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ON CONVOLUTION SEMIGROUPS ON LOCALLY COMPACT NONCOMMUTATIVE GROUPS

BY

MARIAN-DUMITRU PANŢIRUC

Abstract. The purpose of this paper is to study the possibility of recovering a convolution semigroup on a locally compact noncommutative group using an inversion formula. Theorem 2.1 shows how we can do that for the heat kernel on the Heisenberg group, while for a more general case the conditions required to do that are stated in Proposition 3.1.

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1. Preliminaries

For a vaguely continuous convolution semigroup on a locally compact Abelian group it is known that the Fourier transform of its measures can be written using some continuous negative definite function on the dual group (see [3]). In the non-commutative case there are several problems that keeps one from doing the same thing. For instance the dual space of a locally compact noncommutative group is no longer a group. In this paper it will be shown to reconstruct a convolution semigroup in the noncommutative case together with the conditions needed to do that.

Performing a Fourier transform in the non-Abelian case is more difficult that in the commutative one and requires notions of representation theory (see Folland [4]). By a result of Siebert [9] and [10] the Fourier transform of the measures of a vaguely continuous convolution semigroup on a locally compact group at each representation is a strongly continuous contraction semigroup.

Consider a locally compact group G and his dual space $\hat{G}$, formed with equivalence classes of all irreducible representations π of G into some separable Hilbert space H_π. More details can be found in FOLLAND [4] or HEYER [6]. The dual space will be considered endowed with the measurable structure called the Mackey-Borel structure. Further, the group G will be considered a type I group, meaning that the Mackey-Borel structure on $\hat{G}$ is standard.

For such a group, there is a measurable field of representations over $\hat{G}$, $(\rho_\pi)_{[\pi]\in\hat{G}}$ such that $\rho_\pi \in [\pi]$. Thus, we can identify the points in $\hat{G}$ with the representations in this measurable field.

If we fix a measurable field of representations as above, the Fourier transform of a bounded measure $\mu \in M_b(G)$ is defined as the measurable field of operators over $\hat{G}$ given by:

$$\hat{\mu}(\pi) = \int_G \pi(x^{-1}) d\mu(x).$$

The Fourier transform for $f \in L^1(G)$ is the measurable field of operators over $\hat{G}$:

$$\hat{f}(\pi) = \int_G f(x)\pi(x^{-1}) dx.$$

The basic properties of the Fourier transform are still valid. In the next section a Plancherel-type theorem is needed:

Theorem 1.1. *Let G be a locally compact group that is also unimodular, type I, c_2. Denote by $\mathcal{I}_1 = L^1(G) \cap L^2(G)$ and $\mathcal{I}_2 = span\{f * g \mid f, g \in \mathcal{I}_1\}$. There is a measure μ on $\hat{G}$, uniquely determined by the Haar measure on G such that:*

1. *If $f \in \mathcal{I}_1$ then $\hat{f} \in \int_{\hat{G}}^{\oplus} HS_\pi d\mu(\pi)$.*
2. *The map $f \mapsto \hat{f}$ extends to an unitary operator from $L^2(G)$ to*
$$\int_{\hat{G}}^{\oplus} HS_\pi d\mu(\pi).$$
3. *For $f, g \in \mathcal{I}_1$ one has the Parseval formula:*
$$\int f(x)\overline{g(x)} dx = \int tr[\hat{f}(\pi)\hat{g}(\pi)^*] d\mu(\pi).$$

4. *For $h \in \mathcal{I}_2$ we have the Fourier inversion Formula:*

$$h(x) = \int tr[\pi(x)\hat{h}(\pi)]d\mu(\pi),$$

where HS_π is the Hilbert space of the Hilbert-Schmidt operators over H_π, endowed with the Hilbert-Schmidt norm, $\int_{\hat{G}}^{\oplus} HS_\pi d\mu(\pi)$ is the space of square-integrable fields of Hilbert-Schmidt operators over $\hat{G}$, with respect to μ and $tr\ T$ is the trace of the trace-class operator T.

Recall here that a positive operator T on a Hilbert space H is said to be trace-class if T has an orthonormal eigenbasis (e_n) with eigenvalues (λ_n) with $\sum \lambda_n < \infty$ and its trace is $tr(T) = \sum \lambda_n$.

If we consider a convolution semigroup $(\mu_t)_{t>0}$ on G we will have the semigroup property for the family of operators $(\widehat{\mu_t}(\pi))_{t>0}$ on H_π for each π:

$$\widehat{\mu_t * \mu_s}(\pi) = \widehat{\mu}_{t+s}(\pi), \quad \text{for all } s, t > 0.$$

By Siebert, [9], Prop. 3.1 we have even more: $(\widehat{\mu_t}(\pi))_{t>0}$ is a strongly continuous semigroup of operators over H_π for every unitary representation π of G on H_π.

2. A particular case

It is showed next how to reconstruct the heat semigroup on the Heisenberg group, H_n using the Fourier transform of its measures, by means of Theorem 1.1. The Heisenberg group is in fact $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the composition law:

$$(x, \xi, t)(x', \xi', t') = \big(x + x', \xi + \xi', t + t' + \frac{1}{2}(x'\xi - x\xi')\big), \tag{1}$$

where $x\xi$ is the usual scalar product in $\mathbb{R}^n$. H_n is a *non-Abelian* group, has the unit 1=(0,0,0) and the inverse of an element $X = (x, \xi, t) \in H_n$ is $(-x, -\xi, -t)$. H_n is a locally compact group endowed with the usual topology on $\mathbb{R}^{2n+1}$ and the Haar measure for this group is the Lebesgue measure on $\mathbb{R}^{2n+1}$.

H_n has a second order differential operator, Δ_H:

$$\Delta_H = \frac{1}{2}\sum_{i=1}^{n}(X_i^2 + \Xi_i^2),$$

where

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\xi_i\frac{\partial}{\partial t} \quad \text{and} \quad \Xi_i = \frac{\partial}{\partial \xi_i} - \frac{1}{2}x_i\frac{\partial}{\partial t}.$$

The heat kernel for Δ_H was independently computed by Gaveau [5] and Hulanicki [7] as an answer to the requirements:

$$\begin{cases} \dfrac{\partial K_s}{\partial s} = \Delta_H K_s \\ \lim\limits_{s\to 0+} K_s(x,\xi,t) = \delta(x,\xi,t) \end{cases}.$$

We have:

$$K_s(x,\xi,t) = \frac{1}{(4\pi s)^{n+1}}\int_{-\infty}^{+\infty} exp(-f(x,\xi,t,\tau)/2s)V(\tau)d\tau,$$

where

$$f(x,\xi,t,\tau) = -it\tau + (\|x\|^2 + \|\xi\|^2)\frac{\tau}{4}ctgh\frac{\tau}{2}$$

and

$$V(\tau) = \left(\frac{\tau/2}{sh\frac{\tau}{2}}\right)^n.$$

(for further details see also Beals [2]).

In Panţiruc [8] is given an explicit formula for the Fourier transform for the measures of this semigroup. Computations for Fourier transform for a gaussian measure were also made in [1], for different (unitarily equivalent) representations. H_n is a type I group and his dual space, $\widehat{H_n}$, can be identified with the measurable field of irreducible representations $\{\pi_{b,\beta};\rho_h\}_{b,\beta,h}$:

- 1-dimensional:

  $$\pi_{b,\beta} : H_n \to \mathbb{T},\ \pi_{b,\beta}(x,\xi,t) = e^{2\pi i(bx+\beta\xi)}$$

  with $b,\beta \in \mathbb{R}^n$;

- ∞-dimensional:

  $$\rho_h : H_n \to \mathcal{U}(L^2(\mathbb{R}^n)),\ [\rho_h(x,\xi,t)f](y) = e^{2\pi ih(t+\frac{1}{2}x\xi-y\xi)}f(y-x),$$

  where h is a non-zero real number.

Thus the Fourier transform of a bounded measure $\mu \in M_b(H_n)$ is a measurable field over $\widehat{H_n}$ given by:

- the complex values

$$\hat{\mu}(\pi_{b,\beta}) = \int_{H_n} \pi_{b,\beta}(-x,-\xi,-t)d\mu(x,\xi,t) = \int_{H_n} e^{-2\pi i(bx+\beta\xi)}d\mu(x,\xi,t);$$

- and the operators on $L^2(\mathbb{R}^n)$:

$$[\hat{\mu}(\rho_h)\phi](y) = \int_{H_n} [\rho_h(-x,-\xi,-t)\phi](y)d\mu(x,\xi,t) =$$

$$= \int_{H_n} e^{2\pi ih(-t+\frac{1}{2}x\xi+y\xi)}\phi(y+x)d\mu(x,\xi,t).$$

In Panţiruc [8], Proposition 2.1, it is given a formula for the Fourier transform of the measures of the "heat" kernel. This is a measurable field of operators over $\widehat{H_n}$ given by $\widehat{K_s}(\pi_{b,\beta}) = e^{-4\pi^2(\|b\|^2+\|\beta\|^2)}$ for the 1-dimensional representations $\pi_{b,\beta}$ and

$$[\widehat{K_s}(\rho_h)\phi](y) = \left(\frac{h}{sh(4\pi hs)}\right)^{\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \phi(x)e^{-\frac{\pi h}{2}(a\|x-y\|^2+\frac{1}{a}\|x+y\|^2)}dx.$$

for the ∞-dimensional ones, ρ_h. As one can easily see we have $\widehat{K_s}(\rho_h) = \widehat{K_s}(\rho_{-h})$ for each $s,h>0$.

In the same paper, Theorem 2.2 shows that for every $h \neq 0$ the family of operators $(\widehat{K_s}(\rho_h))_{s\geq 0}$ is a C_0 semigroup of operators on $L^2(\mathbb{R}^n)$ whose infinitesimal generator is

$$A_h f(y) = \Delta f(y) - 4\pi^2 h^2 \|y\|^2 f(y), \qquad \text{for} \quad f \in C_c^\infty(\mathbb{R}^n).$$

Putting all these information together we can prove the next:

Theorem 2.1. *Let $(K_s)_{s>0}$ be the heat kernel on H_n as described above. Then*

$$K_s(x,\xi,t) = \int_{\mathbb{R}^n\setminus\{0\}} tr[\rho_h(x,\xi,t)\widehat{K_s}(\rho_h)]d\mu(\rho_h)$$
$$= \int_{\mathbb{R}\setminus\{0\}} [\sum_{j=1}^{\infty} \lambda_j(h) < \rho_h(x,\xi,t)\phi_j(h), \phi_j(h) >] \mid h \mid^n dh,$$

where $(\phi_j(h))_{j\geq 1}$ is an orthonormal basis in $L^2(H_n)$ consisting of eigenvectors (with eigenvalues $\lambda_j(h)$) corresponding to the operators $\widehat{K_s}(\rho_h)$, for $h \in \mathbb{R}\setminus\{0\}$.

Proof. It is enough to verify the conditions from the Plancherel theorem 1.1. It is easy to see that

$$K_s \geq 0,\ K_s(x,\xi,t) = K_s(-x,-\xi,-t) \text{ and } K_{s_1} * K_{s_2} = K_{s_1+s_2}.$$

It follows in particular that the functions K_s are positive definite and therefore bounded. But $K_s \in L^1(H_n)$ and being bounded means that they are also square-integrable. We can then write $K_s = K_{s/2} * K_{s/2}$ to see that $K_s \in \mathcal{I}_2$ from the Plancherel theorem 1.1.

The Plancherel measure for the Heisenberg group is given by:

$$d\mu(\pi_{b,\beta}) = 0, \qquad d\mu(\rho_h) = \mid h \mid^n dh$$

(see Folland [4]). We can write then

$$\begin{aligned} K_s(x,\xi,t) &= \int_{\mathbb{R}^n\setminus\{0\}} tr[\rho_h(x,\xi,t)\widehat{K_s}(\rho_h)]d\mu(\rho_h) \\ &= \int_{\mathbb{R}\setminus\{0\}} [\sum_{j=1}^{\infty} \lambda_j(h) < \rho_h(x,\xi,t)\phi_j(h), \phi_j(h) >] \mid h \mid^n dh \end{aligned}$$

as soon as we prove that the formula makes sense, that is the operators $\widehat{K_s}(\rho_h)$ are trace-class. But

$$[\widehat{K_s}(\rho_h)\phi](y) = \left(\frac{h}{sh(4\pi hs)}\right)^{\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \phi(x) e^{-\frac{\pi h}{2}(a\|x-y\|^2 + \frac{1}{a}\|x+y\|^2)} dx,$$

and in Panţiruc [8], Theorem 2.1 gives some properties for the operators $\widehat{K_s}(\rho_h)$: they are all bounded, self-adjoint and positive operators on $L^2(\mathbb{R}^n)$ and since they are obviously Hilbert-Schmidt it follows that they are also trace-class. Thus, the proof is complete. □

Theorem 2.1 shows how one can reconstruct this convolution semigroup out of the Fourier transforms of its measures but we must emphasize that of great help to do that were the already known properties of the functions K_s.

3. The general case

The next simple example of convolution semigroup on H_n shows that one cannot hope to always apply the inversion formula to recover the initial semigroup.

Consider a translation semigroup on H_1. This has the form $(\varepsilon_{(at,bt,ct)})_{t>0}$ where a, b, c are arbitrary but fixed real numbers and $\varepsilon_{(at,bt,ct)}$ is the Dirac measure concentrated at the point $(at, bt, ct) \in H_1$. In general, the Fourier transform of a Dirac measure ε_x at a representation π is obviously $\widehat{\varepsilon_x}(\pi) = \pi^*(x) = \pi(x^{-1})$ which is a unitary operator on the representation (Hilbert) space H_π. For H_1 and for the representations ρ_h we get $\hat{\varepsilon}_{(at,bt,ct)}(\rho_h) = \rho_h(-at, -bt, -ct)$.

The family $(\rho_h(-at, -bt, -ct))_{t\geq 0}$ is a C_0-semigroup of unitary operators on $L^2(\mathbb{R})$. The infinitesimal generator of this semigroup is given by:

$$\begin{aligned} &A_h : D(A_h) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ &A_h\phi(y) = a\phi'(y) + 2\pi ih(-c + by)\phi(y). \end{aligned}$$

But the inversion formula cannot be applied. For once, the Fourier transform for the semigroups measures are all unitary operators and these are not trace-type operators unless the representation space is finite-dimensional, and here is not the case.

It is known (by Siebert [9]) that if $(\mu_t)_{t>0}$ is a convolution semigroup on G then for any representation π of G in H_π, $[\widehat{\mu_t}(\pi)]_{t\geq 0}$ is a C_0-contraction semigroup on H_π. If we take this observation back to the commutative case we see that the value $\psi(\gamma)$ that appears in the expression of the Fourier transform: $\widehat{\mu_t}(\gamma) = exp(-t\psi(\gamma))$ is the infinitesimal generator of the semigroup of operators on $\mathbb{C}$:

$$-\psi(\gamma) \cdot z = \lim_{t\to 0+} \frac{e^{-t\psi(\gamma)}z - z}{t},\ z \in \mathbb{C}.$$

It seems natural then to ask what conditions should one impose on a family $A_\pi,\ \pi \in \hat{G}$ of infinitesimal generators of C_0 semigroups such that the inversion formula to become applicable and by using it to obtain a convolution semigroup.

For every $\pi \in \hat{G}$ denote by $\mathcal{A}(\pi)$ the set of all infinitesimal generators of C_0 semigroups on the representation space H_π. For every $\pi \in \hat{G}$ let $A(\pi) \in \mathcal{A}(\pi)$ the generator of some semigroup of self adjoint, Hilbert-Schmidt operators. Such a generator can be identified with some sequence of real, negative numbers λ_n having the limit $-\infty$ and such that $\sum \frac{1}{\lambda_n^2} < \infty$). Denote by $(P_t(\pi))_{t\geq 0}$ the semigroup generated by $A(\pi)$. Obviously, we have:

$$P_t(\pi)u = \sum e^{t\lambda_n(\pi)} < u, e_n(\pi) > e_n(\pi),\ u \in H_\pi,$$

where $\lambda_n(\pi)$ are the eigenvalues of the infinitesimal generator $A(\pi)$ and $e_n(\pi)$ is an orthonormal basis formed with the eigenvectors of $P_t(\pi)$. Of course, all the operators $P_t(\pi)$ are trace-type since they are a product of some Hilbert-Schmidt operators. Assume that the function $\pi \mapsto tr(P_t(\pi))$ is μ-integrable:

$$\int_{\hat{G}} tr(P_t(\pi))d\mu(\pi) < \infty \quad \forall\, t > 0.$$

We then have the next:

Proposition 3.1. *Let μ be the Plancherel measure on $\hat{G}$.*
If $\int_{\hat{G}} tr(P_t(\pi))d\mu(\pi) < \infty$ for every $t > 0$ then we have:

i. There is $f_t \in L^2(G)$ such that $\widehat{f_t}(\pi) = P_t(\pi)$ for every $t > 0, \pi \in \hat{G}$ and $f_t * f_s = f_{t+s}$ for every $t, s > 0$

ii. If $f_t \in L^1(G)$ then

- *(a) $f_t(x) = \int_{\hat{G}} tr[\pi(x)P_t(\pi)]d\mu(\pi), t > 0$;*
- *(b) f_t is a positive definite function on G, for every $t > 0$.*
- *(c) If in addition we suppose that for every $t > 0$, $f_t \geq 0$ then $(f_t dx)_{t>0}$ is a convolution semigroup G.*

Proof. i)Since P_t is self-adjoint, we get:

$$tr(P_t(\pi)) = \sum < P_t(\pi)e_n(\pi), e_n(\pi) >= \sum \|P_{\frac{t}{2}}(\pi)e_n(\pi)\|^2 = \|P_{\frac{t}{2}}(\pi)\|^2_{HS}.$$

It follows that $(P_t(\pi))_\pi \in \int^{\oplus} H_\pi \otimes H_{\overline{\pi}} d\mu(\pi)$ and from the Plancherel theorem 1.1 there is some $f_t \in L^2(G)$ such that $\widehat{f_t}(\pi) = P_t(\pi)$. The relation $f_t * f_s = f_{t+s}$ is true by the injectivity of the Fourier transform and the fact that $\hat{f}_{t+s} = P_{s+t} = P_s P_t = \hat{f}_s \hat{f}_t = \widehat{f_t * f_s}$.

ii) If $f_t \in L^1(G)$ then $f_t \in \mathcal{I}_2$ from 1.1 and then we have ii-a). We also have $f_t(x^{-1}) = \overline{f_t(x)}$ and $f_t(e) \geq 0$ and

$$|f_t(x)| \leq \int_{\hat{G}} |tr[\pi(x)P_t(\pi)]| d\mu(\pi) \leq \int_{\hat{G}} tr[P_t(\pi)]d\mu(\pi) = f_t(e).$$

Let $c_1,\ c_2, ...c_n \in \mathbb{C},\ x_1...x_n \in G$. Then:

$$\sum_{i,j=1}^{n} c_i \overline{c_j} f_t(x_i x_j^{-1}) = \int_{\hat{G}} \sum e^{t\lambda_n} \| \sum_{i=1}^{n} c_i \pi(x_i) e_n \| d\mu(\pi) \geq 0$$

which means that f_t is a positive definite function.

Finally, if $f_t \geq 0$ then one can apply Lemma 2.1 from SIEBERT, [9] to see that $f_t dX \to \varepsilon_0$ vaguely. Combining this with *a*) it follows that $(f_t dx)_{t>0}$ is a convolution semigroup on G, completing the proof. $\square$

The assumption on f_t to be positive is a strong one and in the commutative case this is solved by Bochner's theorem on the representation of positive definite functions and for the proof was essential that a LCA-group is "reflexive" (Pontrjagin theorem). In the commutative case the Plancherel measure is the Haar measure on the dual group and so it is positive. Also in the commutative case we have an inversion theorem to help us decide wether the second condition in ii) in the above theorem holds.

In the noncommutative case things are again more complicated. First, one sees that in general there is a problem in identifying $L^1(\hat{G})$. The Fourier transform of a function $f \in L^1(G)$ is defined on $\hat{G}$ but the values are compact operators on (possibly different) H_π's and not complex numbers as in the Abelian case.

Still, for the Heisenberg group we see that the Plancherel measure is positive and we can only take the representations ρ_h, $h \neq 0$ and so the representation spaces can all be taken to be $L^2(\mathbb{R}^n)$. Thus,

$$\int^{\oplus} H_\pi \otimes H_{\overline{\pi}} d\mu\pi = L^2\left(\hat{G}, \mathcal{HS}(L^2(\mathbb{R}^n)); |h|^n dh\right)$$

and $L^1(\hat{G})$ is actually $L^1(\hat{G}, \mathcal{L}(L^2(\mathbb{R}^n)); |h|^n dh)$. Thus, the inversion theorem on the Heisenberg group has the following form:

Conjecture. *Let μ a bounded measure on H_n. If $\hat{\mu} \in L^1(\hat{H}_n, \mathcal{HS}(L^2(\mathbb{R}^n));$ $|h|^n dh)$ and $\hat{\mu}(\rho_h)$ is trace-class for every $h \neq 0$ then μ has a continuous density with respect to the Haar measure on H_n and this density is given by*:

$$\phi(x) = \int_{\hat{G}} tr[\rho_h(x)\hat{\mu}(\rho_h)]|h|^n dh, \ x \in H_n.$$

If this is true then is a first step in reconstructing a whole class of convolution semigroups on (at least) the Heisenberg group out of Fourier transforms of its measures, namely that of convolution semigroups formed by measures that are absolutely continuous with respect to the Haar measure.

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Department of Mathematics and Informatics,
"Gh. Asachi" Technical University of Iaşi,
Bd. Carol I, no. 11, Iaşi-700506,
ROMANIA
mpantiruc@etti.tuiasi.ro