane sections P, N and $(n+1)$ -plane section R , where $P^\perp, N^\perp$ and $R^\perp$ denote the orthogonal complements of P, N and R in $T_p M$, respectively, and

$$a = \{\alpha + \beta + \delta[\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \sum_{i=1}^n D(e_i, e_i)]\}/2,$$

$$b = \{\alpha - \beta + \delta[\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2,$$

$$c = \{\delta[\sum_{i=n+2}^{2n+1} D(e_i, e_i) - \sum_{i=1}^n D(e_i, e_i)] - \beta - \alpha\}/2,$$

where α , β , and δ are scalars.

Proof. Assume M is a $(2n+1)$ -dimensional super quasi-Einstein manifold, such that

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma [A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y), \quad (2.1)$$

where $\alpha, \beta, \gamma, \delta$ are scalars such that β, γ, δ are nonzero and A, B are two nonzero 1-forms. Also $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors of Q which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0$, $\forall X \in \chi(M)$.

Let $P \subseteq T_p M$ be an n -plane orthogonal to ξ_1 and ξ_2 and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of it. Since ξ_1 and ξ_2 are orthogonal to P , we can take an orthonormal basis $\{e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$ of $P^\perp$ such that $e_{2n} = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n+1}\}$ is an orthonormal basis of $T_p M$. Then taking $X = Y = e_i$ in (2.1) we can write

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & 1 \leq i \leq 2n-1 \\ \alpha + \beta, & i = 2n \\ \alpha + \delta D(\xi_2, \xi_2), & i = 2n+1 \end{cases}.$$

By use of (2.1) for any $1 \leq i \leq 2n+1$, we can write

$$\begin{aligned} S(e_1, e_1) &= K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \dots + K(e_1 \wedge e_{2n-1}) \\ &\quad + K(e_1 \wedge \xi_1) + K(e_1 \wedge \xi_2) = \alpha + \delta D(e_1, e_1), \\ S(e_2, e_2) &= K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \dots + K(e_2 \wedge e_{2n-1}) \\ &\quad + K(e_2 \wedge \xi_1) + K(e_2 \wedge \xi_2) \\ &= \alpha + \delta D(e_2, e_2), \\ &\dots\dots\dots \\ S(e_{2n-1}, e_{2n-1}) &= K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) \end{aligned}$$

$$\begin{aligned}
& + \dots + K(e_{2n-1} \wedge \xi_1) + K(e_{2n-1} \wedge \xi_2) \\
& = \alpha + \delta D(e_{2n-1}, e_{2n-1}), \\
S(\xi_1, \xi_1) & = K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \dots + K(\xi_1 \wedge e_{2n-1}) \\
& + K(\xi_1 \wedge \xi_2) = \alpha + \beta, \\
S(\xi_2, \xi_2) & = K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_3) + \dots + K(\xi_2 \wedge e_{2n-1}) \\
& + K(\xi_2 \wedge \xi_1) = \alpha + \delta D(\xi_2, \xi_2).
\end{aligned}$$

Adding first n -equations, we get

$$(2.2) \quad 2\tau(P) + \sum_{1 \leq i \leq n \leq 2n+1} K(e_i \wedge e_j) = n\alpha + \delta \sum_{i=1}^n D(e_i, e_i).$$

By summing up the last $(n+1)$ equation, we also get

$$\begin{aligned}
(2.3) \quad & 2\tau(P^\perp) + \sum_{1 \leq j \leq n+1 \leq i \leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta \\
& + \delta \sum_{i=n+1}^{2n-1} D(e_i, e_i) + \delta D(\xi_2, \xi_2).
\end{aligned}$$

Then, by subtracting the equation (2.2) from (2.3), we obtain

$$\tau(P^\perp) - \tau(P) = \{\alpha + \beta + \delta [\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \sum_{i=1}^n D(e_i, e_i)]\} / 2.$$

Therefore $\tau(P) + a = \tau(P^\perp)$, where, $a = \{\alpha + \beta + \delta [\sum_{i=n+1}^{2n-1} D(e_i, e_i) + D(e_{2n+1}, e_{2n+1}) - \sum_{i=1}^n D(e_i, e_i)]\} / 2$. Similarly, $N \subseteq T_p M$ be an n -plane orthogonal to ξ_2 and let $\{e_1, \dots, e_{n-1}, e_n\}$ be an orthonormal basis of it. Since ξ_2 is orthogonal to N , we can take an orthonormal basis $\{e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$ of $N^\perp$ orthogonal to ξ_1 , such that $e_n = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$ is an orthonormal basis of $T_p M$. Then taking $X = Y = e_i$ in (2.1) we can write

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & 1 \leq i < n \\ \alpha + \beta, & i = n \\ \alpha + \delta D(e_i, e_i), & n < i \leq 2n+1 \end{cases}.$$

Now, by summing up the first n -equations, we get

$$(2.4) \quad 2\tau(N) + \sum_{1 \leq i \leq n < j \leq 2n+1} K(e_i \wedge e_j) = n\alpha + \beta + \delta \sum_{i=1}^{n-1} D(e_i, e_i).$$

Then summing up the last $(n+1)$ equations, we also get

$$(2.5) \quad 2\tau(N^\perp) + \sum_{1 \leq j \leq n < i \leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \delta \sum_{i=n+1}^{2n+1} D(e_i, e_i).$$

Then, by subtracting the equation (2.4) from (2.5), we obtain

$$\tau(N^\perp) - \tau(N) = \{\alpha - \beta + \delta [\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2.$$

Therefore $\tau(N) + b = \tau(N^\perp)$, where, $b = \{\alpha - \beta + \delta [\sum_{i=n+1}^{2n+1} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2$. Analogously, let $R \subseteq T_p M$ be an $(n+1)$ -plane orthogonal to ξ_2 and let $\{e_1, \dots, e_n, e_{n+1}\}$ be an orthonormal basis of it. Since ξ_2 is orthogonal to R , we can take an orthonormal basis $\{e_{n+2}, \dots, e_{2n}, e_{2n+1}\}$ of $R^\perp$ orthogonal to ξ_1 , such that $e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$ is an orthonormal basis of $T_p M$. Then taking $X = Y = e_i$ in (2.1) we can write

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha + \delta D(e_i, e_i), & 1 \leq i \leq n \\ \alpha + \beta, & i = n+1 \\ \alpha + \delta D(e_i, e_i), & n+1 < i \leq 2n+1 \end{cases}.$$

Now, by summing up the first $(n+1)$ -equations, we get

$$(2.6) \quad 2\tau(R) + \sum_{1 \leq i < n+1 < j \leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta + \delta \sum_{i=1}^n D(e_i, e_i).$$

Next, summing the last n -equations, we get

$$(2.7) \quad 2\tau(R^\perp) + \sum_{1 \leq j \leq n < i \leq 2n+1} K(e_i \wedge e_j) = n\alpha + \delta \sum_{i=n+2}^{2n+1} D(e_i, e_i).$$

Then, by subtracting the equation (2.6) from (2.7), we obtain

$$\tau(R^\perp) - \tau(R) = \left\{ \delta \left[\sum_{i=n+2}^{2n+1} D(e_i, e_i) - \sum_{i=1}^n D(e_i, e_i) \right] - \alpha - \beta \right\} / 2.$$

Therefore $\tau(R) + c = \tau(R^\perp)$, where, $c = \{ \delta [\sum_{i=n+2}^{2n+1} D(e_i, e_i) - \sum_{i=1}^n D(e_i, e_i)] - \alpha - \beta \} / 2$.

Conversely, let $g(\ell X, Y) = D(X, Y)$, where ℓ is a symmetric endomorphism of the tangent space $T_p M$ and let V be an arbitrary unit vector of $T_p M$, at $p \in M$, orthogonal to ξ_1 and ξ_2 . Now D is symmetric as g is so, trace of D is zero and $D(X, \xi_1) = 0$ for all $X \in \chi(M)$. We take an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$ of $T_p M$ such that $V = e_1, e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$. We consider n -plane section N and $(n+1)$ -plane section R in $T_p M$ as follows $N = \text{span}\{e_2, \dots, e_{n+1}\}$ and $R = \text{span}\{e_1, \dots, e_{n+1}\}$ respectively. Then we have $N^\perp = \text{span}\{e_1, e_{n+2}, \dots, e_{2n}, e_{2n+1}\}$ and $R^\perp = \text{span}\{e_{n+2}, \dots, e_{2n}, e_{2n+1}\}$ respectively. Now

$$\begin{aligned} S(V, V) &= [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \dots + K(e_1 \wedge e_{n+1})] \\ &\quad + [K(e_1 \wedge e_{n+2}) + \dots + K(e_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(R) - \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j)] + [\tau(N^\perp) - \sum_{2+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)] \\ &= [\tau(R) - c - \tau(N)] + [\tau(N) + b - \tau(R^\perp)] = (b - c). \end{aligned}$$

Therefore $S(V, V) = b - c$, for any unit vector $V \in T_p M$, orthogonal to ξ_1 and ξ_2 . Then we can write for any $1 \leq i \leq 2n+1$, $S(e_i, e_i) = b - c$. Since $S(V, V) = (b - c)g(V, V)$ for any unit vector $V \in T_p M$ orthogonal to ξ_1 and ξ_2 , it follows that $S(X, X) = (b - c)g(X, X) + K_1 A(X)A(X) + K_2 D(X, X)$ and $S(Y, Y) = (b - c)g(Y, Y) + K_3 [A(Y)B(Y) + A(Y)B(Y)]$ for any $X \in [\text{span}\{\xi_1\}]^\perp$ and $Y \in [\text{span}\{\xi_2\}]^\perp$, where A and B denote dual forms of ξ_1 and ξ_2 with respect to g , respectively and K_1, K_2, K_3 are scalars, such that $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0$.

Now from the above equations, we get from their symmetry that S with tensors $(b - c)g + K_1(A \otimes A) + K_2 D$ and $(b - c)g + K_3[(A \otimes B) + (A \otimes B)]$ must coincide on the complement of ξ_1 and ξ_2 , respectively, that is $S(X, Y) = (b - c)g(X, Y) + K_1 A(X)A(Y) + K_3 [A(X)B(Y) + A(Y)B(X)] + K_2 D(X, Y)$, for any $X, Y \in [\text{span}\{\xi_1, \xi_2\}]^\perp$. Since ξ_1 and ξ_2 are eigenvector fields of Q , we also have $S(X, \xi_1) = 0$ and $S(Y, \xi_2) = 0$, for any $X, Y \in T_p M$ orthogonal

to ξ_1 and ξ_2 . Thus, we can extend the above equation to

$$(2.8) \quad S(X, Z) = (b - c)g(X, Z) + K_1 A(X)A(Z) + K_3[A(X)B(Z) + A(Z)B(X)] \\ + K_2 D(X, Z),$$

for any $X \in [\text{span}\{\xi_1, \xi_2\}]^\perp$ and $Z \in T_p M$, where K_1, K_2, K_3 are scalars and $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0$. Now, let us consider the n -plane section P and $(n+1)$ -plane section R in $T_p M$ as follows $P = \text{span}\{e_1, \dots, e_n\}$ and $R = \text{span}\{e_1, \dots, e_n, \xi_1\}$, respectively. Then we have, $P^\perp = \text{span}\{\xi_1, e_{n+2}, \dots, e_{2n+1}\}$ and $R^\perp = \text{span}\{e_{n+2}, \dots, e_{2n}, e_{2n+1}\}$. Similarly after some calculations, we obtain

$$\begin{aligned} S(\xi_1, \xi_1) &= [K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \dots + K(\xi_1 \wedge e_n)] \\ &\quad + [K(\xi_1 \wedge e_{n+2}) + K(\xi_1 \wedge e_{2n}) + K(\xi_1 \wedge e_{2n-1})] \\ &= [\tau(R) - \sum_{1 \leq i < n+1 < j \leq n} K(e_i \wedge e_j)] \\ &\quad + [\tau(P^\perp) - \sum_{n+2 \leq i < n+1 < j \leq 2n+1} K(e_i \wedge e_j)] \\ &= [\tau(R^\perp) - c - \tau(P)] + [\tau(P) + a - \tau(R^\perp)] = a - c. \end{aligned}$$

Therefore we can write

$$(2.9) \quad S(\xi_1, \xi_1) = (b - c)g(\xi_1, \xi_1) + (a - b)A(\xi_1)A(\xi_1) + K_2 D(\xi_1, \xi_1).$$

Analogously, let us consider n -plane sections P and $N \in T_p M$ as follows $P = \text{span}\{e_1, \dots, e_n\}$ and $N = \text{span}\{e_{n+1}, \dots, e_{2n}\}$ respectively. Therefore we have $P^\perp = \text{span}\{e_{n+1}, \dots, e_{2n}, \xi_2\}$ and $N^\perp = \text{span}\{e_1, \dots, e_n, \xi_2\}$. Similarly after some calculations, we get $S(\xi_2, \xi_2) = a + b$. Then we may write

$$(2.10) \quad S(\xi_2, \xi_2) = (a + b)g(\xi_2, \xi_2) + K_3[A(\xi_2)B(\xi_2) + A(\xi_2)B(\xi_2)].$$

Now from (2.8), (2.9) and (2.10) we can write the Ricci tensor S be

$$(2.11) \quad S(X, Y) = \mu_1 g(X, Y) + K_1 A(X)A(Y) + K_3[A(X)B(Y) + A(Y)B(X)] \\ + K_2 D(X, Y),$$

for any $X, Y \in T_p M$. From (2.11) it follows that, M is a super quasi-Einstein manifold, where μ_1, K_1, K_2, K_3 are scalars and $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0$. Hence the theorem is proved.

Theorem 2.2. Let (M, g) be a Riemannian manifold of dimension $2n$, with $n \geq 2$. Then M is super quasi-Einstein if and only if the Ricci operator Q has eigenvector fields ξ_1 and ξ_2 such that at any $p \in M$, there exist three real numbers a , b and c satisfying

$$\begin{aligned}\tau(P) + a &= \tau(P^\perp); & \xi_1, \xi_2 &\in T_p P^\perp, \\ \tau(N) + b &= \tau(N^\perp); & \xi_1 &\in T_p N, \xi_2 \in T_p N^\perp, \\ \tau(R) + c &= \tau(R^\perp); & \xi_1 &\in T_p R, \xi_2 \in T_p R^\perp,\end{aligned}$$

for any n -plane sections P, R and $(n-1)$ -plane section N , where $P^\perp, N^\perp$ and $R^\perp$ denote the orthogonal complements of P, N and R in $T_p M$, respectively, and

$$\begin{aligned}a &= \{\beta + \delta[\sum_{i=n+1}^{2n-2} D(e_i, e_i) + D(e_{2n}, e_{2n}) - \sum_{i=1}^n D(e_i, e_i)]\}/2, \\ b &= \{2\alpha - \beta + \delta[\sum_{i=n}^{2n} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)]\}/2, \\ c &= \{\delta[\sum_{i=n+1}^{2n} D(e_i, e_i) - \sum_{i=1}^{n-1} D(e_i, e_i)] - \beta\}/2,\end{aligned}$$

where α , β , and δ are scalars.

Proof. Let P and R be n -plane sections and N be an $(n-1)$ -plane section such that, $P = \text{span}\{e_1, \dots, e_n\}$, $R = \text{span}\{e_{n+1}, \dots, e_{2n}\}$, and $N = \text{span}\{e_2, \dots, e_n\}$ respectively. Therefore the orthogonal complements of these sections can be written as $P^\perp = \text{span}\{e_{n+1}, \dots, e_{2n}\}$, $R^\perp = \text{span}\{e_1, \dots, e_n\}$ and $N^\perp = \text{span}\{e_1, e_{n+1}, \dots, e_{2n}\}$.

Then rest of the proof is similar to the proof of Theorem 2.1.

Corollary 1. For mixed super quasi-Einstein manifolds the characterization will be same but the values of a , b , c will be different for both odd and even dimensional manifolds. For mixed generalized quasi-Einstein manifolds the characterization will be same but the values of a , b , c will be different for odd and even dimensional manifolds. For odd dimensional manifold, $a = \{\alpha + \beta + \gamma\}/2$, $b = \{\alpha - \beta + \gamma\}/2$, $c = \{\gamma - \alpha - \beta\}/2$ and for even dimensional manifold, $a = \{\beta + \gamma\}/2$, $b = \{2\alpha - \beta + \gamma\}/2$, $c = \{\gamma - \beta\}/2$, respectively.

Acknowledgement. The authors are grateful to referee for valuable suggestions in improving the paper.

REFERENCES

1. BEJAN, C.-L. – *Characterization of quasi-Einstein manifolds*, An. Ştiinţ. Univ. "Al.I. Cuza" Iaşi. Mat. (N.S.), 53 (2007), suppl. 1, 67–72.
2. BHATTACHARYYA, A.; TARAFDAR, M.; DEBNATH, D. – *On mixed super quasi-Einstein manifolds*, Differ. Geom. Dyn. Syst., 10 (2008), 44–57.
3. BHATTACHARYYA, A.; DE, T. – *On mixed generalized quasi-Einstein manifolds*, Differ. Geom. Dyn. Syst., 9 (2007), 40–46 (electronic).
4. CHAKI, M.C. – *On super quasi Einstein manifolds*, Publ. Math. Debrecen, 64 (2004), 481–488.
5. CHEN, B.-Y. – *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan. J. Math. (N.S.), 26 (2000), 105–127.
6. DESZCZ, R.; GŁOGOWSKA, M.; HOTŁOŚ, MARIAN; ŞENTÜRK, Z. – *On certain quasi-Einstein semisymmetric hypersurfaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 41 (1998), 151–164 (1999).
7. DUMITRU, D. – *On Einstein spaces of odd dimension*, Bull. Transilv. Univ. Braşov Ser. B (N.S.), 14 (2007), suppl., 95–96.
8. SINGER, I.M.; THORPE, J.A. – *The curvature of 4-dimensional Einstein spaces*, 1969 Global Analysis, pp. 355–365 Univ. Tokyo Press, Tokyo.

Received: 26.VIII.2011

Revised: 5.XII.2011

Revised: 23.XII.2011

Accepted: 30.01.2012

*Department of Mathematics,
Jadavpur University,
Kolkata 700032, West Bengal,
INDIA*

*halderk@rediffmail.com
buddha.pal@rediffmail.com
bhattachar1968@yahoo.co.in
tapu2027@yahoo.co.in*