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ALMOST α-COSYMPLECTIC f-MANIFOLDS

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Abstract. The purpose of this paper is to study a new class of framed manifolds. Such manifolds are called almost α-cosymplectic f-manifolds. For some special cases of α and s, one obtains (almost) α-cosymplectic, (almost) C-manifolds, and (almost) Kenmotsu f-manifolds. Moreover, several tensor conditions are studied. We conclude our results with a general example on α-cosymplectic f-manifolds.

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1. Introduction and preliminaries

Let M a real $(2n + s)$-dimensional smooth manifold. M admits an f-structure ([1], [7]) if there exists a non null smooth (1,1) tensor field ϕ, of the tangent bundle TM, satisfying $\phi^3 + \phi = 0$, $\operatorname{rank}\phi = 2n$. An f-structure is a generalization of almost complex ($s = 0$) and almost contact ($s = 1$) structure ([5], [7]). In the latter case, M is orientable ([6]). Corresponding to two complementary projection operators P and Q applied to TM, defined by $P = -\phi^2$ and $Q = \phi^2 + I$, where I is the identity operator, there exist two complementary distributions D and $D^\perp$ such that $\dim(D) = 2n$ and $\dim(D^\perp) = s$. The following relations hold $\phi P = P\phi = \phi$, $\phi Q = Q\phi = 0$, $\phi^2 P = -P$, $\phi^2 Q = 0$. Thus, we have an almost complex distribution $\left(D, J = \phi_{|_D}, J^2 = -I\right)$ and ϕ acts on $D^\perp$ as a null operator. It follows that $TM = D \oplus D^\perp$, $D \cap D^\perp = \{0\}$. Assume that $D_p^\perp$ is spanned by s globally defined orthonormal vectors $\{\xi_i\}$ at each point $p \in M$, $(1 \le i, j, ... \le s)$, with its dual set $\{\eta^i\}$. Then one obtains $\phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i$. In the above case, M is called a globally framed manifold (or simply an f-manifold) ([1], [4] and [5]) and we denote its framed structure by $M(\phi, \xi_i)$. From the above conditions one has $\phi\xi_i = 0, \eta^i \circ \phi = 0, \eta^i(\xi_j) = \delta_i^j$. Now, we consider a Riemannian metric g on M that is compatible with an f-structure such that

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \;\; g(X, \xi_i) = \eta^i(X).$$

In the above case, we say that M is a metric f-manifold and its associated structure will be denoted by $M(\phi, \xi_i, \eta^i, g)$.

A framed structure $M(\phi, \xi_i)$ is said to be normal ([4]) if the torsion tensor N_ϕ of ϕ is zero i.e., if $N_\phi = N + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$, where N denotes the Nijenhuis tensor field of ϕ.

Define a 2-form Ω on M by $\Omega(X, Y) = g(X, \phi Y)$, for any $X, Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f-manifold satisfies the following formula ([1]):

$$\begin{aligned} 2g\left((\nabla_X \phi)Y, Z\right) &= 3d\Omega(X, \phi Y, \phi Z) - 3d\Omega(X, Y, Z) \\ &\quad + g\left(N(Y, Z), \phi X\right) + N_j^2(Y, Z)\eta^j(X) \\ &\quad + 2d\eta^j(\phi Y, X)\eta^j(Z) - 2d\eta^j(\phi Z, X)\eta^j(Y), \end{aligned}$$

where the tensor field N_j^2 is defined by $N_j^2(X, Y) = (L_{\phi X}\eta^j)Y - (L_{\phi Y}\eta^j)X = 2d\eta^j(\phi X, Y) - 2d\eta^j(\phi Y, X)$, for each $j \in \{1, ..., s\}$.

Following the terminology introduced by Blair ([1]), we say that a normal metric f-manifold is a K-manifold if its 2-form Ω closed (i.e., $d\Omega = 0$). Since $\eta^1 \wedge ... \wedge \eta^s \wedge \Omega^n \neq 0$, a K-manifold is orientable. Furthermore, we say that a K-manifold is a C-manifold if each η^i is closed, an S-manifold if $d\eta^1 = d\eta^2 = ... = d\eta^s = \Omega$.

Note that, if $s = 1$, namely if M is an almost contact metric manifold, the condition $d\Omega = 0$ means that M is quasi-Sasakian. M is said a K-contact manifold if $d\eta = \Omega$ and ξ is Killing.

Falcitelli and Pastore [3] introduced and studied a class of manifolds which is called almost Kenmotsu f-manifold. Such manifolds admit an f-structure with s-dimensional parallelizable kernel. A metric $f.pk$-manifold of dimension $(2n + s)$, $s \geq 1$, with $f.pk$-structure (ϕ, ξ_i, η^i, g), is said to be a almost Kenmotsu $f.pk$-manifold if the 1-forms η^i's are closed and $d\Omega =$

$2\eta^1 \wedge \Omega$. Several foliations canonically associated with an almost Kenmotsu $f.pk$-manifold are studied and locally conformal almost Kenmotsu $f.pk$-manifolds are characterized by Falcitelli and Pastore.

In this paper, we consider a wide subclass of f-manifolds called almost α-cosymplectic f-manifolds. Firstly, we give the concept of almost α-cosymplectic f-manifold and state general curvature properties. We derive several important formulas on almost α-cosymplectic f-manifolds. These formulas enable us to find the geometrical properties of almost α-cosymplectic f-manifolds with η-parallel tensors h_i and φh_i. We also examine the tensor fields τ_i's which are defined by $g(\tau_i X, Y) = (\mathcal{L}_{\xi_i} g)(X, Y)$, for arbitrary vector fields X, Y on M. Then we give some results on η-parallelity, cyclic parallelity, Codazzi condition. Finally, we give an explicit example of almost α-cosymplectic f-manifold.

Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi^1 + \xi^2 + ... + \xi^s$ and $\overline{\delta}_i^j = \delta_i^1 + \delta_i^2 + ... + \delta_i^s$.

2. Almost α-cosymplectic f-manifolds

We introduce a notion of an almost α-cosymplectic f-manifold for any real number α which is defined as metric f-manifold with f-structure $(\varphi, \xi_i, \eta^i, g)$ satisfying the conditions $d\eta^i = 0$, $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$. The manifold is called generalized almost Kenmotsu f-manifold for $\alpha = 1$.

Let M be an almost α-cosymplectic f-manifold. Since the distribution D is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by:

$$2g\left(\left(\nabla_X \varphi\right) Y, Z\right) = 2\alpha g\left(\sum_{j=1}^{s} \left(g\left(\varphi X, Y\right)\xi_j - \eta^j(Y)\varphi X\right), Z\right) + g\left(N(Y, Z), \varphi X\right), \tag{2.1}$$

for any $X, Y, Z \in \Gamma(TM)$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i}\varphi = 0$ which implies $\nabla_{\xi_i}\xi_j \in D^{\perp}$ and then $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$.

We put $A_i X = -\nabla_X \xi_i$ and $h_i = \frac{1}{2}(L_{\xi_i}\varphi)$, where L denotes the Lie derivative operator.

Proposition 1. *For any $i \in \{1, ..., s\}$ the tensor field A_i is a symmetric operator such that*

1) $A_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$
2) $A_i \circ \varphi + \varphi \circ A_i = -2\alpha\varphi$.
3) $tr(A_i) = -2\alpha n$

Proof. $d\eta^i = 0$ implies that A_i is symmetric.

1) For any $i, j, k \in \{1, ..., s\}$ deriving $g(\xi_i, \xi_j) = \delta_i^j$ with respect to ξ_k, using $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$, we get $2g(\xi_k, A_i(\xi_j)) = 0$. Since $\nabla_{\xi_i}\xi_j \in D^{\perp}$, we conclude $A_i(\xi_j) = 0$.

2) For any $Z \in \Gamma(TM)$, we have $\varphi(N(\xi_i, Z)) = (L_{\xi_i}\varphi) Z$ and, on the other hand, since $\nabla_{\xi_i}\varphi = 0$,

$$L_{\xi_i}\varphi = A_i \circ \varphi - \varphi \circ A_i \tag{2.2}$$

One can easily obtain from (2.2)

$$-A_i X = -\alpha\varphi^2 X - \varphi h_i X \tag{2.3}$$

Applying (2.1) with $Y = \xi_i$, we have $2g(\varphi A_i X, Z) = -2\alpha g(\varphi X, Z) - g(\varphi N(\xi_i, Z), X)$, which implies the desired result.

3) Considering local adapted orthonormal frame $\{X_1, ..., X_n, \varphi X_1, ..., \varphi X_n, \xi_1, ..., \xi_s\}$, by 1) and 2), one has

$$trA_i = \sum_{j=1}^{n} g(A_i X_j, X_j) + g(A_i\varphi X_j, \varphi X_j) = -2\alpha \sum_{j=1}^{n} g(\varphi X_j, \varphi X_j) = -2\alpha n.$$

□

Proposition 2 ([1]). *For any $i \in \{1, ..., s\}$ the tensor field h_i is a symmetric operator and satisfies*

1) $h_i\xi_j = 0$, *for any* $j \in \{1, ..., s\}$
2) $h_i \circ \varphi + \varphi \circ h_i = 0$
3) $trh_i = 0$
4) $tr\varphi h_i = 0$

Proposition 3. $\nabla\varphi$ *satisfies the following relation:*

$$(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = \sum_{i=1}^{s}[-\alpha(\eta^i(Y)\varphi X + 2g(X, \varphi Y)\xi_i) - \eta^i(Y)h_i X].$$

Proof. By direct computations, we get $\varphi N(X, Y) + N(\varphi X, Y) = 2\sum_{i=1}^{s}\eta^i(X)h_i Y$, and $\eta^i(N(\varphi X, Y)) = 0$. From (2.1) and the equations above, the proof is completed. □

Proposition 4. *Let M be an almost α-cosymplectic f-manifold. The integral manifolds of D are almost Kaehler manifolds with mean curvature vector field $H = -\alpha\overline{\xi}$.*

Proof. Let $\widetilde{M}$ be an integral manifold of D. We know that $(D, J = \varphi_{|_D}, J^2 = -I)$ is an almost complex distribution and the induced metric $\widetilde{g}$ on $\widetilde{M}$ is a Hermitian metric. Therefore, for any $X, Y \in \Gamma(\widetilde{M})$, we have the induced 2-form on $\widetilde{M}$ such that $\widetilde{\Omega}(X,Y) = \widetilde{g}(X, JY) = g(X, \varphi Y) = \Omega(X,Y)$ and $d\widetilde{\Omega} = 0$ on $\widetilde{M}$. In this manner, $\widetilde{M}$ is an almost Keahler manifold. Computing the second fundamental form B, since, A_i's are the Weingarten operators in the directions ξ_i, we get,

$$B(X,Y) = \sum_{i=1}^{s} g(A_i X, Y)\xi_i = \sum_{i=1}^{s} \left[-\alpha g(X,Y)\xi_i + g(\varphi h_i X, Y)\xi_i\right]. \tag{2.4}$$

Using the Proposition 2 and (2.3). Now, we choose a local orthonormal frame $\{e_1, e_2, ..., e_{2n}\}$ such that $e_{l+n} = \varphi e_l$, for $l = 1, 2, ..., n$, in $T\widetilde{M}$. Taking $X = Y = e_p$ in (2.4) and summing over $p = 1, 2, ..., 2n$, we get $H = \frac{1}{2n}\sum_{i=1}^{s}(trA_i)\xi_i = -\alpha\overline{\xi}$. □

Proposition 5. *Let M be an almost α-cosymplectic f-manifold and $\widetilde{M}$ be an integral manifold of D. Then*

1) *when $\alpha = 0$, $\widetilde{M}$ is totally geodesic if and only if all the operators h_i vanish;*
2) *when $\alpha \neq 0$, $\widetilde{M}$ is totally umbilic if and only if all the operators h_i vanish.*

Proof. The proof is obvious through (2.4). □

Proposition 6. *Under the same situation as in Proposition 5, M is α-cosymplectic f-manifold with structure f-structure $(\varphi, \xi_i, \eta^i, g)$ if and only if the integral manifolds of D are tangentially Kaehler and all the operators h_i vanish.*

Proof. If the structure is normal, for any $X \in \Gamma(TM)$, one obtains that

$$\begin{aligned} 0 = N(X, \xi_j) &= N_\varphi(X, \xi_j) + 2\sum_{i=1}^{s} d\eta^i(X, \xi_j)\xi_i \\ &= -\varphi\left[\varphi X, \xi_j\right] + \varphi^2\left[X, \xi_j\right] + 2\sum_{i=1}^{s} d\eta^i(X, \xi_j)\xi_i = 2\varphi h_j X. \end{aligned} \tag{2.5}$$

Hence, all the operators h_i vanish. On the other hand, for each $X, Y \in \Gamma(D)$ we have

$$N_{\varphi}(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]-[X,Y]=N_{J=\varphi_{|D}}(X,Y). \tag{2.6}$$

It is obvious that $N_J = 0$ if and only if almost complex structure J is integrable. Therefore, the proof is completed by (2.5) and (2.6). □

Theorem 1 ([1]). *A C-manifold M^{2n+s} is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

3. Curvature properties

Proposition 7. *Let M be an almost α-cosymplectic f-manifold. Then we have*

$$\begin{aligned} R(X,Y)\xi_i &= \alpha^2 \sum_{k=1}^{s} \left[\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y\right] \\ &- \alpha \sum_{k=1}^{s} \left[\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X\right] + \left(\nabla_Y \varphi h_i\right) X - \left(\nabla_X \varphi h_i\right) Y. \end{aligned} \tag{3.1}$$

Proof. Using the Riemannian curvature tensor and (2.3), we obtain (3.1). □

Using (2.3) and (3.1), by simple computations, we have the following proposition.

Proposition 8. *For an almost α-cosymplectic f-manifold with the f-structure $(\varphi, \xi_i, \eta^i, g)$, the following relations hold*

$$\begin{aligned} R(X,\xi_j)\xi_i &= \sum_{k=1}^{s} \delta_j^k \left[\alpha^2 \varphi^2 X + \alpha \varphi h_k X\right] \\ &+ \alpha \varphi h_i X - h_i h_j X + \varphi\left(\nabla_{\xi_j} h_i\right) X \end{aligned} \tag{3.2}$$

$$R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2\left[-\alpha^2 \varphi^2 X + h_i h_j X\right], \tag{3.3}$$

$$\begin{aligned} \left(\nabla_{\xi_j} h_i\right) X &= -\varphi R(X,\xi_j)\xi_i + \sum_{k=1}^{s} \delta_j^k \left[-\alpha^2 \varphi X - \alpha h_k X\right] \\ &- \alpha h_i X - \varphi h_i h_j X, \end{aligned} \tag{3.4}$$

(3.5) $$S(X,\xi_i) = -2n\alpha^2 \sum_{k=1}^{s} \eta^k(X) - (div\,\varphi h_i)\,X,$$

(3.6) $$S(\xi_i,\xi_j) = -2n\alpha^2 - tr\,(h_j h_i)\,,$$

Corollary 1. *Let M be an almost α-cosymplectic f-manifold. The Ricci tensor satisfies the following conditions:*

1) *$S(\xi_i,\xi_i)$ always takes negative value when $\alpha \neq 0$,*

2) *If all the values $S(\xi_i,\xi_i)$ vanish then any leaf of D is totally geodesic.*

3) *If all the values $S(\xi_i,\xi_i)$ vanish and M is normal then M is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

Proof. The proof is clear through (3.6). □

The tensor τ was introduced by Chern and Hamilton [2] and is defined by $g(\tau X,Y) = (L_\xi g)(X,Y)$ for arbitrary vector fields X,Y on a contact metric manifold. Now, we define and examine this tensor field for an almost α-cosymplectic f-manifold

Proposition 9. *An almost α-cosymplectic f-manifold with f-structure (φ,ξ_i,η^i,g) has tensor fields τ_i such that $\tau_i X = 2\nabla_X \xi_i$, where τ_i's are defined by $g(\tau_i X,Y) = (L_{\xi_i} g)(X,Y)$ for arbitrary vector fields X,Y on M.*

Proof. Using the definition of the tensor fields τ_i, we get

$$\begin{aligned}(L_{\xi_i} g)(X,Y) &= g(\nabla_X \xi_i, Y) + g(X, \nabla_Y \xi_i)\\ &= 2g\left(-\alpha\varphi^2 X - \varphi h_i X, Y\right)\end{aligned}$$

for arbitrary vector fields X,Y on M. Applying the formula (2.3), the proof is completed. □

Proposition 10. *Let M be a locally symmetric almost α-cosymplectic f-manifold. Then, $\nabla_{\xi_\gamma} h_i = 0$, for any $\gamma \in \{1,..,s\}$.*

Proof. Notice that (3.3) can be written as $\frac{1}{2}\left(R(\xi_j,.)\xi_i - \varphi R(\xi_j,\varphi.)\xi_i\right) = -\alpha^2\varphi^2 + h_i h_j$ and since the operator $R(\xi_j,\cdot)\xi_i$ is parallel with respect to ξ_k, we get $\nabla_{\xi_k} h_i h_j = 0$. Applying ∇_{ξ_γ} to (3.4), we obtain $\nabla_{\xi_\gamma}\left(\nabla_{\xi_j} h_i\right) = -\alpha\nabla_{\xi_\gamma} h_i - \alpha\nabla_{\xi_\gamma} h_j$. Moreover, $\nabla_{\xi_k} h_i h_j = 0$ implies that $(\nabla_{\xi_k} h_i)\,h_j + h_i\,(\nabla_{\xi_k} h_j) = 0$, and applying ∇_{ξ_γ} to this equation, we get $\nabla_{\xi_\gamma} h_i = 0$. □

Theorem 2. *Let M be a locally symmetric generalized almost Kenmotsu f-manifold. Then the following conditions are equivalent:*

1) *M is a generalized α-Kenmotsu f-manifold;*

2) *all the operators h_i vanish.*

Moreover, if any of the conditions above holds, then M cannot have constant sectional curvature.

Proof. Assuming that M is a generalized α-Kenmotsu f-manifold, we have $\nabla\xi_i = -\alpha\varphi^2$ and, by (2.3) , all the operators h_i vanish. Now, supposing all the operators h_i vanish, it follows that $\nabla\xi_i = -\alpha\varphi^2$ and $\nabla\eta^i = \alpha\left(g - \sum_{k=1}^{s}\eta^k \otimes \eta^k\right)$ and by (3.1), $R(X,Y)\xi_i = \alpha^2\sum_{k=1}^{s}[\eta^k(Y)\varphi^2 X -\eta^k(X)\varphi^2 Y]$. So, M is a generalized α-Kenmotsu f-manifold. Moreover, The sectional curvature of any 2-plane spaned by $\{Y,\xi_i\}$ is $K(Y,\xi_i) = -\alpha^2\left\|\varphi Y\right\|^2$,for all vector fields Y on M. So, the sectional curvature of any 2-plane spaned by $\{\xi_i,\xi_j\}$, for any $i,\ j \in \{1,2,...,s\}$, vanishes and one gets that the sectional curvature of any plane spaned by $Y \in D$ and ξ_i is equal to $-\alpha^2$. □

4. Some tensor conditions

For any vector field X on M, we can take $X = X^T + \sum_{i=1}^{s}\eta^i(X)\xi_i$ where X^T is the tangential part of X and $\sum_{i=1}^{s}\eta^i(X)\xi_i$ is the normal part of X. We can rewrite η-parallel condition for a given almost α-cosymplectic f-manifold. We say that any $(1,1)$- type tensor field B is η-parallel if and only if $g((\nabla_{X^T}B)\,Y^T, Z^T) = 0$, for $X^T, Y^T, Z^T \in \mathcal{D}$.

The starting point of the investigation of almost α-cosymplectic f-manifolds with η-parallel tensors h_i and φh_i is the following propositions:

Proposition 11. *Let M be an almost α-cosymplectic f-manifold and h_i's are $(1,1)$-type tensor fields. If the tensor fields h_i's are η-parallel, then*

$$(\nabla_X h_i)Y = -\sum_{k=1}^{s}\eta^k(X)\left[\varphi l_{ki}Y + \sum_{\gamma=1}^{s}\delta_k^{\gamma}[\alpha^2\varphi Y + \alpha h_\gamma Y] + \varphi h_i h_k Y + \alpha h_i Y\right]$$

$$-\sum_{k=1}^{s}\eta^k(Y)[\alpha h_i X + \varphi h_i h_k X] - \sum_{k=1}^{s} g(\alpha h_i X + \varphi h_i h_k X, Y)\xi_k, \tag{4.1}$$

for all vector fields X,Y on M, where the tensor $l_{ki} = R(.,\xi_k)\xi_i$ is the Jacobi operator with respect to the characteristic vector fields and h_i's are $(1,1)$-type tensor fields.

Proof. Suppose that each h_i is η-parallel. Denoting the component of X orthogonal to ξ by X^T, we obtain

$$\begin{aligned}0 &= g((\nabla_{X^T} h_i)\, Y^T, Z^T)\\ &= g(\left(\nabla_{X-\sum_{k=1}^{s}\eta^k(X)\xi_k} h_i\right)(Y - \sum_{k=1}^{s}\eta^k(Y)\xi_k), Z - \sum_{k=1}^{s}\eta^k(Z)\xi_k)\\ &= g((\nabla_X h_i)\, Y, Z) - \sum_{k=1}^{s}\eta^k(X) g((\nabla_{\xi_k} h_i)\, Y, Z) - \sum_{k=1}^{s}\eta^k(Y) g((\nabla_X h_i)\, \xi_k, Z)\\ &- \sum_{k=1}^{s}\eta^k(Z) g((\nabla_X h_i)\, Y, \xi_k) = g((\nabla_X h_i)\, Y, -\varphi^2 Z)\\ &- \sum_{k=1}^{s}\eta^k(X) g((\nabla_{\xi_k} h_i)\, Y, Z) - \sum_{k=1}^{s}\eta^k(Y) g((\nabla_X h_i)\, \xi_k, Z),\end{aligned}$$

for all vector fields X, Y, Z on M. Using (2.3) and (3.4), the proof is completed. □

Proposition 12. *Let M be an almost α-cosymplectic f-manifold. If the tensor fields φh_i's are η-parallel, then*

$$\begin{aligned}(\nabla_X \varphi h_i)Y &= \sum_{k=1}^{s}\eta^k(X)\left[l_{ki}Y - \sum_{\gamma=1}^{s}\delta_k^\gamma[\alpha^2\varphi^2 Y + \alpha\varphi h_\gamma Y] + h_i h_k Y - \alpha\varphi h_i Y\right]\\ &- \sum_{k=1}^{s}\eta^k(Y)[\alpha\varphi h_i X - h_i h_k X] - \sum_{k=1}^{s} g(\alpha\varphi h_i X - h_i h_k X, Y)\xi_k. \qquad (4.2)\end{aligned}$$

Proof. We consider that φh_i is η-parallel. Thus,

$$\begin{aligned}0 &= g((\nabla_{X^T}\varphi h_i)\, Y^T, Z^T)\\ &= g(\left(\nabla_{X-\sum_{k=1}^{s}\eta^k(X)\xi_k}\varphi h_i\right)(Y - \sum_{k=1}^{s}\eta^k(Y)\xi_k), Z - \sum_{k=1}^{s}\eta^k(Z)\xi_k)\\ &= g((\nabla_X\varphi h_i)\, Y, Z) - \sum_{k=1}^{s}\eta^k(X) g((\nabla_{\xi_k}\varphi h_i)\, Y, Z)\\ &- \sum_{k=1}^{s}\eta^k(Y) g((\nabla_X\varphi h_i)\, \xi_k, Z) - \sum_{k=1}^{s}\eta^k(Z) g((\nabla_X\varphi h_i)\, Y, \xi_k)\end{aligned}$$

for all vector fields X, Y on M. If we simplify the equation above, then

$$g((\nabla_X \varphi h_i)\, Y, Z) = \sum_{k=1}^{s} \eta^k(X) g((\nabla_{\xi_k} \varphi h_i)\, Y, Z) + \sum_{k=1}^{s} \eta^k(Y) g((\nabla_X \varphi h_i)\, \xi_k, Z) + \sum_{k=1}^{s} \eta^k(Z) g\left((\nabla_X \varphi h_i)\, Y, \xi_k\right).$$

Using (2.3) and $(\nabla_{\xi_k} \varphi h_i)\, Y = \varphi(\nabla_{\xi_k} h_i) Y$, the proof is completed. □

Theorem 3. *An almost α-cosymplectic f-manifold with the η-parallel tensor fields φh_i's satisfy the following relation:*

$$R(X, Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y) l_{ki} X - \eta^k(X) l_{ki} Y, \tag{4.3}$$

where $l_{ki} = R(., \xi_k)\xi_i$ is the Jacobi operator with respect to the characteristic vector fields ξ_k and ξ_i.

Proof. Using (3.1) and (4.2), we get

$$\begin{aligned} R(X, Y)\xi_i &= \alpha^2 \sum_{k=1}^{s} \left[\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y\right] \\ &- \alpha \sum_{k=1}^{s} \left[\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X\right] \\ &+ \sum_{k=1}^{s} \eta^k(Y) \left[l_{ki} X - \sum_{\gamma=1}^{s} \delta_k^\gamma \left[\alpha^2\varphi^2 X + \alpha\varphi h_\gamma X\right] + h_i h_k X - \alpha\varphi h_i X\right] \\ &- \sum_{k=1}^{s} \eta^k(X) \left[\alpha\varphi h_i Y - h_i h_k Y\right] - \sum_{k=1}^{s} g(\alpha\varphi h_i Y - h_i h_k Y, X)\xi_k \\ &- \sum_{k=1}^{s} \eta^k(X) \left[l_{ki} Y - \sum_{\gamma=1}^{s} \delta_k^\gamma \left[\alpha^2\varphi^2 Y + \alpha\varphi h_\gamma Y\right] + h_i h_k Y - \alpha\varphi h_i Y\right] \\ &+ \sum_{k=1}^{s} \eta^k(Y) \left[\alpha\varphi h_i X - h_i h_k X\right] + \sum_{k=1}^{s} g(\alpha\varphi h_i X - h_i h_k X, Y)\xi_k. \end{aligned}$$

Then, we can easily write (4.3) by simplifying the equation above. □

Theorem 4. *An almost α-cosymplectic f-manifold has negative pointwise constant ξ_i-sectional curvature.*

Proof. Let M be an almost α-cosymplectic f-manifold with a pointwise constant ξ_i-sectional curvature $K(p), p\in M$. It means that$g(R(X^T, \xi_i)\xi_i, X^T) = K_i(p)g(X^T, X^T)$ for all tangent vectors X^T orthogonal to ξ_i at the point $p \in M$,i.e, $X^T \in \mathcal{D}$. Putting $X^T = X - \sum_{k=1}^{s} \eta^k(X)\xi_k$ and using the symmetries of curvature tensor R, we see that the equation above is equivalent to$\varphi l_{ii}X = K_i\varphi X$, for any vector field X, where K_i is a smooth function in M. From the equation (3.4), we get

$$(\nabla_{\xi_i} h_i)\, X = -K_i\varphi X + \sum_{k=1}^{s} \delta_i^k \left[-\alpha^2\varphi X - \alpha h_k X\right] - \alpha h_i X - \varphi h_i^2 X$$

Seperating the equation above to symmetric and skew-symmetric parts, we obtain

$$(\nabla_{\xi_i} h_i)X = -\alpha \left[\sum_{k=1}^{s} \delta_i^k h_k X + h_i X\right]$$

and

$$-K_i\varphi X - \alpha^2\varphi X - \varphi h_i^2 X = 0. \tag{4.4}$$

Let $\{E_1, E_2, ..., E_{2n}, \xi_1, ..., \xi_s\}$ be an orthonormal basis of the tangent space at any point. Firstly, we apply inner product with φX both two sides in (4.4). Then, the sum for $1 \le j \le 2n$ of the relation (4.4) with $X = E_j$ yields $K_i = -(\alpha^2 + \frac{\|h_i\|^2}{2n})$. □

Remark 1. The conditions "h_i is a Codazzi tensor " and "φh_i is a Codazzi tensor" are equivalent.

Proposition 13. *Let M be an almost α-cosymplectic f-manifold. If the tensor field φh_i's (or h_i's) are Codazzi, then the following conditions hold:*

1) *If $\alpha = 0$ then the integral manifolds of D are totally geodesic.*

2) *If $\alpha = 0$ and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

3) *The integral manifolds of D are totally umbilic when $\alpha \neq 0$.*

Proof. Let the tensor field φh_i be Codazzi. Taking $X = \xi_j$, $Y \in D$, we get $(\nabla_{\xi_j} h_i)Y - (\nabla_Y h_i)\xi_j = 0$. By using (3.4), we obtain $-\varphi l_{ji}Y = \alpha^2\varphi Y + \alpha h_j Y$. By (3.3), we have $h_i h_j Y = 0$, for any i, j, so $h_i = 0$, for any i, and the statement follows by Proposition 5. □

Theorem 5. *Let M be an almost α-cosymplectic f-manifold. If the tensors τ_i's are parallel and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

Proof. *Let the tensor fields τ_i's are the parallel tensor field. It means that $(\nabla_X \tau_i)Y = 0$, for all $i \in \{1, 2, ..., s\}$ and $X, Y \in \Gamma(TM)$. Putting $Y = \xi_j$ for any $j \in \{1, 2, ..., s\}$ and contracting the equation above with respect to X, we get $-2n\alpha^2 + \alpha \operatorname{trace}(\varphi h_j) + \alpha \operatorname{trace}(\varphi h_i) - \operatorname{trace}(h_i h_j) = 0$. If we examine the last equation for all values of i and j and , we see that suffices $\alpha = 0$ and $h_\varsigma = 0$ for all $\varsigma \in \{1, 2, ..., s\}$. Hence, the proof is obvious by Theorem 1.* □

Proposition 14. *Let M be an almost α-cosymplectic f-manifold. If the tensor fields τ_i's are η-parallel, then*

$$\begin{aligned}(\nabla_X \varphi h_i) Y = \sum_{k=1}^{s} \Big[& \eta^k(X) (\nabla_{\xi_k} \varphi h_i) Y - \eta^k(Y) \varphi h_i \nabla_X \xi_k \\ & + g((\nabla_X \varphi h_i) \xi_k, Y) \xi_k \big]. \end{aligned} \tag{4.5}$$

Proof. Suppose that τ_i is η-parallel. It satisfies equation $g((\nabla_{X^T} \tau_i)Y^T, Z^T) = 0$ for any vector fields X^T, Y^T, Z^T on D. By simple computations, we get

$$\begin{aligned}(\nabla_X \tau_i) Y = \sum_{k=1}^{s} \big[& g((\nabla_X \tau_i) Y, \xi_k) \xi_k \\ & + \eta^k(Y) (\nabla_X \tau_i) \xi_k + \eta^k(X) (\nabla_{\xi_k} \tau_i) Y \Big]. \end{aligned} \tag{4.6}$$

On the other hand, one can easily obtain that

$$(\nabla_X \tau_i)Y = \sum_{\upsilon=1}^{s} [-2\alpha\eta^\upsilon(Y)\nabla_X \xi_\upsilon - 2\alpha g(\nabla_X \xi_\upsilon, Y)\xi_\upsilon] - 2(\nabla_X \varphi h_i)Y. \tag{4.7}$$

From (4.6) and (4.7) we have the desired result. □

Theorem 6. *Let M be an almost α-cosymplectic f-manifold. If the tensor fields τ_i's are η-parallel, then* $R(X,Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y) l_{ki}X - \eta^k(X) l_{ki}Y$.

Proof. Using equation (4.5), we obtain the following difference:

$$\begin{aligned}(\nabla_Y \varphi h_i) X - (\nabla_X \varphi h_i) Y &= \sum_{k=1}^{s} \eta^k(Y) (\nabla_{\xi_k} \varphi h_i) X - \sum_{k=1}^{s} \eta^k(X) (\nabla_{\xi_k} \varphi h_i) Y \\ &+ \sum_{k=1}^{s} \eta^k(Y) \varphi h_i \nabla_X \xi_k - \sum_{k=1}^{s} \eta^k(X) \varphi h_i \nabla_Y \xi_k. \end{aligned} \tag{4.8}$$

Using (3.4) and (4.8), we get, $R(X,Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y) l_{ki}X - \eta^k(X) l_{ki}Y$. Hence, the proof is completed. □

Proposition 15. *Let M be an almost α-cosymplectic f-manifold. If the tensor field φh_i's are cyclically parallel, then the following conditions hold:*

1) *If $\alpha = 0$ then the integral manifolds of D are totally geodesic.*

2) *If $\alpha = 0$ and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

3) *The integral manifolds of D are totally umbilic when $\alpha \neq 0$.*

Proof. The hypothesis can be written

$$g((\nabla_X \varphi h_i) Y, \xi_j) + g((\nabla_Y \varphi h_i) \xi_j, X) + g(\left(\nabla_{\xi_j} \varphi h_i\right) X, Y) = 0$$

for all vector fields X, Y on M. From this equation, we get the following equation $\left(\nabla_{\xi_j} h_i\right) X = 2\alpha h_i X + \varphi(h_i \circ h_j + h_j \circ h_i)X$. Making use of (3.2), we obtain $R(X,\xi_i)\xi_i = \sum_{k=1}^{s} \delta_i^k [\alpha^2 \varphi^2 X + \alpha \varphi h_k X] + 3\alpha\varphi h_i X - 3h_i^2 X$ Applying φ to the last equation , substituting φX for X and using (3.3), we get $h_i^2 = 0$. So, we obtain trace$(h_i^2) = 0$, for any i, and apply Proposition 5. □

Theorem 7. *Let M be an almost α-cosymplectic f-manifold. If the tensors τ_i's are cyclically parallel , then the following conditions hold:*

1) *The integral manifolds of D are totally geodesic*

2) *If M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s.*

Proof. As $\tau_i X = -2\alpha\varphi^2 X - 2\varphi h_i X$, the hypothesis can be written $g((\nabla_X \tau_i)Y, Z) + g((\nabla_Y \tau_i)Z, X) + g((\nabla_Z \tau_i)X, Y) = 0$, for arbitrary vector fields X, Y, Z on M. Using (2.3) and replacing Z by ξ_j, we reduce the following relation:

$$\varphi\left(\nabla_{\xi_j} h_i\right) X = 2\alpha^2\varphi^2 X + 2\alpha\varphi h_j X + 2\alpha\varphi h_i X - h_i h_j X - h_j h_i X. \tag{4.9}$$

Substitution of the (4.9) into (3.2), we get

$$l_{ji} X - \varphi l_{ji}\varphi X = 6\alpha^2\varphi^2 X - 4h_i h_j X - 2h_j h_i X. \tag{4.10}$$

From equality of (3.3) and (4.10), we have $2\alpha^2\varphi^2 X - h_j h_i X - h_i h_j X = 0$. Hence, the proof is clear. □

Example 1. Let, $n = 1$ and $s = 2$. We consider the 4-dimensional manifold $M = \left\{(x, y, z_1, z_2) \in \mathbb{R}^4\right\}$, where (x, y, z_1, z_2) are the standart coordinates in $\mathbb{R}^4$. The vector fields$e_1 = f_1(z_1, z_2)\frac{\partial}{\partial x} + f_2(z_1, z_2)\frac{\partial}{\partial y}$, $e_2 = -f_2(z_1, z_2)\frac{\partial}{\partial x} + f_1(z_1, z_2)\frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z_1}$, $e_4 = \frac{\partial}{\partial z_2}$, where f_1 and f_2 are given by

$$\begin{aligned} f_1(z_1, z_2) &= c_2 e^{-\alpha(z_1+z_2)}\cos(z_1+z_2) - c_1 e^{-\alpha(z_1+z_2)}\sin(z_1+z_2), \\ f_2(z_1, z_2) &= c_1 e^{-\alpha(z_1+z_2)}\cos(z_1+z_2) + c_2 e^{-\alpha(z_1+z_2)}\sin(z_1+z_2) \end{aligned}$$

for constant $c_1, c_2, \alpha \in \mathbb{R}$. It is obvious that $\{e_1, e_2, e_3, e_4\}$ are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

for all $i, j \in \{1, 2, 3, 4\}$and given by the tensor product $g = \frac{1}{f_1^2+f_2^2}(dx \otimes dx + dy \otimes dy) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2$. Let η^1 and η^2 be the 1-form defined by $\eta^1(X) = g(X, e_3)$ and $\eta^2(X) = g(X, e_4)$, respectively, for any vector field X on M and φ be the $(1, 1)$ tensor field defined by $\varphi(e_1) = e_2, \varphi(e_2) = -e_1, \varphi(e_3 = \xi_1) = 0$, $\varphi(e_4 = \xi_2) = 0$. Also, let h_i's be the $(1, 1)$ tensor fields defined by $h_i(e_1) = -e_1, h_i(e_2) = e_2, h_i(e_3) = 0$ and $h_i(e_4) = 0$. Then using linearity of g and φ, we have

$$\begin{aligned} &\varphi^2 X = -X + \eta^1(X)e_3 + \eta^2(X)e_4 \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y) \\ &\eta^1(e_3) = 1 \text{ and } \eta^2(e_4) = 1 \end{aligned}$$

for any vector fields on M

It remains to prove that $d\Omega = 2\overline{\eta}\wedge\Omega$ and Nijenhuis torsion tensor of φ is zero. It follows that $\Omega(e_1, e_2) = -1$ and otherwise $\Omega(e_i, e_j) = 0$ for $i \leq j$. Therefore, the essential non-zero component of Ω is $\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -\frac{1}{f_1^2+f_2^2} = -\frac{e^{2\alpha(z_1+z_2)}}{c_1^2+c_2^2}$, and hence

$$\Omega = -\frac{2e^{2\alpha(z_1+z_2)}}{c_1^2+c_2^2}dx\wedge dy. \tag{4.11}$$

Consequently, the exterior derivative $d\Omega$ is given by

$$d\Omega = -\frac{4\alpha e^{2\alpha(z_1+z_2)}}{c_1^2+c_2^2}dx\wedge dy\wedge(dz_1+dz_2). \tag{4.12}$$

Since $\eta^1 = dz_1$ and $\eta^2 = dz_2$, by (4.11) and (4.12), we find $d\Omega = 2\alpha(\eta^1 + \eta^2)\wedge\Omega$. Let ∇ be the Levi-Civita connection with respect to the metric g. Then, we obtain $[e_1, e_3] = [e_1, e_4] = \alpha e_1 - e_2$, $[e_2, e_3] = [e_2, e_4] = e_1 + \alpha e_2$, $[e_1, e_2] = 0$, $[e_3, e_4] = 0$. In conclusion, it can be noted that Nijenhuis torsion tensor of φ is zero. Thus, the manifold is an α-cosymplectic f-manifold.

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