ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LX, 2014, f.2
DOI: 10.2478/aicu-2013-0015

WEAKLY (τ_q, m)-CONTINUOUS FUNCTIONS

BY

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Abstract. In this paper we introduce a new class of functions called weakly (τ_q, m)-continuous functions. Some characterizations and several properties concerning weak (τ_q, m)-continuity are obtained.

Mathematics Subject Classification 2010: 54C08, 54A05, 54D25.
Key words: m-structure, (τ_q, m)-continuity, weak (τ_q, m)-continuity, strongly *clp*-m-closed graph, m_X-regular set, ultra Hausdorff space, ultraregular space.

1. Introduction

Semi-open sets, preopen sets, α-sets, b-open sets, β-open sets play an important role for generalization of continuity in topological spaces. By using these sets several authors introduced and studied various modifications of continuity such as weak continuity, almost s-continuity ([22]), $p(\theta)$-continuity ([7]). Popa and Noiri [30] introduced the notions of minimal structures. After this work, various mathematicians turned their attention in introducing and studying diverse classes of sets and functions defined on an structure, because this notions are a natural generalization of many well known results related with generalized sets and several weaker forms of continuity such as ([20], [21], [32], [33], [40]). The notion of weakly M-continuous and weakly (τ, m)-continuous functions are introduced and studied by Popa and Noiri ([28], [29]) for unifying weak continuity types using minimal conditions. They also defined weakly (τ, β)-continuous functions as a special case of weak (τ, m)-continuity. Weak (τ, β)-continuity is also studied by present author [39] and by Basu and Ghosh [4] (under the name of (θ, β)-continuous functions). Recently Son, Park and Lim introduced and studied weakly clopen functions ([36]). In fact this type of functions can be unified as weakly M-continuous function from a space with quasi-topology τ_q, to a space with an m-structure, that is a function $f : (X, \tau_q) \to (Y, m_Y)$ which can be named as weakly (τ_q, m)-continuous functions. The purpose of this paper is to introduce and investigate the notion of weakly (τ_q, m)-continuous functions. In addition we also discuss possible generalizations of the concept of almost clopen functions due to Ekici [11], which is recently studied in detail by various authors ([15],[16]).

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset S of (X, τ), $cl(S)$ and $int(S)$ represent, the closure of S and the interior of S, respectively. A subset S of a space (X, τ) is said to be regular open ([38]) (resp. regular closed ([38])) if $S = int(cl(S))$ (resp. $S = cl(int(S))$). A point x of X is called a θ-cluster ([41]) point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure ([41]) of A and is denoted by $cl_\theta(A)$. A set A is said to be θ-closed if $A = cl_\theta(A)$. The complement of a θ-closed set is said to be θ-open. A subset S of a space (X, τ) is said to be semi-open ([17]) (resp. preopen ([19]), α-open ([27]), semi-preopen ([2]) or β-open ([1]), b-open ([3])) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(S))$, $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$, $S \subset cl(int(S)) \cup int(cl(S))$). The family of all semi-open (resp.preopen, α-open, β-open, b-open) sets of X is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$). The complement of a semi-open (resp.preopen, α-open, β-open, b-open) set is said to be semi-closed (resp. preclosed, α-closed, β-closed, b-closed). If S is a subset of a space X, then the b-closure of S, denoted by $bcl(S)$, is the smallest b-closed set containing S. The semiclosure (resp. preclosure, α-closure, b-closure) of S is similarly defined and is denoted by $scl(S)$ (resp. $pcl(S)$, $\alpha Cl(S)$, $bCl(S)$). A point $x \in X$ is said to be in the semi-θ-closure ([9]) (resp. β-θ-closure or sp-θ-closure ([24])) of A, denoted by $scl_\theta(A)$ (resp. by $\beta cl_\theta(A)$), if $A \cap scl(V) \neq \emptyset$ (resp. $A \cap \beta cl(V) \neq \emptyset$) for every $V \in SO(X, x)$ (resp. $V \in \beta O(X, x)$). If $scl_\theta(A) = A$ (resp. $\beta cl_\theta(A) = A$), then A is said to be semi-θ-closed (resp. β-θ-closed or sp-θ-closed ([24])). The complement of a semi-θ-closed (resp. β-θ-closed) set is said to be semi-θ-open (resp.

β-θ-open).

The quasi-component ([10]) of a point $x \in X$ is the intersection of all clopen subsets of X which contain the point x. The quasi-topology τ_q on X is the topology having as base clopen subsets of (X, τ). The closure of each point in quasi-topology is precisely the quasi-component of that point. The open (resp. closed) subsets of the quasi-topology is called quasi-open ([10]) (resp. quasi-closed ([10])). For a space (X, τ) the space (X, τ_q) is called by Staum [37] the ultraregular kernel of X and denoted by X_q for simplicity. A space (X, τ) is called ultraregular ([37]) if $\tau = \tau_q$. For a subset A of a space X, we define the quasi-interior (resp. quasi-closure) of A, denoted by $int_q(A)$ (resp. $cl_q(A)$), defined by $int_q(A) = \cup\{U$ is quasi-open: $U \subset A\}$,(resp. $cl_q(A) = \cap\{F$ is quasi-closed: $A \subset F\}$).

Definition 1. A subfamily m_X of the power set $\wp(X)$ of a nonempty set X is called a minimal structure (briefly m-structure) ([30]) on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty subset X with a minimal structure m_X on X. Each member of m_X is said to be m_X-open and the complement of m_X-open set is said to be m_X-closed.

Definition 2. A subset S is said to be m_X-regular if it is m_X-open and m_X-closed. The family of all m_X-regular sets of X is denoted by $mR(X)$ and the family of all m_X-open (resp. m_X-regular or m_X-clopen) sets of X containing a point $x \in X$ is denoted by $mO(X, x)$ (resp. $mR(X, x)$).

Remark 1. Let (X, τ) be a topological space. Then the families τ, τ_q, $SO(X), PO(X), \alpha O(X)$, $\beta O(X)$, $SR(X)$, $\beta R(X)$ are all m-structures on X.

Definition 3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined in [18] as follows:

(a) m_X-$Cl(V) = \cap\{F : A \subset F, X - F \in m_X\}$
(b) m_X-$Int(V) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$, (resp. τ_q, $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $SR(X)$, $\beta R(X)$) then we have:

(a) m_X-$Cl(V) = cl(V)$ (resp. $cl_q(V)$, $scl(A)$, $pcl(A)$,$\alpha Cl(A)$, $\beta cl(A)$, $scl_\theta(X)$, $\beta cl_\theta(X)$).

(b) m_X-$Int(V) = int(V)$ (resp. $int_q(V)$, $sint(A)$, $pint(A)$,$\alpha Int(A)$, $\beta int(A)$, $sint_\theta(X)$, $\beta int_\theta(X)$).

Lemma 1 ([18])**.** *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following hold:*

(a) m_X-$Cl(X - A) = X - (m_X$-$Int(A))$ *and* m_X-$Int(X - A) = X - (m_X - Cl(A))$.

(b) *If* $X - A \in m_X$, *then* m_X-$Cl(A) = A$ *and if* $A \in m_X$, *then* $m_X - Int(A) = A$.

(c) m_X-$Cl(\emptyset) = \emptyset$, m_X-$Cl(X) = X$,m_X-$Int(\emptyset) = \emptyset$ *and* m_X-$Int(X) = X$.

(d) *If* $A \subset B$, *then* m_X-$Cl(A) \subset m_X$-$Cl(B)$ *and* m_X-$Int(A) \subset m_X$-$Int(B)$.

(e) *If* $A \subset m_X$-$Cl(A)$ *and* m_X-$Int(A) \subset A$.

(f) m_X-$Cl(m_X$-$Cl(A)) = m_X$-$Cl(A)$ *and* m_X-$Int(m_X$-$Int(A)) = m_X$-$Int(A)$.

Lemma 2 ([30])**.** *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in m_X$-$Cl(A)$ if and only if $U \cap A \neq \emptyset$, for every $U \in m_X$ containing x.*

A point $x \in X$ is called a m_θ-adherent point ([29]) of S if m_X-$Cl(U) \cap S \neq \emptyset$ for every m_X-open set U containing x. The set of all m_θ-adherent points of S is denoted by $mCl_\theta(S)$. A subset S is said to be m_θ-closed if $S = mCl_\theta(S)$. The complement of a m_θ-closed set is said to be m_θ-open.

Definition 4 ([18])**.** A minimal structure m_X on a nonempty set X is said to have property $(\mathcal{B})$ if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3 ([30])**.** *Let X be a nonempty set and m_X a minimal structure on X satisfying the property $(\mathcal{B})$. For a subset A of X, the following properties hold:*

(a) $A \in m_X$ *if and only if* m_X-$Int(A) = A$.

(*b*) *A is m_X-closed if and only if m_X-$Cl(A) = A$.*

(*c*) *m_X-$Int(A) \in m_X$ and m_X-$Cl(A)$ is m_X-closed.*

Definition 5. A minimal structure m_X on a nonempty set X satisfying the property $(\mathcal{B})$ is said to have *property* (mR) if for any subset A of X the following two conditions are true:

(a) m_X-$Cl(m_X$-$Int(m_X$-$Cl(A))) = m_X$-$Int(m_X$-$Cl(A))$.

(b) m_X-$Int(m_X$-$Cl(m_X$-$Int(A))) = m_X$-$Cl(m_X$-$Int(A))$.

Remark 3. Let X be a nonempty set and m_X a minimal structure on X. For the case $m_X = SO(X)$, m_X satisfies equalities in Definition 5 by [1], for $m_X \in \{BO(X),\ \beta O(X)\}$ m_X-$Cl(m_X$-$Int(A)) = m_X$-$Int(m_X$-$Cl(A))$ is true. This statement implies conditions of Definition 5.

Lemma 4. *Let X be a nonempty set and m_X a minimal structure on X satisfying the property $(\mathcal{B})$ and have* property (mR), *then the following properties hold:*

(*a*) *If $V \in m_X$ then m_X-$Cl(V)$ is m_X-regular.*

(*b*) *If F is m_X-closed then m_X-$Int(A)$ is m_X-regular.*

Proof. (a) If $V \in m_X$ then by *property* (mR) (b), m_X-$Int(m_X$-$Cl(V)) = m_X$-$Cl(V)$, that is m_X-$Cl(V)$ is both m_X-open and m_X-closed.

That is the m_X-closure of every m_X-open set is m_X-open, then m_X is m-extremely disconnected (see [40] Definition 3.14).

(b) If F is m_X-closed then by *property* (mR) (a), m_X-$Cl(m_X$-$Int(F)) = m_X$-$Int(F)$, that is m_X-$Int(F)$ is both m_X-open and m_X-closed. □

Definition 6. A function $f : (X, m_X) \to (Y, m_Y)$, where X and Y are nonempty sets with minimal structures m_X and m_Y, respectively, is said to be weakly M-continuous ([28]) (M-continuous, ([30]), almost M-continuous ([5])) at $x \in X$ if for each $V \in m_Y$ containing $f(x)$ there exist $U \in m_X$ containing x such that $f(U) \subset m_Y$-$Cl(V)$ (resp. $f(U) \subset V$, $f(U) \subset m_Y$-$Int(m_Y$-$Cl(V)))$. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be weakly M-continuous (resp. M-continuous, almost M-continuous) if it has the property at each point $x \in X$.

Definition 7. A function $f : (X, m_X) \to (Y, m_Y)$, is said to be M^*-continuous ([20]) if for every $V \in m_Y$, $f^{-1}(V) \in m_X$.

Remark 4. Let Y be a nonempty set and m_Y a minimal structure on Y for which satisfying the property (mR). For a function $f:(X,m_X)\to(Y,m_Y)$ the following properties are equivalent:

(a) f is weakly M-continuous.

(b) f is almost M-continuous.

(c) For $m_Y^* = mR(Y)$, $f:(X,m_X)\to(Y,m_Y^*)$ *is* M-continuous.

Proof. Let $V\in m_Y$, then by property (mR)(b), m_X-$Int(m_X$-$Cl(V))=m_X$-$Cl(V)$, that is m_X-$Int(m_X$-$Cl(V))$ and m_X-$Cl(V)$ are both m_X-regular and equal. □

Lemma 5 ([30]). *For a function $f:(X,m_X)\to(Y,m_Y)$ the following properties are equivalent:*

(*a*) *f is M-continuous.*

(*b*) *$f^{-1}(V)=m_X$-$Int(f^{-1}(V))$ for every $V\in m_Y$.*

(*c*) *$f(m_X$-$Cl(A))\subset m_Y$-$Cl(f(A))$ for every subset A of X.*

(*d*) *m_X-$Cl(f^{-1}(B))\subset f^{-1}(m_Y$-$Cl(B))$ for every subset B of Y.*

(*e*) *$f^{-1}(m_X$-$Int(B))\subset m_X$-$Int(f^{-1}(B))$ for every subset B of Y.*

(*f*) *m_X-$Cl(f^{-1}(K))=f^{-1}(K)$ for every m_Y-closed set K of Y.*

Definition 8. A function $f:X\to Y$ is (τ,m)-continuous ([29]), (resp. weakly (τ,m)-continuous ([29]), almost (τ,m)-continuous) for each $x\in X$ and each m_Y-open set V containing $f(x)$, there exists an open set U containing x, such that $f(U)\subset V$ (resp. $f(U)\subset m_Y$-$cl(V)$, $f(U)\subset m_Y$-$Int(m_Y$-$cl(V)))$.

Definition 9. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be;

(a) almost continuous ([35]) (resp. (θ,b)-continuous, almost s-continuous ([22]), weakly (τ,β)-continuous ([29]), $p(\theta)$-continuous ([7])), if for each $x\in X$ and each open (resp. b-open, semiopen, β-open, preopen) set V containing $f(x)$, there exists an open set U containing x such that $f(U)\subset int(cl(V))$ (resp. $f(U)\subset bcl(V)$, $f(U)\subset scl(V)$, $f(U)\subset \beta cl(V)$, $f(U)\subset pcl(V)$),

(b) almost clopen ([11]) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset int(cl(V))$.

Remark 5. Let (X, τ) and (Y, σ) be topological spaces.

(a) We put $m_X = \tau$ and $m_Y = \tau$ (resp. $SO(Y), PO(Y), \beta O(Y)$). Then, a weakly M-continuous function $f : (X, \tau) \to (Y, m_Y)$ is weakly continuous (resp. almost s-continuous, $p(\theta)$-continuous, weakly (τ, β)-continuous).

(b) We put $m_X = \tau_q$ and $m_Y = \tau$. Then, an almost M-continuous $f : (X, \tau_q) \to (Y, m_Y)$ is almost clopen.

Definition 10. Let Y be a nonempty set and m_Y a minimal structure on Y. A function $f : X \to Y$, is said to be (τ_q, m)-continuous (resp. weakly (τ_q, m)-continuous, almost (τ_q, m)-continuous) at $x \in X$, if for each $V \in m_Y$ containing $f(x)$ there exists a clopen set U containing x such that $f(U) \subset V$ (resp. $f(U) \subset m_Y\text{-}Cl(V)$, $f(U) \subset m_Y\text{-}Int(m_Y\text{-}Cl(V))$). A function $f : (X, \tau_q) \to (Y, m_Y)$ is said to be (τ_q, m)-continuous (resp. weakly (τ_q, m)-continuous, almost (τ_q, m)-continuous) if it has the property at each point $x \in X$.

We will write for (τ_q, m)-continuous (resp. weakly (τ_q, m)-continuous, almost (τ_q, m)-continuous) briefly (τ_q, m).c (resp. w.(τ_q, m).c, a.(τ_q, m).c).

Proposition 1. *A function $f : (X, \tau_q) \to (Y, m_Y)$ is (τ_q, m)-continuous (resp. weakly (τ_q, m)-continuous, almost (τ_q, m)-continuous) if and only if $f : (X, \tau_q) \to (Y, m_Y)$ is M-continuous (resp. weakly M-continuous, almost M-continuous).*

Proof. $(\Rightarrow)$ Let $x \in X$ and V be a m_Y-open set in Y containing $f(x)$. Then by definition there exists a clopen set U containing x such that $f(U) \subset V$ (resp. $f(U) \subset m_Y\text{-}Cl(V)$, $f(U) \subset m_Y\text{-}Int(m_Y\text{-}Cl(V))$). Since every clopen set is quasi-open we have $f : (X, \tau_q) \to (Y, m_Y)$ is M-continuous (resp. weakly M-continuous, almost M-continuous).

$(\Leftarrow)$ Let $x \in X$ and V is a m_Y-open set containing $f(x)$ then there exists a quasi-open set U containing x, such that $f(U) \subset V$ (resp. $f(U) \subset m_Y\text{-}Cl(V)$, $f(U) \subset m_Y\text{-}Int(m_Y\text{-}Cl(V))$). Since U is quasi open there exists a clopen set W in U containing x such that $f(W) \subset V$ (resp. $f(W) \subset m_Y\text{-}Cl(V)$, $f(W) \subset m_Y\text{-}Int(m_Y\text{-}Cl(V))$) and by Definition 10, f is (τ_q, m).c. (resp. w.(τ_q, m).c., a.(τ_q, m).c.). $\square$

Theorem 1. *For a function $f : (X, \tau_q) \to (Y, m_Y)$ the following properties are equivalent:*

(a) *f is (τ_q, m)-continuous.*

(b) *$f^{-1}(V) = int_q(f^{-1}(V))$ for every $V \in m_Y$.*

(c) *$f(cl_q(A)) \subset m_Y\text{-}Cl(f(A))$ for every subset A of X.*

(d) *$cl_q(f^{-1}(B)) \subset f^{-1}(m_Y\text{-}Cl(B))$ for every subset B of Y.*

(e) *$f^{-1}(m_Y\text{-}Int(B)) \subset int_q(f^{-1}(B))$ for every subset B of Y.*

(f) *$cl_q(f^{-1}(K)) = f^{-1}(K)$ for every m_Y-closed set K of Y.*

Proof. Here we will use same techniques with the proof of Lemma 5 ([30]).

(a)⇒(b) Let $V \in m_Y$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. There exists $U \in \tau_q$ containing x such that $f(U) \subset V$. Thus $x \in U \subset f^{-1}(V)$. This implies that $x \in int_q(f^{-1}(V))$. This shows that $f^{-1}(V) \subset int_q(f^{-1}(V))$. Hence we have $f^{-1}(V) = int_q(f^{-1}(V))$.

(b)⇒(c) Let A be any subset of X. Let $x \in cl_q(A)$ and $V \in m_Y$ containing $f(x)$. Then $x \in f^{-1}(V) = int_q(f^{-1}(V))$. There exists $U \in \tau_q$ such that $x \in U \subset f^{-1}(V)$. Since $x \in cl_q(A)$, $U \cap A \neq \emptyset$ and $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$. Since $V \in m_Y$ containing $f(x)$, $f(x) \in m_Y\text{-}Cl(f(A))$ and hence $f(cl_q(A)) \subset m_Y\text{-}Cl(f(A))$.

(c)⇒(d) Let B be any subset of Y. Then, we have $f(cl_q(f^{-1}(B))) \subset m_Y\text{-}Cl(f(f^{-1}(B))) \subset m_Y\text{-}Cl(B)$. Therefore we obtain $cl_q(f^{-1}(B)) \subset f^{-1}(m_Y\text{-}Cl(B))$.

(d)⇒(e) Let B be any subset of of Y. Then, we have $X - int_q(f^{-1}(B)) = cl_q\left(f^{-1}\left(Y - B\right)\right) \subset f^{-1}(m_Y\text{-}Cl(Y - B)) = f^{-1}(Y - m_Y\text{-}Int(B)) = X - f^{-1}(m_Y\text{-}Int(B))$. Therefore, we obtain $f^{-1}(m_Y\text{-}Int(B)) \subset int_q(f^{-1}(B))$.

(e)⇒(f) Let K be any subset of Y such that $Y - K \in m_Y$. By (e), we have $X - f^{-1}(K) = f^{-1}(m_Y\text{-}int(Y - K)) \subset int_q(f^{-1}(Y - K)) = int_q(X - f^{-1}(K)) = X - cl_q\left(f^{-1}(K)\right)$. Therefore, we have $cl_q(f^{-1}(K)) \subset f^{-1}(K) \subset cl_q(f^{-1}(K))$. Thus we obtain $cl_q(f^{-1}(K)) = f^{-1}(K)$.

(f)⇒(a) Let $x \in X$ and $V \in m_Y$ containing $f(x)$. By (f) we have $X - f^{-1}(V) = f^{-1}(Y - V) = cl_q(f^{-1}(Y - V)) = cl_q(X - f^{-1}(V)) = X - int_q\left(f^{-1}(V)\right)$. Hence, we have $x \in f^{-1}(V) = int_q\left(f^{-1}(V)\right)$. Therefore, there exists $U \in \tau_q$ such that $x \in U \subset f^{-1}(V)$. Therefore, $x \in U \in \tau_q$ and $f(U) \subset V$. This shows that f is (τ_q, m)-*continuous.* □

Definition 11 ([28])**.** Let X be a nonempty set with a minimal structure m_X is said to be m-regular if for each m_X-closed set F and each $x \notin F$, there exist disjoint m_X-open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 6 ([28])**.** *Let X be a nonempty set with a minimal structure m_X and m_X satisfy property $\mathcal{B}$. Then (X, m_X) is said to be m-regular if and only if for each $x \in X$ and each m_X-open set U containing x, there exists an m_X-open set V such that $x \in V \subset m_X$-$Cl(V) \subset U$.*

Theorem 2. *Let (Y, m_Y) be m-regular and satisfying property $(\mathcal{B})$. Then for a function $f : (X, \tau_q) \to (Y, m_Y)$ the following properties are equivalent:*

(*a*) *f is $(\tau_q, m).c.$*

(*b*) *$f^{-1}(mCl_\theta(B)) \subset cl_q(f^{-1}(mCl_\theta(B)))$ for every subset B of Y.*

(*c*) *f is weakly (τ_q, m)-continuous.*

(*d*) *$f^{-1}(F) = cl_q(f^{-1}(F))$ for every m_θ-closed set F of Y.*

(*e*) *$f^{-1}(V) = int_q(f^{-1}(V))$ for every m_θ-open set V of Y.*

Proof. Consider a function $f : (X, m_X) \to (Y, m_Y)$, where X and Y are nonempty sets with minimal structures m_X and m_Y , respectively, and let (Y, m_Y) be m-regular and satisfying the property $(\mathcal{B})$. Then put $m_X = \tau_q$ in Theorem 4.2 of [28]. □

Theorem 3. *Let (Y, m_Y) satisfy the property $(\mathcal{B})$. For a function $f : (X, \tau_q) \to (Y, m_Y)$, the following are equivalent:*

(*a*) *f is w.$(\tau_q, m).c.$*

(*b*) *For each $x \in X$ and each m_Y-open set V of Y containing $f(x)$, there exists a quasi-open set U of X containing x such that $f(U) \subset m_Y$-$Cl(V)$.*

(*c*) *$f^{-1}(V) \subset int_q(f^{-1}(m_Y\text{-}Cl(V)))$ every m_Y-open set V of Y.*

(*d*) *$cl_q(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(B))) \subset f^{-1}(m_Y\text{-}Cl(B))$ for every subset B of Y.*

(*e*) *$cl_q(f^{-1}(m_Y\text{-}Int(F))) \subset f^{-1}(F)$ every m_Y-closed set F of Y.*

(*f*) $cl_q(f^{-1}(V)) \subset f^{-1}(m_Y\text{-}Cl(V))$ *every* m_Y*-open set* V *of* Y.

(*g*) $f(cl_q(A)) \subset mCl_\theta(f(A))$ *for each subset* A *of* X.

(*h*) $cl_q(f^{-1}(B)) \subset f^{-1}(mCl_\theta(B))$ *for each subset* B *of* Y.

Proof. (a)⇔(b): These implications are clear from the definition of quasi topology.

(b)⇒(c): Let V be a m_Y-open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and by (b), there exists a quasi-open set U of X containing x such that $f(U) \subset m_Y\text{-}Cl(V)$. Then $x \in U \subset f^{-1}(m_Y\text{-}Cl(V))$ and hence $x \in int_q(f^{-1}(m_Y\text{-}Cl(V)))$.

(a)⇔(c): It follows from Theorem 3.2 of [28].

(a)⇒(d)⇒(e)⇒(a): It follows from Theorem 2.1 of [25].

(f)⇔(a): It follows from Theorem 3.4 of [28].

(a)⇒(g)⇒(h)⇒(a): It follows from Theorem 3.3 of [28].

The above proofs do not use the property $(\mathcal{B})$, except for the case (f)⇔(a). □

Definition 12. Let Y be a nonempty set and m_Y a minimal structure on Y. A function $f : X \to Y$, is said to be (τ_q, m^*)-continuous if for each $V \in m_Y$, $f^{-1}(V)$ is clopen in X.

Remark 6. Let Y be a nonempty set and m_Y a minimal structure on Y. We put $m_Y = \tau$ (resp. $RO(Y), SR(Y)$). Then, a (τ_q, m^*)-continuous function $f : (X, \tau_q) \to (Y, m_Y)$ is perfectly continuous ([23]) (resp. regular set connected ([10]), almost s-continuous)

Proposition 2. *Let* Y *be a nonempty set and* m_Y *a minimal structure on* Y*. Then, the following are quivalent:*

(*a*) $f : X \to Y$ *is* (τ_q, m^*)*-continuous.*

(*b*) *For each* $Y - F \in m_Y$, $X - f^{-1}(F)$ *is clopen in* X.

Proof. (a)⇒(b) Let $Y - F \in m_Y$, since $f : X \to Y$ *is* (τ_q, m^*)-*continuous* $f^{-1}(Y - F) = X - f^{-1}(F)$ is clopen in X and hence $f^{-1}(F)$ is clopen in X

(b)⇒(a) Let $U \in m_Y$ then $Y - (Y - U) \in m_Y$ and by (b), $f^{-1}(Y - (Y - U)) = X - f^{-1}(Y - U) = X - \big(X - f^{-1}(U)\big) = f^{-1}(U)$ *is clopen in* X, *hence* $f : X \to Y$ *is* (τ_q, m^*)*-continuous.* □

Note that *property* (mR) requires *property* $(\mathcal{B})$.

Theorem 4. *Let Y be a nonempty set and m_Y a minimal structure on Y for which satisfying the* property (mR). *Then the following properties are equivalent for a function $f : (X, \tau_q) \to (Y, m_Y)$:*

(a) *f is weakly (τ, m)-continuous.*

(b) *For each $x \in X$ and each $V \in mR(Y, f(x))$, there exists an open set U containing x such that $f(U) \subset V$.*

(c) *For each $x \in X$ and each $V \in mR(Y, f(x))$, there exists an α-open set U containing x such that $f(U) \subset V$.*

(d) *$f^{-1}(V)$ is α-open in X for every $V \in mR(Y)$.*

(e) *$f^{-1}(V)$ is clopen in X for every $V \in mR(Y)$.*

(f) *f is w.(τ_q, m).c.*

(g) *For each $x \in X$ and each $V \in mR(Y, f(x))$, there exists an clopen set U containing x such that $f(U) \subset V$.*

(h) *For each $x \in X$ and each $V \in mR(Y, f(x))$, there exists a quasi-open set U of X containing x such that $f(U) \subset V$.*

(i) *$f : (X, \tau_q) \to (Y, m_Y)$ is weakly M-continuous.*

Proof. (a)⇒(b): Let $x \in X$ and $V \in mR(Y, f(x))$. There exists an open set U containing x such that $f(U) \subset m_Y\text{-}Cl(V) = V$.

(b)⇒(c): This is clear.

(c)⇒(d): Let $V \in mR(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V \in m_Y$. There exists an α-open set U_x containing x such that $f(U_x) \subset m_Y\text{-}Cl(V) = V$. Therefore, $x \in U_x \subset f^{-1}(V)$ and hence $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$ is α-open in X.

(d)⇒(e): Let $V \in mR(Y)$ Since $Y - V \in mR(Y)$ by (d) $X - f^{-1}(V) = f^{-1}(Y - V)$ is α-open. Therefore $f^{-1}(V)$ α-closed and α-open in X. Hence by Lemma 3.1 of [14], $f^{-1}(V)$ is clopen. Note that (e) can be rephrased as, $f : (X, \tau_q) \to (Y, m^1_Y)$ is (τ_q, m^1)-continuous or M^*-continuous, where $m^1_Y = mR(Y)$.

(e)⇒(f): Let $x \in X$ and V be any m_Y-open set of Y containing $f(x)$. By Lemma 4, $m_Y\text{-}Cl(V)$ is m_Y-clopen and hence $f^{-1}(m_Y\text{-}Cl(V))$ is clopen in X. Put $U = f^{-1}(m_Y\text{-}Cl(V))$, then U is clopen set containing x and $f(U) \subset m_Y\text{-}Cl(V)$.

(f)⇒(g): Let $x \in X$ and $V \in mR(Y, f(x))$. There exists a clopen set U containing x such that $f(U) \subset m_Y\text{-}Cl(V) = V$.

(g)⇒(h): It follows from the definition of quasi topology.

(h)⇒(i): Let $x \in X$ and $V \in mR(Y, f(x))$.Then by (h), there exists a quasi-open set U containing x such that $f(U) \subset V$. Since every m_Y-regular set is m_Y-open, f is (τ_q, m)-*continuous.Then* $f : (X, \tau_q) \to (Y, m_Y)$ is M-continuous, hence weakly M-continuous.

(i)⇒(a): This is clear. □

Remark 7. Let Y be a nonempty set and m_Y a minimal structure on Y if $m_Y \in \{SO(X), BO(X), \beta O(X)\}$ then m_Y has property (mR), so a *weakly* (τ_q, m)-continuous function $f : (X, \tau_q) \to (Y, m_Y)$ is a generalization and unification of almost s-continuity (resp. (θ, b)-continuity, weakly (τ, β)-continuity).

Remark 8. We have the following implications for a function $f : X \to Y$:

(τ, m)-continuous ⇒	almost (τ, m)-continuous ⇒	weakly (τ, m)-continuous
⇑	⇑	⇑
(τ_q, m)-continuous ⇒	almost (τ_q, m)-continuous ⇒	weakly (τ_q, m)-continuous
⇑		
(τ_q, m^*)-continuous		

Note that these implications cannot be reversed in general as the following examples shows:

Example 1. Let X be the real numbers with the upper limit topology τ and Y be the real numbers with the usual topology σ. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is (τ_q, σ)-continuous function but not (τ_q, σ^*)-continuous since $f^{-1}((0, 1))$ is not clopen in (X, τ).

Example 2 ([11])**.** Let X be the real numbers with the usual topology τ and $f : X \to X$ be the identity function. Then f is an almost (τ, τ)-continuous function which is not almost (τ_q, τ)-continuous.

Example 3 ([11])**.** Let $\mathbb{R}$ and $\mathbb{Q}$ be the real and rational numbers, respectively. Let $A = \{x \in \mathbb{R} : x \text{ is rational and } 0 < x < 1\}$. We define two topologies on $\mathbb{R}$ as $\tau = \{\mathbb{R}, \emptyset, A, \mathbb{R} - A\}$ and $\nu = \{\mathbb{R}, \emptyset, \{0\}\}$. Let $f : (\mathbb{R}, \tau) \to$

$(\mathbb{R}, \nu)$ be a function which is defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is almost (τ_q, ν)-continuous and weakly (τ_q, ν)-continuous, but f is not (τ_q, ν)-continuous since for $f(x) = 0 \in \{0\} \in \nu(x \notin \mathbb{Q})$, there is no clopen set U containing x such that $f(U) \subset \{0\}$.

Example 4 ([36])**.** Let $X = \{a, b, c, d\}$,$\tau = \{X, \emptyset, \{d\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is weakly (τ_q, σ)-continuous but not almost (τ, σ)-continuous (hence not almost (τ_q, σ)-continuous) since there exists a regular open set $\{a, c\}$ of (X, σ) such that $f^{-1}(\{a, c\})$ is not clopen in (X, τ).

Example 5 ([36])**.** Let X be the real numbers and τ be the usual topology on X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is almost (τ, τ)-continuous (hence weakly (τ, τ)-continuous) but not weakly (τ_q, τ)-continuous since the only clopen set of X is itself.

Definition 13. A filter base $\mathcal{F}$ is said to be:

(a) m_X-θ-convergent ([21]) to a point x in X, if for any m_X–open set U containing x there exist $B \in F$ such that $B \subset m_X$-$Cl(U)$.

(b) clopen convergent ([12]) to a point x in X, if for any clopen set U containing x, there exist $B \in F$ such that $B \subset U$.

Definition 14. A net (x_λ) in a space X, θ-converges ([8]) (resp. clopen converges ([15]), m_X-θ-converges ([21])) to x if and only if for each open (resp. clopen, m_X-open) set U containing x, there exists a λ_0 such that $x_\lambda \in cl(U)$ (resp. $x_\lambda \in U$, $x_\lambda \in m_X$-$Cl(U)$) for all $\lambda \geqq \lambda_0$.

Lemma 7. *For a net (x_λ) in a space X:*

(a) if (x_λ) converges to x, then (x_λ) θ-converges to x ([6]).

(b) if (x_λ) converges or θ-converges to x, then (x_λ) clopen converges to x ([15]).

Theorem 5. *A function $f : X \to Y$ is weakly (τ_q, m)-continuous if and only if for each point $x \in X$ and each filter base $\mathcal{F}$ in X that clopen converging to x the filter base $f(\mathcal{F})$ is m_Y-θ-convergent to $f(x)$.*

Proof. Suppose that $x \in X$ and $\mathcal{F}$ is any filter base in X that clopen converges to x. By hypothesis for any m_Y-open set V containing $f(x)$ there exists a clopen set U containing x in X such that $f(U) \subset m_Y\text{-}Cl(V)$. Since $\mathcal{F}$ is clopen convergent to x in X then there exists $B \in \mathcal{F}$ such that $B \subset U$. It follows that $f(B) \subset m_Y\text{-}Cl(V)$. This means that $f(\mathcal{F})$ is m_Y-θ-convergent to $f(x)$.

Conversely, let x be a point in X and V be a m_Y-open set containing $f(x)$. If we set $\mathcal{F} = \{U : U$ is clopen and $x \in U\}$, then $\mathcal{F}$ will be a filter base which clopen converges to x. So there exists $U \in \mathcal{F}$ such that $f(U) \subset m_Y\text{-}Cl(V)$. This completes the proof. □

Theorem 6. *The implications* $(a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d) \Leftrightarrow (e)$ *hold for the following properties of a function* $f : (X, \tau_q) \to (Y, m_Y)$ *:*

(a) *f is w.*(τ_q, m)*.c.*

(b) *For each $x \in X$ and each net (x_λ) in X which clopen converges to x, the net $(f(x_\lambda))$ m_Y-θ-converges to $f(x)$.*

(c) *For each $x \in X$ and each net (x_λ) in X which θ-converges to x, the net $(f(x_\lambda))$ m_Y-θ-converges to $f(x)$.*

(d) *For each $x \in X$ and each net (x_λ) in X which converges to x, the net $(f(x_\lambda))$ m_Y-θ-converges to $f(x)$.*

(e) *f is weakly (τ, m)-continuous.*

Proof. $(a) \Rightarrow (b)$: Let $x \in X$ and let (x_λ) be a net in X such that (x_λ) clopen converges to x. Let V be a m_Y-open set containing $f(x)$. Since f is weakly (τ_q, m)-continuous, there exists a clopen set U containing x such that $f(U) \subset m_Y\text{-}Cl(V)$. Since (x_λ) clopen converges to x, there exists λ_0 such that $x_\lambda \in U$ for all $\lambda \geqq \lambda_0$. Hence $f(x_\lambda) \in m_Y\text{-}Cl(V)$ for all $\lambda \geqq \lambda_0$.

$(b) \Rightarrow (a)$: Suppose that f is not weakly (τ_q, m)-continuous. Then there exists $x \in X$ and a m_Y open set V containing $f(x)$ such that $f(U) \not\subseteq m_Y\text{-}Cl(V)$ for all clopen neighborhoods U of x. Thus, for every clopen neighborhood U of x we can find $x_U \in U$ such that $f(x_U) \notin m_Y\text{-}Cl(V)$. Let $\mathcal{N}(x)$ be the set of clopen neighborhoods of x in X. The set $\mathcal{N}(x)$ with the relation of inverse inclusion (that is $U_1 \leq U_2$ if and only if $U_2 \subseteq U_1$) forms a directed set (Theorem 1.1 of [12]). Clearly the net $\{x_U : U \in \mathcal{N}(x)\}$

clopen converges to x in X but $(f(x_U))_{U\in\mathcal{N}(x)}$ does not m_Y-θ–converge to $f(x)$.

$(b) \Rightarrow (c)$: Let $x \in X$ and let (x_λ) be a net in X such that (x_λ) θ-converges to x. By Lemma 7, (x_λ) clopen converges to x. By (b), $(f(x_\lambda))$ m_Y-θ-converges to $f(x)$.

$(c) \Rightarrow (d)$: Let $x \in X$ and let (x_λ) be a net in X such that (x_λ) converges to x. By Lemma 7, (x_λ) θ-converges to x. By (c), $(f(x_\lambda))$ m_Y-θ-converges to $f(x)$.

$(d) \Rightarrow (e)$: Suppose that f is not weakly (τ, m)-continuous. Then there exists $x \in X$ and a m_Y-open set V containing $f(x)$ such that $f(U) \nsubseteq m_Y$-$Cl(V)$ for all open U containing x. Consider the set $\{x_U : U$ is open set containing $x\}$. Then (x_U) converges to x but $(f(x_U))$ does not m_Y-θ-converges to $f(x)$.

$(e) \Rightarrow (d)$: This can be proved similar to $(a) \Rightarrow (b)$.

If m_Y satisfies *property* (mR), $(e) \Rightarrow (a)$ is true by Theorem 4. □

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 15. A function $f : (X, m_X) \to (Y, m_Y)$ is said to have a strongly M-closed graph ([29]) if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists an m_X-open set U containing x and an m_Y-open set V containing y such that $(U \times m_Y\text{-}Cl(V)) \cap G(f) = \emptyset$.

Lemma 8 ([29]). *A function $f : (X, m_X) \to (Y, m_Y)$ has a strongly M-closed graph $G(f)$ if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists an m_X-open set U containing x and m_Y-open set V containing y such that $f(U) \cap m_Y\text{-}Cl(V) = \emptyset$.*

Definition 16. A graph $G(f)$ of a function $f : (X, \tau_q) \to (Y, m_Y)$ is said to be strongly *clp*-*m*-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exists a clopen set U in X containing x and m_Y-open set V containing y such that $(U \times m_Y\text{-}Cl(V)) \cap G(f) = \emptyset$.

Remark 9. If a function $f : (X, m_X) \to (Y, m_Y)$ has the strongly M-closed graph, then for the special case $m_X = \tau_q$, $G(f)$ has strongly *clp*-*m*-closed graph.

Note that the concepts of strongly M-closed graph and strongly *clp*-*m*-closed graph are generalizations of the the following notions.

Definition 17. A function $f:(X,\tau)\to(Y,\sigma)$ has a strongly-closed ([13]) (strongly *clp*-closed =strongly *clopen* ([36])) graph if for each $(x,y)\notin G(f)$, there exists open sets $U\in\tau$ $(U\in\tau_q)$ and $V\in\sigma$ containing x and y, respectively, such that $(U\times cl(V))\cap G(f)=\emptyset$.

Note that for a graph $G(f)$ strongly *clp*-closednes imply strongly closedness, but the reverse implication is not true in general as the following example shows.

Example 6 ([13])**.** Let $X=[0,1]$ have the usual topology $\tau_{\mathbb{R}^1|_X}$ $(=m_X)$ and let $Y=[0,1]$ have the topology σ $(=m_Y)$ generated by the usual open sets together with the set $A=\{r:r\in\mathbb{Q}\text{ ve }\frac{1}{4}<r<\frac{3}{4}\}$ as subbase. The identity function $i:\left(X,\tau_{\mathbb{R}^1|_X}\right)\to(Y,\sigma)$, has a strongly-closed graph $G(i)$. But $G(i)$ is not strongly *clp*-closed.

Theorem 7. *The following properties are equivalent for a graph $G(f)$ of a function:*

(*a*) *$G(f)$ is strongly clp-m-closed.*

(*b*) *For each point $(x,y)\in(X\times Y)-G(f)$, there exists a clopen set U containing x in X and m_Y open set V containing y such that $f(U)\cap m_Y\text{-}Cl(V)=\emptyset$.*

(*c*) *For each point $(x,y)\in(X\times Y)-G(f)$, there exists a quasi-open set U containing x in X and m_Y-open set in Y containing y such that. $f(U)\cap m_Y\text{-}Cl(V)=\emptyset$.*

Proof. (a)⇒(b) It follows from Lemma 8.

(b)⇒(c) It is clear since every clopen set is quasi-open.

(c)⇒(a) If (c) holds, then the set U in the statement of (c) is quasi open. Then, there exists a clopen set W such that $W\subset U$ and we have $f(W)\cap m_Y\text{-}Cl(V)\subset f(U)\cap m_Y\text{-}Cl(V)=\emptyset$. By Lemma 8 result follows. □

Definition 18. A nonempty set X with a minimal structure m_X, (X,m_X), is said to be m-T_2 ([30]) (resp. m-Urysohn ([28])) if for each distinct points x, $y\in X$, there exist U, $V\in m_X$ containing x and y, respectively, such that $U\cap V=\emptyset$ (resp. $m_X\text{-}Cl(U)\cap m_X\text{-}Cl(V)=\emptyset$).

See ([34]) for a study on minimal structures and separation properties.

Theorem 8. *If $f : X \to Y$ is (τ_q, m).c. function and Y is m-T_2, then $G(f)$ is strongly clp-m-closed in $X \times Y$.*

Proof. If the condition holds and $(x, y) \in (X \times Y) - G(f)$ then, it is true that $f(x) \neq y$ and there exists V, $W \in m_Y$ containing y and $f(x)$, respectively, such that $V \cap W = \emptyset$. Then, by Lemma 2, m_X -$Cl(V) \cap W = \emptyset$. Now, as f is (τ_q, m).c. there exists a clopen set U in X containing x such that $f(U) \subset W$. Therefore $f(U) \cap m_Y$-$Cl(V) = \emptyset$ and $G(f)$ is strongly *clp*-*m*-closed. □

Theorem 9. *If $f : X \to Y$ is w.(τ_q, m).c. function and Y is m-Urysohn , then $G(f)$ is strongly clp-m-closed in $X \times Y$.*

Proof. If the condition holds and $(x, y) \in (X \times Y) - G(f)$ then, it is true that $f(x) \neq y$ and there exists V, $W \in m_Y$ containing y and $f(x)$, respectively, such that m_Y-$Cl(V) \cap m_Y$-$Cl(W) = \emptyset$. Now, as f is w.(τ_q, m).c. there exists a clopen set U in X containing x such that $f(U) \subset m_Y$-$Cl(W)$. Therefore $f(U) \cap m_Y$-$Cl(V) = \emptyset$ and $G(f)$ is strongly *clp*-*m*-closed. □

Theorem 10 ([39])**.** *If $f : (X, \tau_q) \to (Y, m_Y)$ is a w.(τ_q, m).c. and (Y, m_Y) is m-T_2, then f has quasi-closed point inverses in X.*

Theorem 11. *If $f, g : X \to Y$ is w.(τ_q, m).c. function and Y is m-Urysohn , then $A = \{x \in X : f(x) = g(x)\}$ is quasi-closed in X.*

Proof. If $x \in X - A$, then it follows that $f(x) \neq g(x)$. Since Y is m-Urysohn, there exists m_Y-open set U in Y containing $f(x)$ and m_Y-open set V in Y containing $g(x)$ such that m_Y-$Cl(U) \cap m_Y$-$Cl(V) = \emptyset$. Since f and g are w.(τ_q, m).c. there exists clopen sets G and H with $x \in G$ and $x \in H$ such that $f(G) \subset m_Y$-$Cl(U)$ and $g(H) \subset m_Y$-$Cl(W)$, set $O = G \cap H$. Then O is clopen, $f(O) \cap g(O) = \emptyset$ and $A \cap O = \emptyset$. Thus every point of $X - A$ has a clopen neighborhood disjoint from A. Hence $X - A$ is a union of clopen sets or equivalently A is quasi-closed. □

Theorem 12. *If $f : (X, \tau_q) \to (Y, m_Y)$ is w.(τ_q, m).c. function and Y is m-Urysohn, then $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is quasi-closed in $X \times X$.*

Proof. Let $(x, y) \in (X \times X) - A$, then it follows that $f(x) \neq f(y)$. Since Y is m-Urysohn, there exist m_Y-open set U containing $f(x)$ and m_Y-open set V containing $f(y)$ such that m_Y-$Cl(U) \cap m_Y$-$Cl(V) = \emptyset$. Since f is w.(τ_q, m).c., there exists clopen sets W and O with $x \in O$ and $y \in W$

such that $f(O) \subset m_Y\text{-}Cl(U)$ and $f(W) \subset m_Y\text{-}Cl(V)$. Then, we have $(x, y) \in O \times W \subset f^{-1}(m_Y\text{-}Cl(U)) \times f^{-1}(m_Y\text{-}Cl(V))$. Thus we have $O\times W$ is a clopen set containing (x, y) and $O\times W\subset(X\times X)-A$. Hence $(X\times X)-A$ is union of clopen sets or equivalently A is quasi-closed in $X\times X$. □

Definition 19. A space X is said to be *ultra Hausdorff* ([36]) if every two distinct points of X can be separated by disjoint clopen sets. Note that if a space X is ultra Hausdorff then it is totally disconnected.

Theorem 13. *Let $f : (X, \tau_q) \to (Y, m_Y)$ have a strongly clp-m-closed graph. Then the following properties hold:*

(*a*) *If f is injective then X is ultra Hausdorff.*

(*b*) *If f is surjective then Y is m-T_2.*

Proof. (a) Suppose that x and y are any two distinct points of X by the injectivity of f, $(x, f(y)) \notin G(f)$. Since $G(f)$ is *strongly clp-m-closed*, by Theorem 7, there exist a clopen set U containing x and m_Y-open set V containing $f(y)$ such that $f(U) \cap m_Y\text{-}Cl(V) = \emptyset$. We have $U \cap f^{-1}(m_Y\text{-}Cl(V)) = \emptyset$. Therefore $y \notin U$. Then U and $X - U$ are disjoint clopen sets containing x and y, respectively. Hence X is ultra Hausdorff.

(b) Let y_1 and y_2 be any two distinct points of Y. Since f is surjective there exists a point $x \in X$ such that $f(x) = y_2$. Since $G(f)$ is *strongly clp-m-closed* and $(x, y_1) \notin G(f)$ there exists a clopen set U containing x and m_Y-open set V in Y containing y_1 such that $f(U) \cap m_Y\text{-}Cl(V) = \emptyset$. Therefore we have $y_2 \in f(U) \subset Y - (m_Y\text{-}Cl(V))$ and hence by Lemma 2, Y is m-T_2. □

Definition 20. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be M-closed ([26]) if for each m_X-closed set F, $f(F)$ is m_Y-closed in Y.

Definition 21. A space X is said to be:

(a) *ultraregular* ([37]) if for each closed set F and each $x \notin F$, there exist disjoint clopen sets U and V such that $x \in U$ and $F \subset V$.

(b) *ultranormal* ([37]) if disjoint closed sets contained in disjoint clopen sets.

Note that if a space X is ultraregular then it has a basis consisting of clopen sets.

Theorem 14. *Let Y be a nonempty set and m_Y a minimal structure on Y for which satisfying the property (mR). If $f : X \to Y$, is a $w.(\tau_q, m).c.$ and M-closed injection and Y is m-regular, then X is ultraregular.*

Proof. Let F be any quasi-closed set of X and $x \in X - F$. Since f is M-closed, $f(F)$ is m_Y-closed and $f(x) \in Y - f(F)$. Since (Y, m_Y) is m-regular, there exist disjoint m_Y-open sets U and V such that $f(x) \in U$ and $f(F) \subset V$. Since $U \cap V = \emptyset$ by Lemma 2, we have m_Y-$Cl(V) \cap U = \emptyset$. Since m_Y has property (mR), m_Y-$Cl(V)$ is a m-regular set containing $f(F)$ and disjoint from $f(x)$. Since f is w.(τ_q, m).c. by Theorem 4, the inverse image of m_Y-$Cl(V)$, under f is a clopen subset of X containing F and disjoint from x. This shows X is ultraregular. □

Definition 22. An m-space (X, m_X) is said to be m-normal ([26]) if for each pair of disjoint m-closed sets F_1, F_2 of X, there exist $U_1, U_2 \in m_X$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Theorem 15. *Let Y be a nonempty set and m_Y a minimal structure on Y for which satisfying the property (mR). If $f : X \to Y$, is a w.(τ_q, m).c. and M-closed injection and Y is m-normal, then X is ultranormal.*

Proof. Let A and B be disjoint closed sets of X. Since f is M-closed injection, $f(A)$ and $f(B)$ are disjoint m_Y-closed sets of Y. By the m-normality of (Y, m_Y), there exist disjoint m_Y-open sets U and V such that $f(A) \subset U$ and $f(B) \subset V$. Since $U \cap V = \emptyset$ by Lemma 2 , we have m_Y-$Cl(U) \cap V = \emptyset$. Since m_Y has property (mR), m_Y-$Cl(U)$ is a m-regular set containing $f(A)$ and disjoint from $f(B)$. Since f is w.(τ_q, m).c. by Theorem 4, the inverse image of m_Y-$Cl(U)$ under f is a clopen subset of X containing A and disjoint from B. Thus X is ultranormal. □

Definition 23. A subset K of a space X is said to be mildly compact ([37]), relative to X if for every cover $\{V_\alpha : \alpha \in I\}$ of K by clopen sets of X, there exists a finite subset I_0 of I such that $K \subset \cup\{V_\alpha : \alpha \in I_0\}$.

Definition 24. A subset K of a nonempty set X with a minimal structure m_X is said to be m-compact ([30]) (m-closed ([21])) relative to (X, m_X) if any cover $\{U_i : i \in I\}$ of K by m_X-open sets, there exists a finite subset I_0 of I such that $K \subseteq \cup\{U_i : i \in I_0\}$ $(K \subseteq \cup\{m_X\text{-}Cl(U_i) : i \in I_0\})$.

It is clear that (X, m_X) is m-closed if X is m-closed relative to (X, m_X). Let (X, τ) be a topological space. Note that, if $m_X = \tau$ (resp. $SO(X)$) the definition of m-closed sets gives the definitions of quasi H-closed ([31]) (resp. of s-closed ([9])) sets.

Theorem 16. *Let $f : (X, \tau_q) \to (Y, m_Y)$ be a w.(τ_q, m).c. surjection. If X is mildly compact, then Y is m-closed.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a cover of Y by m_Y-open sets of Y. For each point $x \in X$, there exists $\alpha(x) \in I$ such that $f(x) \in V_{\alpha(x)}$. Since f is w.(τ_q, m).c., there exists a clopen set U_x of X containing x such that $f(U_x) \subset m_X\text{-}Cl(V_{\alpha(x)})$. The family $\{U_x : x \in X\}$is a cover of X by clopen sets of X and hence there exists a finite subset X_0 of X such that $X \subset \bigcup_{x\in X_0} U_x$. Therefore, we obtain $Y = f(X) \subset \bigcup_{x\in X_0} m_X\text{-}Cl(V_{\alpha(x)})$. This shows that Y is m-closed. □

Theorem 17 ([21]). *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Assume that m_X is a base for a topology. If the graph $G(f)$ is strongly M-closed, then $m_X\text{-}Cl(f^{-1}(K)) = f^{-1}(K)$ whenever the set $K \subseteq Y$ is m-closed relative to (Y, m_Y).*

Corollary 1 ([39]). *If a function $f : (X, \tau_q) \to (Y, m_Y)$ has a strongly clp-m-closed graph, then $f^{-1}(K)$ is quasi-closed in (X, τ_q) for each set K which is m-closed relative to (Y, m_Y).*

Theorem 18. *If a function $f : (X, \tau_q) \to (Y, m_Y)$ has a strongly clp-m-closed graph and Y is m-closed, m_Y has property (mR) then f is w.(τ_q, m).c.*

Proof. Let V be a m_Y-open set then by Lemma 4, $m_Y\text{-}Cl(V) \in mR(Y)$ and $Y - (m_Y\text{-}Cl(V)) \in mR(Y)$. By the m-closedness of Y , $Y - (m_Y\text{-}Cl(V))$ is m-closed. By Corollary 1, $f^{-1}(Y - (m_Y\text{-}Cl(V))) = X - f^{-1}(m_Y\text{-}Cl(V))$ is quasi-closed, hence $f^{-1}(m_Y\text{-}Cl(V))$ is quasi open. Then $f^{-1}(V) \subset int_q(f^{-1}(m_Y\text{-}Cl(V)))$ and by Theorem 3, f is w.(τ_q, m).c. □

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Received: 14.X.2011
Revised: 6.IX.2012
Revised: 26.XI.2012
Accepted: 11.II.2013

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