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BACKWARD STOCHASTIC VARIATIONAL INEQUALITIES WITH LOCALLY BOUNDED GENERATORS*

BY

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Abstract. The paper deals with the existence and uniqueness of the solution of the backward stochastic variational inequality:

$$\begin{cases} -dY_t + \partial\varphi\left(Y_t\right)dt \ni F\left(t, Y_t, Z_t\right)dt - Z_t dB_t,\ 0 \le t < T \\ Y_T = \eta, \end{cases}$$

where F satisfies a local boundedness condition.

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1. Introduction

We consider the following backward stochastic variational inequality (BSVI):

$$\begin{cases} -dY_t + \partial\varphi\left(Y_t\right)dt \ni F\left(t, Y_t, Z_t\right)dt - Z_t dB_t,\ 0 \le t < T \\ Y_T = \eta \end{cases}, \tag{1}$$

where $\{B_t : t \ge 0\}$ is a standard Brownian motion, $\partial\varphi$ is the subdifferential of a convex l.s.c. function φ, and $T > 0$ is a fixed deterministic time.

The study of the backward stochastic differential equations (BSDE) (equation of type (1) without the subdifferential operator) was initiated

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by Pardoux and Peng in [11] (see also [12]) where is proved the existence and the uniqueness of the solution for the BSDE under the assumption of Lipschitz continuity of F with respect to y and z and square integrability of η and $F(t, 0, 0)$.

The more general case of scalar BSDE with one-sided reflection and associated optimal control problems was considered by El Karoui et al. [8] and with two-sided reflection associated with stochastic game problem by Cvitanic, Karatzas [6] (see also [3] and [7] for the investigation of zero-sum two-player stochastic differential games whose cost functionals are given by controlled reflected BSDE).

On the other hand, it is worth to mention the backward in time problems in mechanics of continua, since a large number of physical phenomena leads to these new non-standard problems. Specify that for improperly posed problems the solutions will not exist for arbitrary data and not depend continuously on the data (see e.g. [5], [4] and references therein).

The standard work on BSVI is that of Pardoux and Răşcanu [13], which give a proof of existence and uniqueness of the solution for (1) under the following assumptions on F: monotonicity with respect to y (in the sense that $\langle y'-y, F(t,y',z)-F(t,y,z)\rangle \le \alpha|y'-y|^2$), Lipschitzianity with respect to z and a sublinear growth for $F(t,y,0)$, $|F(t,y,0)| \le \beta_t + L|y|$, $\forall (t,y) \in [0,T]\times\mathbb{R}^m$. It is proved that there exists a unique triple (Y,Z,K) such that

$$Y_t+K_T-K_t=\eta+\int_t^T F(s,Y_s,Z_s)\,ds-\int_t^T Z_s dB_s, \text{ a.s., with } dK_t\in\partial\varphi(Y_t)\,dt.$$

Moreover the process K is absolute continuous with respect to dt. In [14] the same authors extend the results from [13] to the Hilbert spaces framework. Using a mixed Euler-Yosida scheme, Maticiuc, Rotenstein provided in [9] numerical results concerning the multi-valued stochastic differential equation (1).

Our paper generalize the previous existence and uniqueness results for (1) by assuming a local boundedness condition (instead of sublinear growth of F), i.e.

$$\mathbb{E}\Big(\int_0^T F_\rho^{\#}(s)ds\Big)^p < \infty, \text{ where } F_\rho^{\#}(t) \stackrel{def}{=} \sup_{|y|\le\rho} |F(t,y,0)|.$$

Concerning to this requirement on F we remark that a similar one was considered by Pardoux in [10] for the study of BSDE. More precisely, his result is the following: if $\eta \in L^2(\Omega;\mathbb{R}^m)$, $F(t,0,0) \in L^2(\Omega \times [0,T];\mathbb{R}^m)$, F is monotone with respect to y, Lipschitz with respect to z and there exists a deterministic continuous increasing function ψ such that $\forall (t,y) \in [0,T] \times \mathbb{R}^m$, $|F(t,y,0)| \leq |F(t,0,0)| + \psi(|y|)$, $\mathbb{P}$-a.s, then there exist a unique solution for BSDE (1) with $\varphi \equiv 0$. This result was generalized by Briand et al. [2].

The article is organized as follows: in the next Section we prove some a priori estimates and the uniqueness result for the solution of BSVI (1). Section 3 is concerned on the existence result under two alternative assumptions (which allow to obtain the absolute continuity of the process K) and Section 4 establishes the general existence result. In the Appendix we presents, following [15], some results useful throughout the paper.

2. Preliminaries; a priori estimates and the uniqueness result

Let $\{B_t : t \geq 0\}$ be a k-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t : t \geq 0\}$ the natural filtration generated by $\{B_t : t \geq 0\}$ and augmented by $\mathcal{N}$, the set of $\mathbb{P}$- null events of $\mathcal{F}$, $\mathcal{F}_t = \sigma\{B_r : 0 \leq r \leq t\} \vee \mathcal{N}$. We suppose that the following assumptions holds:

(A$_1$) $\eta : \Omega \to \mathbb{R}^m$ is a $\mathcal{F}_T$-measurable random vector,

(A$_2$) $F : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ satisfies that, for all $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$, $(\omega,t) \longmapsto F(\cdot,\cdot,y,z) : \Omega \times [0,T] \to \mathbb{R}^m$ is progressively measurable stochastic process, and there exist $\mu : \Omega \times [0,T] \to \mathbb{R}$ and $\ell : \Omega \times [0,T] \to \mathbb{R}_+$ progressively measurable stochastic processes with $\int_0^T \left(|\mu_t| + \ell_t^2\right) dt < \infty$, such that, for all $t \in [0,T]$, $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times k}$, $\mathbb{P}$-a.s.:

$$
\begin{aligned}
&(C_y) \quad y \longmapsto F(t,y,z) : \mathbb{R}^m \to \mathbb{R}^m \text{ is continuous,} \\
&(M_y) \quad \langle y' - y, F(t,y',z) - F(t,y,z)\rangle \leq \mu_t |y' - y|^2, \\
&(L_z) \quad |F(t,y,z') - F(t,y,z)| \leq \ell_t |z' - z|, \\
&(B_y) \quad \int_0^T F_\rho^{\#}(s)\, ds < \infty,\ \forall\, \rho \geq 0, \\
&\qquad\qquad \text{where, for } \rho \geq 0,\ F_\rho^{\#}(t) \overset{def}{=} \sup_{|y|\leq\rho} |F(t,y,0)|,
\end{aligned}
$$

(A$_3$) $\varphi : \mathbb{R}^m \to (-\infty, +\infty]$ is a proper, convex l.s.c. function.

The subdifferential of φ is given by $\partial\varphi(y) = \{\hat{y} \in \mathbb{R}^m : \langle \hat{y}, v - y\rangle + \varphi(y) \leq \varphi(v),\ \forall\ v \in \mathbb{R}^m\}$.

We define $\mathrm{Dom}(\varphi) = \{y \in \mathbb{R}^m : \varphi(y) < \infty\}$, $\mathrm{Dom}(\partial\varphi) = \{y \in \mathbb{R}^m : \partial\varphi(y) \neq \emptyset\} \subset \mathrm{Dom}(\varphi)$ and by $(y, \hat{y}) \in \partial\varphi$ we understand that $y \in \mathrm{Dom}\,(\partial\varphi)$ and $\hat{y} \in \partial\varphi\,(y)$.

Recall that $\overline{\mathrm{Dom}(\varphi)} = \overline{\mathrm{Dom}(\partial\varphi)}$, $\mathrm{Int}(\mathrm{Dom}(\varphi)) = \mathrm{Int}(\mathrm{Dom}(\partial\varphi))$. Let $\varepsilon > 0$ and the Moreau-Yosida regularization of φ :

$$\begin{aligned} \varphi_\varepsilon\,(y) &\overset{def}{=} \inf\left\{\frac{1}{2\varepsilon}\,|y - v|^2 + \varphi\,(v) : v \in \mathbb{R}^m\right\} \\ &= \frac{1}{2\varepsilon}\,|y - J_\varepsilon\,(y)|^2 + \varphi\,(J_\varepsilon\,(y))\,, \end{aligned} \tag{2}$$

where $J_\varepsilon\,(y) = (I_{m\times m} + \varepsilon\partial\varphi)^{-1}\,(y)$. Remark that φ_ε is a C^1 convex function and J_ε is a 1-Lipschitz function.

We mention some properties (see Brézis [1], and Pardoux, Răşcanu [13] for the last one): for all $x, y \in \mathbb{R}^m$

$$\begin{aligned} &(a) \quad \nabla\varphi_\varepsilon(y) = \partial\varphi_\varepsilon\,(y) = \frac{y - J_\varepsilon\,(y)}{\varepsilon} \in \partial\varphi(J_\varepsilon y), \\ &(b) \quad |\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)| \leq \frac{1}{\varepsilon}\,|x - y|\,, \\ &(c) \quad \langle\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y), x - y\rangle \geq 0, \\ &(d) \quad \langle\nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y), x - y\rangle \geq -(\varepsilon + \delta)\,\langle\nabla\varphi_\varepsilon(x), \nabla\varphi_\delta(y)\rangle\,. \end{aligned} \tag{3}$$

We denote by $\mathcal{S}^p_m[0, T]$ the space of (equivalent classes of) progressively measurable and continuous stochastic processes $X : \Omega \times [0, T] \to \mathbb{R}^m$ such that $\mathbb{E}\sup_{t\in[0,T]} |X_t|^p < \infty$, if $p > 0$, and by $\Lambda^p_m\,(0, T)$ the space of (equivalent classes of) progressively measurable stochastic process $X : \Omega \times [0, T] \to \mathbb{R}^m$ such that

$$\begin{aligned} &\int_0^T |X_t|^2\,dt < \infty,\ \mathbb{P}\text{-a.s. } \omega \in \Omega, \quad \text{if } p = 0, \\ &\mathbb{E}\left(\int_0^T |X_t|^2\,dt\right)^{p/2} < \infty, \qquad\quad \text{if } p > 0. \end{aligned}$$

For a function $g : [0, T] \to \mathbb{R}^m$, let us denote by $\updownarrow g \updownarrow_T$ the total variation of g on $[0, T]$ i.e.

$$\updownarrow g \updownarrow_T \overset{def}{=} \sup\left\{\sum_{i=0}^{n-1} |g\,(t_{i+1}) - g\,(t_i)| : n \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots t_n = T\right\},$$

and by $BV\left([0,T];\mathbb{R}^m\right)$ the space of the functions $g:[0,T]\to\mathbb{R}^m$ such that $\updownarrow g\updownarrow_T<\infty$ ($BV\left([0,T];\mathbb{R}^m\right)$ equipped with the norm $||g||_{BV([0,T];\mathbb{R}^m)}\overset{def}{=}|g(0)|+\updownarrow g\updownarrow_T$ is a Banach space).

Definition 1. A pair $(Y,Z)\in S_m^0[0,T]\times\Lambda_{m\times k}^0(0,T)$ of stochastic processes is a solution of backward stochastic variational inequality (1) if there exists $K\in S_m^0[0,T]$ with $K_0=0$, such that

$$\begin{array}{ll}(a) & \updownarrow K\updownarrow_T+\int_0^T|\varphi(Y_t)|\,dt+\int_0^T|F(t,Y_t,Z_t)|\,dt<\infty,\ \text{a.s.},\\ (b) & dK_t\in\partial\varphi(Y_t)\,dt,\ \text{a.s. that is: }\mathbb{P}\text{-a.s.},\\ & \int_t^s\langle y(r)-Y_r,dK_r\rangle+\int_t^s\varphi(Y_r)dr\le\int_t^s\varphi(y(r))dr,\\ & \qquad\forall y\in C([0,T];\mathbb{R}^d),\ \forall 0\le t\le s\le T,\end{array}$$

and, $\mathbb{P}$-a.s., for all $t\in[0,T]$:

$$Y_t+K_T-K_t=\eta+\int_t^T F(s,Y_s,Z_s)\,ds-\int_t^T Z_s dB_s \tag{4}$$

(we also say that triplet (Y,Z,K) is solution of equation (1)).

Remark 2. If K is absolute continuous with respect to dt, i.e. there exists a progressively measurable stochastic process U such that

$$\int_0^T|U_t|\,dt<\infty,\ \text{a.s. and } K_t=\int_0^t U_s ds,\ \text{for all } t\in[0,T],$$

then $dK_t\in\partial\varphi(Y_t)\,dt$ means $U_t\in\partial\varphi(Y_t)$, dt-a.e., a.s.

If $dK_t\in\partial\varphi(Y_t)\,dt$ and $d\tilde{K}_t\in\partial\varphi(\tilde{Y}_t)dt$ then we clearly have

$$\int_0^T|\varphi(Y_t)|\,dt+\int_0^T|\varphi(\tilde{Y}_t)|dt<\infty,\ \text{a.s.}$$

and, using the subdifferential inequalities

$$\begin{aligned}&\int_t^s\langle\tilde{Y}_r-Y_r,dK_r\rangle+\int_t^s\varphi(Y_r)dr\le\int_t^s\varphi(\tilde{Y}_r)dr,\\ &\int_t^s\langle Y_r-\tilde{Y}_r,d\tilde{K}_r\rangle+\int_t^s\varphi(\tilde{Y}_r)dr\le\int_t^s\varphi(Y_r)dr,\end{aligned}$$

we infer that, for all $0 \le t \le s \le T$

$$\int_t^s \langle Y_r - \tilde{Y}_r, dK_r - d\tilde{K}_r \rangle \ge 0, \text{ a.s.} \tag{5}$$

Let $a, p > 1$ and

$$V_t = V_t^{a,p} \overset{def}{=} \int_0^t \big(\mu_s + \frac{a}{2n_p}\ell_s^2\big) ds, \tag{6}$$

where $n_p = (p-1) \wedge 1$.

Denote

$$S_m^{1^+,p}[0,T] \overset{def}{=} \left\{ Y \in S_m^0[0,T] : \exists\ a > 1,\ \mathbb{E} \sup_{s\in[0,T]} |e^{V_s^{a,p}} Y_s|^p < \infty \right\}.$$

Remark that if μ_s and ℓ_s^2 are deterministic functions then, for all $p > 1$, $S_m^{1^+,p}[0,T] = S_m^p[0,T]$.

Proposition 3. *Let $(u_0, \hat{u}_0) \in \partial\varphi$ and assumptions $(A_1 - A_3)$ be satisfied. Then for every $a, p > 1$ there exists a constant $C_{a,p}$ such that for every (Y, Z) solution of BSDE (1) satisfying*

$$\mathbb{E} \sup_{s\in[0,T]} e^{pV_s} |Y_s - u_0|^p + \mathbb{E}\Big(\int_0^T e^{V_s} (|\hat{u}_0| + |F(s, u_0, 0)|) ds \Big)^p < \infty,$$

the following inequality holds $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t} \left[\sup_{s\in[t,T]} \left| e^{V_s}(Y_s - u_0) \right|^p + \left(\int_t^T e^{2V_s} |Z_s|^2 ds \right)^{p/2} \right] \\
&+ \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{2V_s} |\varphi(Y_s) - \varphi(u_0)| ds \right)^{p/2} \\
&+ \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{pV_s} |Y_s - u_0|^{p-2} \mathbf{1}_{Y_s \ne u_0} |Z_s|^2 ds \\
&+ \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{pV_s} |Y_s - u_0|^{p-2} \mathbf{1}_{Y_s \ne u_0} |\varphi(Y_s) - \varphi(u_0)| ds \\
&\le C_{a,p} \mathbb{E}^{\mathcal{F}_t} \Big[e^{pV_T} |\eta - u_0|^p \\
&+ \left(\int_t^T e^{V_s} |\hat{u}_0| ds \right)^p + \left(\int_t^T e^{V_s} |F(s, u_0, 0)| ds \right)^p \Big]
\end{aligned} \tag{7}$$

and, for every $R_0 > 0$ *and* $p \geq 2$,

$$
\begin{aligned}
& R_0^{p/2}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|F\left(s,Y_s,Z_s\right)\right|ds\right)^{p/2} \\
& +\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{pV_s}\left|Y_s-u_0\right|^{p-2}\mathbf{1}_{Y_s\neq u_0}\left|F\left(s,Y_s,Z_s\right)\right|ds \\
& \leq C_{a,p}\left[\mathbb{E}^{\mathcal{F}_t}e^{pV_T}\left|\eta-u_0\right|^p+R_0^{p/2}\right. \\
& \cdot\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\mathbf{1}_{p\geq 2}\left(F^{\#}_{u_0,R_0}\left(s\right)+R_0\gamma_s^+\right)ds\right)^{p/2} \\
& \left.+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{V_s}\left(F^{\#}_{u_0,R_0}\left(s\right)+2R_0\left|\gamma_s\right|\right)ds\right)^{p}\right],
\end{aligned}
\tag{8}
$$

where

$$F^{\#}_{u_0,R_0}\left(t\right)\overset{def}{=}\sup_{\left|y-u_0\right|\leq R_0}\left|F\left(t,y,0\right)\right|.$$

Proof. We can write

$$Y_t-u_0=\eta-u_0+\int_t^T\left[F\left(s,Y_s,Z_s\right)ds-dK_s\right]-\int_t^T Z_s dB_s.$$

Let $R_0 \geq 0$. The monotonicity property of F implies that, for all $|v| \leq 1$:

$$\left\langle F\left(t,u_0+R_0v,z\right)-F\left(t,y,z\right),u_0+R_0v-y\right\rangle\leq\mu_t\left|u_0+R_0v-y\right|^2,$$

and, consequently

$$
\begin{aligned}
& R_0\left\langle F\left(t,y,z\right),-v\right\rangle+\left\langle F\left(t,y,z\right),y-u_0\right\rangle \\
& \leq\mu_t\left|u_0+R_0v-y\right|^2+\left|F\left(t,u_0+R_0v,z\right)\right|\left|y-R_0v-u_0\right| \\
& \leq\mu_t\left|u_0+R_0v-y\right|^2+\left[F^{\#}_{u_0,R_0}\left(t\right)+\ell_t\left|z\right|\right]\left|y-R_0v-u_0\right| \\
& \leq\mu_t\left|u_0+R_0v-y\right|^2+F^{\#}_{u_0,R_0}\left(t\right)\left|y-R_0v-u_0\right| \\
& +\frac{a}{2n_p}\ell_t^2\left|y-R_0v-u_0\right|^2+\frac{n_p}{2a}\left|z\right|^2\leq F^{\#}_{u_0,R_0}\left(t\right)\left(\left|y-u_0\right|+R_0\right) \\
& +\gamma_t\left[\left|y-u_0\right|^2-2R_0\left\langle v,y-u_0\right\rangle+R_0^2\left|v\right|^2\right]+\frac{n_p}{2a}\left|z\right|^2\leq\left[R_0F^{\#}_{u_0,R_0}\left(t\right)+R_0^2\gamma_t^+\right] \\
& +\left[F^{\#}_{u_0,R_0}\left(t\right)+2R_0\left|\gamma_t\right|\right]\left|y-u_0\right|+\gamma_t\left|y-u_0\right|^2+\frac{n_p}{2a}\left|z\right|^2.
\end{aligned}
$$

Taking $\sup_{|v|\leq 1}$, we have

$$\begin{aligned}&R_0\left|F\left(t,Y_t,Z_t\right)\right|dt+\left\langle Y_t-u_0,F\left(t,Y_t,Z_r\right)\right\rangle dt\\&\leq\left[R_0F^{\#}_{u_0,R_0}\left(t\right)+R_0^2\gamma_t^+\right]+\left[F^{\#}_{u_0,R_0}\left(t\right)+2R_0\left|\gamma_t\right|\right]\left|Y_t-u_0\right|\\&+\left|Y_t-u_0\right|^2dV_t+\frac{n_p}{2a}\left|Z_t\right|^2.\end{aligned}$$

From the subdifferential inequalities we have $|\varphi(t,Y_t)-\varphi(t,u_0)|\leq[\varphi(t,Y_t)-\varphi(t,u_0)]+2|\hat{u}_0||Y_t-u_0|$ and $[\varphi(t,Y_t)-\varphi(t,u_0)]dt\leq\langle Y_t-u_0,dK_t\rangle$. Therefore $|\varphi(t,Y_t)-\varphi(t,u_0)|dt\leq\langle Y_t-u_0,dK_t\rangle+2|\hat{u}_0||Y_t-u_0|dt$. From the above it follows that

$$\begin{aligned}&\left[R_0\left|F\left(t,Y_t,Z_t\right)\right|+\left|\varphi(Y_t)-\varphi\left(u_0\right)\right|\right]dt+\left\langle Y_t-u_0,F\left(t,Y_t,Z_t\right)dt-dK_t\right\rangle\\&\leq\left[R_0F^{\#}_{u_0,R_0}\left(t\right)+R_0^2\gamma_t^+\right]dt+\left[F^{\#}_{u_0,R_0}\left(t\right)+2R_0\left|\gamma_t\right|+2\left|\hat{u}_0\right|\right]\left|Y_t-u_0\right|dt\\&+\left|Y_t-u_0\right|^2dV_t+\frac{n_p}{2a}\left|Z_t\right|^2.\end{aligned}\tag{9}$$

For $R_0=0$, inequality (7) clearly follows from (9) applying Proposition 11 from Appendix.

For $R_0>0$ we moreover deduce, using once again Proposition 11, inequality (8). □

Remark 4. Denoting

$$\Theta=e^{V_T}\left|\eta-u_0\right|+\int_0^T e^{V_s}\left|\hat{u}_0\right|ds+\int_0^T e^{V_s}\left|F\left(s,u_0,0\right)\right|ds$$

we deduce that, for all $t\in[0,T]$:

$$\left|Y_t\right|\leq\left|u_0\right|+C_{a,p}^{1/p}\ e^{-V_t}\left(\mathbb{E}^{\mathcal{F}_t}\Theta^p\right)^{1/p},\ \text{a.s.}\tag{10}$$

Corollary 5. *Let $p\geq 2$. We suppose moreover that there exist $r_0,c_0>0$ such that*

$$\varphi^{\#}_{u_0,r_0}\overset{def}{=}\sup\left\{\varphi\left(u_0+r_0v\right):|v|\leq 1\right\}\leq c_0\ .$$

Then

$$\begin{aligned}&r_0^{p/2}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}d\updownarrow K\updownarrow_s\right)^{p/2}\\&\leq C_{a,p}\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T}\left|\eta-u_0\right|^p+\left(\varphi^{\#}_{u_0,r_0}-\varphi\left(u_0\right)\right)\left(\int_t^T e^{2V_s}ds\right)^{p/2}\\&\quad+\left(\int_t^T e^{V_s}\left|\hat{u}_0\right|ds\right)^p+\left(\int_t^T e^{V_s}\left|F\left(s,u_0,0\right)\right|ds\right)^p\Big].\end{aligned}\tag{11}$$

Proof. Let an arbitrary function $v \in C\left([0,T];\mathbb{R}^m\right)$ such that $\|v\|_T \leq 1$. From the subdifferential inequality $\langle u_0 + r_0 v\left(t\right) - Y_t, dK_t\rangle + \varphi(Y_t)dt \leq \varphi\left(u_0 + r_0 v\left(t\right)\right)dt$, we deduce that $r_0 d\updownarrow K\updownarrow_t + \varphi(Y_t)dt \leq \langle Y_t - u_0, dK_t\rangle + \varphi^{\#}_{u_0,r_0}dt$. Since $\langle Y_t - u_0, \hat{u}_0\rangle + \varphi\left(u_0\right) \leq \varphi(Y_t)$, then

$$r_0 d\updownarrow K\updownarrow_t \leq \langle Y_t - u_0, dK_t\rangle + |\hat{u}_0|\,|Y_t - u_0|\,dt + \left[\varphi^{\#}_{u_0,r_0} - \varphi\left(u_0\right)\right]dt.$$

Therefore

$$\begin{aligned}
&r_0 d\updownarrow K\updownarrow_t + \langle Y_t - u_0, F\left(t, Y_t, Z_t\right)dt - dK_t\rangle \\
&\leq \left(\varphi^{\#}_{u_0,r_0} - \varphi\left(u_0\right)\right)dt + |Y_t - u_0|\left(|\hat{u}_0| + |F\left(t, u_0, 0\right)|\right)dt \\
&+ |Y_t - u_0|^2\,dV_t + \frac{n_p}{2a}\,|Z_t|^2\,dt.
\end{aligned}$$

The inequality (11) follows using Proposition 11. □

Proposition 6 (Uniqueness). *Let assumptions $(A_1 - A_3)$ be satisfied. Let $a, p > 1$. If $\left(Y,Z\right), (\tilde{Y},\tilde{Z}) \in S^0_m\left[0,T\right] \times \Lambda^0_{m\times k}\left(0,T\right)$ are two solutions of BSDE (1) corresponding respectively to η and $\tilde{\eta}$ such that $\mathbb{E}\sup_{s\in[0,T]} e^{pV_s}|Y_s - \tilde{Y}_s|^p < \infty$, then for all $t \in [0,T]$, $e^{pV_t}|Y_s - \tilde{Y}_s|^p \leq \mathbb{E}^{\mathcal{F}_t}\left(e^{pV_T}\left|\eta - \tilde{\eta}\right|^p\right)$, $\mathbb{P}$-a.s. and there exists a constant $C_{a,p}$ such that $\mathbb{P}$-a.s., for all $t \in [0,T]$:*

$$\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{s\in[t,T]} e^{pV_s}|Y_s - \tilde{Y}_s|^p + \Big(\int_t^T e^{2V_s}|Z_s - \tilde{Z}_s|^2 ds\Big)^{p/2}\Big] \leq C_{a,p}\mathbb{E}^{\mathcal{F}_t} e^{pV_T}\left|\eta - \tilde{\eta}\right|^p. \tag{12}$$

Moreover, the uniqueness of solution $\left(Y,Z\right)$ of BSDE (1) holds in $S^{1^+,p}_m\left[0,T\right] \times \Lambda^0_{m\times k}\left(0,T\right)$.

Proof. Let (Y,Z), $(\tilde{Y},\tilde{Z}) \in S^0_m\left[0,T\right] \times \Lambda^0_{m\times k}\left(0,T\right)$ be two solutions corresponding to η and $\tilde{\eta}$ respectively. Then there exists $p > 1$ such that $Y, \tilde{Y} \in S^p_m\left[0,T\right]$ and $Y_t - \tilde{Y}_t = \eta - \tilde{\eta} + \int_t^T dL_s - \int_t^T (Z_s - \tilde{Z}_s)dB_s$ where $L_t = \int_0^t [(F(s,Y_s,Z_s) - F(s,\tilde{Y}_s,\tilde{Z}_s))ds - (dK_s - d\tilde{K}_s)]$. Since by (5) $\langle Y_s - \tilde{Y}_s, dK_s - d\tilde{K}_s\rangle \geq 0$, then, for all $a > 1$,

$$\begin{aligned}
&\langle Y_t - \tilde{Y}_t, dL_t\rangle \leq |Y_t - \tilde{Y}_t|^2\mu_t dt + |Y_t - \tilde{Y}_t||Z_t - \tilde{Z}_t|\ell_t dt \\
&\leq |Y_t - \tilde{Y}_t|^2\Big(\mu_t + \frac{a}{2n_p}\ell_t^2\Big)dt + \frac{n_p}{2a}|Z_t - \tilde{Z}_t|^2 dt.
\end{aligned}$$

By Proposition 11, from Appendix, inequality (12) follows.

Let now $p > 1$ be such that $(Y, Z), (\tilde{Y}, \tilde{Z}) \in S_m^{1^+,p}[0,T] \times \Lambda_{m\times k}^0(0,T)$ are two solutions of BSDE (1) corresponding respectively to η and $\tilde{\eta}$. From the definition of space $S_m^{1+,p}[0,T]$ there exists $a > 1$ such that

$$\mathbb{E} \sup_{t\in[0,T]} |e^{V_t^{a,p}} Y_t|^p < \infty, \quad \mathbb{E} \sup_{t\in[0,T]} |e^{V_t^{a,p}} \tilde{Y}_t|^p < \infty.$$

Consequently estimate (12) follows and uniqueness too. □

3. BSVI - an existence result

Using Proposition 3 we can prove now the existence of a triple (Y, Z, K) which is a solution, in the sense of Definition 1, for BSVI (1). In order to obtain the absolute continuity with respect to dt for the process K it is necessary to impose a supplementary assumption.

Let $(u_0, \hat{u}_0) \in \partial\varphi$ be fixed and

$$\begin{aligned}\Theta_T^{a,p} &\overset{def}{=} C_{a,p} e^{2p\|V\|_T} \Big[|\eta - u_0|^p + \Big(\int_0^T |\hat{u}_0|\, ds \Big)^p \\ &\quad + \Big(\int_0^T |F(s, u_0, 0)|\, ds \Big)^p \Big], \end{aligned} \tag{13}$$

where $a, p > 1$, $C_{a,p}$ is the constant given by Proposition 3 and $V_t^{a,p}$ is defined by (6).

If there exists a constant M such that $|\eta| + \int_0^T |F(s, u_0, 0)|\, ds \le M$, a.s. then $\Theta_T^{a,p} \le C_{a,p} e^{2p\|V\|_T} [(M + |u_0|)^p + |\hat{u}_0|^p T^p]$ and by (10)

$$|Y_t| \le |u_0| + \left(\mathbb{E}^{\mathcal{F}_t} \Theta_T^{a,p}\right)^{1/p} \le |u_0| + C_{a,p}^{1/p}\, e^{2\|V\|_T} [M + |u_0| + |\hat{u}_0| T], \text{ a.s.}$$

We will make the following assumptions:

(A_4) There exist $p \ge 2$, a positive stochastic process $\beta \in L^1(\Omega \times (0,T))$, a positive function $b \in L^1(0,T)$ and a real number $\kappa \ge 0$, such that

(i) $\mathbb{E}\varphi^+(\eta) < \infty$,
(ii) for all $(u, \hat{u}) \in \partial\varphi$ and $z \in \mathbb{R}^{m\times k}$:

$$\langle \hat{u}, F(t, u, z) \rangle \le \frac{1}{2} |\hat{u}|^2 + \beta_t + b(t) |u|^p + \kappa |z|^2$$
$$d\mathbb{P} \otimes dt\text{-a.e.}, \ (\omega, t) \in \Omega \times [0, T],$$

and

(A_5) There exist $M, L > 0$ and $(u_0, \hat{u}_0) \in \partial\varphi$ such that:

$$\begin{array}{ll} (i) & \mathbb{E}\varphi^+(\eta) < \infty, \\ (ii) & \ell_t \leq L, \text{ a.e., } t \in [0,T], \\ (iii) & |\eta| + \int_0^T |F(s, u_0, 0)|\, ds \leq M, \text{ a.s., } \omega \in \Omega, \\ (iv) & \exists R_0 \geq |u_0| + C_{a,p}^{1/p} e^{2\|V\|_T} [M + |u_0| + |\hat{u}_0| T] \\ & \qquad \text{such that } \mathbb{E} \int_0^T \left(F_{R_0}^{\#}(s)\right)^2 ds < \infty. \end{array}$$

We note that, if $\langle \hat{u}, F(t,u,z) \rangle \leq 0$, for all $(u,\hat{u}) \in \partial\varphi$, then condition $(\mathrm{A}_4\text{-}ii)$ is satisfied with $\beta_t = b(t) = \kappa = 0$. For example, if $\varphi = I_{\bar{D}}$ (the convex indicator of closed convex set $\bar{D}$) and $\mathbf{n}_y$ denotes the unit outward normal vector to $\bar{D}$ at $y \in Bd(\bar{D})$, then condition $\langle \mathbf{n}_y, F(t,y,z) \rangle \leq 0$ for all $y \in Bd(\bar{D})$ yields $(\mathrm{A}_4\text{-}ii)$ with $\beta_t = b(t) = \kappa = 0$. In this last case the Itô's formula for $\psi(y) = [dist_{\bar{D}}(y)]^2$ and the uniqueness yields $K = 0$.

We also remark that if $F(t,y,z) = F(y,z)$ then assumptions (A_5) becomes

$$|\eta| + \mathbb{E}\varphi^+(\eta) \leq M, \text{ a.s., } \omega \in \Omega.$$

Theorem 7 (Existence). *Let $p \geq 2$ and assumptions $(A_1 - A_3)$ be satisfied with $s \to \mu_s = \mu(s)$ and $s \to \ell_s = \ell(s)$ deterministic processes. Suppose moreover that, for all $\rho \geq 0$,*

$$\mathbb{E}|\eta|^p + \mathbb{E}\left(\int_0^T F_\rho^{\#}(s) ds\right)^p < \infty,$$

and one of assumptions (A_4) or (A_5) is satisfied. Then there exists a unique pair $(Y,Z) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T)$ and a unique stochastic process $U \in \Lambda_m^2(0,T)$ such that

$$\begin{array}{ll} (a) & \int_0^T |F(t,Y_t,Z_t)|\, dt < \infty, \ \mathbb{P}\text{-}a.s., \\ (b) & Y_t(\omega) \in \mathrm{Dom}(\partial\varphi), \ d\mathbb{P} \otimes dt\text{- } a.e. \ (\omega,t) \in \Omega \times [0,T], \\ (c) & U_t(\omega) \in \partial\varphi(Y_t(\omega)), \ d\mathbb{P} \otimes dt \text{ - } a.e. \ (\omega,t) \in \Omega \times [0,T] \end{array}$$

and for all $t \in [0,T]$:

$$(14) \qquad Y_t + \int_t^T U_s ds = \eta + \int_t^T F(s,Y_s,Z_s)\, ds - \int_t^T Z_s dB_s, \ a.s.$$

Moreover, uniqueness holds in $S_m^{1^+}[0,T] \times \Lambda_{m\times k}^0(0,T)$, *where*

$$S_m^{1^+}[0,T] \stackrel{def}{=} \bigcup_{p>1} S_m^p[0,T].$$

Proof. Let (Y,Z), $(\tilde{Y},\tilde{Z}) \in S_m^{1^+}[0,T] \times \Lambda_{m\times k}^0(0,T)$ be two solutions. Then $\exists p_1,p_2>1$ such that $Y\in S_m^{p_1}[0,T]$, $\tilde{Y}\in S_m^{p_2}[0,T]$ and it follows that $Y,\tilde{Y}\in S_m^p[0,T]$, where $p=p_1\wedge p_2$. Applying Proposition 6 we obtain the uniqueness.

To prove existence of a solution we can assume, without loss of generality, that there exists $u_0\in \mathrm{Dom}(\varphi)$ such that

$$0=\varphi(u_0)\le\varphi(y),\ \forall y\in\mathbb{R}^m, \tag{15}$$

hence $0\in\partial\varphi(u_0)$, since, in the sense of Definition 1, we can replace BSVI (1) by

$$\begin{cases} -dY_t+\partial\tilde{\varphi}(Y_t)\,dt \ni \tilde{F}(t,Y_t,Z_t)\,dt - Z_t dB_t,\ 0\le t<T\\ Y_T=\eta,\end{cases}$$

where, for $(u_0,\hat{u}_0)\in\partial\varphi$ fixed,

$$\begin{aligned}&\tilde{\varphi}(y)\stackrel{def}{=}\varphi(y)-\varphi(u_0)-\langle\hat{u}_0,y-u_0\rangle,\ y\in\mathbb{R}^d\\ &\tilde{F}(t,y,z)\stackrel{def}{=}F(t,y,z)-\hat{u}_0,\ y\in\mathbb{R}^d,\ t\in[0,T].\end{aligned}$$

Step 1. *Approximating problem.*

Let $\varepsilon\in(0,1]$ and the approximating equation

$$\begin{aligned} Y_t^\varepsilon+\int_t^T\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,ds=\eta+\int_t^T F(s,Y_s^\varepsilon,Z_s^\varepsilon)\,ds\\ -\int_t^T Z_s^\varepsilon dB_s,\ \text{a.s.},\ t\in[0,T],\end{aligned}\tag{16}$$

$\nabla\varphi_\varepsilon$ is the gradient of the Yosida's regularization φ_ε of the function φ.

Using (15) we obtain

$$0=\varphi(u_0)\le\varphi(J_\varepsilon y)\le\varphi_\varepsilon(y)\le\varphi(y),\ J_\varepsilon(u_0)=u_0,\ \nabla\varphi_\varepsilon(u_0)=0.\tag{17}$$

It follows from [2], Theorem 4.2 (see also [15], Chapter 5) that equation (16) has an unique solution $(Y^\varepsilon,Z^\varepsilon)\in S_m^p[0,T]\times\Lambda_{m\times k}^p(0,T)$.

Step 2. *Boundedness of* Y^{ε} *and* Z^{ε}, *without supplementary assumptions* (A_4) *or* (A_5).

From Proposition 3, applied for (16), we obtain, for all $a > 1$,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{s\in[t,T]}\left|e^{V_s}\left(Y_s^{\varepsilon}-u_0\right)\right|^p\\ &+\Big(\int_t^T e^{2V_s}\varphi_{\varepsilon}(Y_s^{\varepsilon})ds\Big)^{p/2}+\Big(\int_t^T e^{2V_s}\left|Z_s^{\varepsilon}\right|^2 ds\Big)^{p/2}\Big]\\ &\leq C_{a,p}\mathbb{E}^{\mathcal{F}_t}\Big[e^{pV_T}\left|\eta-u_0\right|^p+\Big(\int_t^T e^{V_s}\left|F\left(s,u_0,0\right)\right|ds\Big)^p\Big].\end{aligned} \tag{18}$$

In particular there exists a constant independent of ε such that

$$\begin{aligned}&(a)\quad \mathbb{E}\left\|Y^{\varepsilon}\right\|_T^2\leq\left(\mathbb{E}\left\|Y^{\varepsilon}\right\|_T^p\right)^{2/p}\leq C,\\ &(b)\quad \mathbb{E}\int_0^T\left|Z_s^{\varepsilon}\right|^2ds\leq\Big[\mathbb{E}\Big(\int_0^T\left|Z_s^{\varepsilon}\right|^2ds\Big)^{p/2}\Big]^{2/p}\leq C.\end{aligned} \tag{19}$$

Moreover, from (10) we obtain

$$|Y_t^{\varepsilon}|\leq|u_0|+\left(\mathbb{E}^{\mathcal{F}_t}\Theta_T^{a,p}\right)^{1/p}, \tag{20}$$

where $\Theta_T^{a,p}$ is given by (13) with $\hat{u}_0=0$ (since $\nabla\varphi_{\varepsilon}(u_0)=0$).

Throughout the proof we shall fix $a=2$ (and then V_t defined by (6), with $n_p=1\wedge(p-1)=1$, becomes $V_t=\int_0^t\left[\mu\left(s\right)+\ell^2\left(s\right)\right]ds$).

Step 3. *Boundedness of* $\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon})$.

Using the following stochastic subdifferential inequality (for proof see Proposition 2.2, [13])

$$\varphi_{\varepsilon}(Y_t^{\varepsilon})+\int_t^T\left\langle\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon}),\,dY_s^{\varepsilon}\right\rangle\leq\varphi_{\varepsilon}(Y_T^{\varepsilon})=\varphi_{\varepsilon}(\eta)\leq\varphi(\eta),$$

we deduce that, for all $t\in[0,T]$,

$$\begin{aligned}&\varphi_{\varepsilon}(Y_t^{\varepsilon})+\int_t^T\left|\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon})\right|^2ds\leq\varphi(\eta)\\ &+\int_t^T\left\langle\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon}),F\left(s,Y_s^{\varepsilon},Z_s^{\varepsilon}\right)\right\rangle ds-\int_t^T\left\langle\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon}),\,Z_s^{\varepsilon}dB_s\right\rangle.\end{aligned} \tag{21}$$

Since

$$\mathbb{E}\Big(\int_0^T |\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)|^2|Z_s^\varepsilon|^2ds\Big)^{1/2} \leq \frac{1}{\varepsilon}\mathbb{E}\Big[\Big(\sup_{s\in[0,T]}|Y_s^\varepsilon|\Big)\Big(\int_0^T |Z_s^\varepsilon|^2ds\Big)^{1/2}\Big]$$
$$\leq \frac{1}{\varepsilon^2}\mathbb{E}\Big(\sup_{s\in[0,T]}|Y_s^\varepsilon|^2\Big) + \mathbb{E}\Big(\int_0^T |Z_s^\varepsilon|^2ds\Big) < \infty,$$

then $\mathbb{E}\int_t^T \langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), Z_s^\varepsilon dB_s\rangle = 0$. Under assumption ($\mathrm{A}_4$), since $\nabla\varphi_\varepsilon(Y_s^\varepsilon) \in \partial\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right)$, then

$$\begin{aligned}
&\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\rangle\\
&= \frac{1}{\varepsilon}\left\langle Y_s^\varepsilon - J_\varepsilon\left(Y_s^\varepsilon\right), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) - F\left(s, J_\varepsilon\left(Y_s^\varepsilon\right), Z_s^\varepsilon\right)\right\rangle\\
&+ \langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, J_\varepsilon\left(Y_s^\varepsilon\right), Z_s^\varepsilon\right)\rangle\\
&\leq \frac{1}{\varepsilon}\mu^+\left(s\right)\left|Y_s^\varepsilon - J_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 + \frac{1}{2}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 + \beta_s + b\left(s\right)\left|J_\varepsilon\left(Y_s^\varepsilon\right)\right|^p + \kappa\left|Z_s^\varepsilon\right|^2.
\end{aligned}$$

From (2) and inequality $\left|J_\varepsilon\left(Y_s^\varepsilon\right)\right| \leq \left|J_\varepsilon\left(Y_s^\varepsilon\right) - J_\varepsilon\left(u_0\right)\right| + |u_0| \leq |Y_s^\varepsilon - u_0| + |u_0|$ we have, for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\varphi_\varepsilon(Y_t^\varepsilon) + \frac{1}{2}\mathbb{E}\int_t^T \left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds \leq \mathbb{E}\varphi(\eta) + 2\int_t^T \mu^+\left(s\right)\mathbb{E}\varphi_\varepsilon(Y_s^\varepsilon)ds\\
&+\mathbb{E}\int_t^T \left(\beta_s + b\left(s\right)\left(|Y_s^\varepsilon - u_0| + |u_0|\right)^p + \kappa\left|Z_s^\varepsilon\right|^2\right)ds
\end{aligned}$$

that yields, via estimate (18) and the backward Gronwall's inequality, that there exists a constant $C > 0$ independent of $\varepsilon \in (0,1]$ such that

$$\begin{aligned}
&(a)\quad \mathbb{E}\varphi_\varepsilon(Y_t^\varepsilon) + \mathbb{E}\int_0^T \left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 ds \leq C,\\
&(b)\quad \mathbb{E}\left|Y_t^\varepsilon - J_\varepsilon\left(Y_t^\varepsilon\right)\right|^2 \leq C\varepsilon.
\end{aligned}\tag{22}$$

If we suppose (A_5) then, from (20), we infer that

$$|Y_t^\varepsilon| \leq |u_0| + (\mathbb{E}^{\mathcal{F}_t}\Theta_T^{2,p})^{1/p} \leq |u_0| + C_{2,p}^{1/p}\; e^{2\|V\|_T}[M + |u_0| + |\hat{u}_0|T] \overset{def}{=} R_0.\tag{23}$$

Now

$$\begin{aligned}
&\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\rangle\\
&= \langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, 0\right)\rangle + \langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) - F\left(s, Y_s^\varepsilon, 0\right)\rangle\\
&\leq \frac{1}{2}\left|\nabla\varphi_\varepsilon(Y_s^\varepsilon)\right|^2 + |F_{R_0}^{\#}\left(s\right)|^2 + L^2\left|Z_s^\varepsilon\right|^2.
\end{aligned}$$

Hence from (21) it follows that, for all $t \in [0,T]$,

$$\begin{aligned} &\mathbb{E}\varphi(J_\varepsilon(Y_t^\varepsilon)) + \frac{1}{2}\mathbb{E}\int_t^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \\ &\leq \mathbb{E}\left(\varphi(\eta) + \int_t^T |F_{R_0}^{\#}(s)|^2 ds + L^2 \int_t^T |Z_s^\varepsilon|^2 ds\right) \end{aligned} \tag{24}$$

and from (19) we obtain boundedness inequalities (22).

Step 4. *Cauchy sequence and convergence.*

Let $\varepsilon, \delta \in (0,1]$.

We can write $Y_t^\varepsilon - Y_t^\delta = \int_t^T dK_s^{\varepsilon,\delta} - \int_t^T Z_s^\varepsilon dB_s$, where

$$K_t^{\varepsilon,\delta} = \int_0^t \left[F(s, Y_s^\varepsilon, Z_s^\varepsilon) - F(s, Y_s^\delta, Z_s^\delta) - \nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\varphi_\delta(Y_s^\delta)\right] ds.$$

Then

$$\langle Y_t^\varepsilon - Y_t^\delta, dK_t^{\varepsilon,\delta}\rangle \leq (\varepsilon+\delta)\langle\nabla\varphi_\varepsilon(Y_t^\varepsilon), \nabla\varphi_\delta(Y_t^\delta)\rangle dt + |Y_t^\varepsilon - Y_t^\delta|^2 dV_t + \frac{1}{4}|Z_t^\varepsilon - Z_t^\delta|^2 dt,$$

and by Proposition 11, with $p = 2$,

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,T]} |Y_s^\varepsilon - Y_s^\delta|^2 + \mathbb{E}\int_0^T |Z_s^\varepsilon - Z_s^\delta|^2 ds \\ &\leq C\mathbb{E}\int_0^T (\varepsilon+\delta)\langle\nabla\varphi_\varepsilon(Y_s^\varepsilon), \nabla\varphi_\delta(Y_s^\delta)\rangle ds \\ &\leq \frac{1}{2}C(\varepsilon+\delta)\left[\mathbb{E}\int_0^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds + \mathbb{E}\int_0^T |\nabla\varphi_\delta(Y_s^\delta)|^2 ds\right] \leq C'(\varepsilon+\delta). \end{aligned}$$

Hence there exist $(Y, Z, U) \in S_m^2[0,T] \times \Lambda_{m\times k}^2(0,T) \times \Lambda_m^2(0,T)$ and a sequence $\varepsilon_n \searrow 0$ such that

$$\begin{aligned} &Y^{\varepsilon_n} \to Y, \text{ in } S_m^2[0,T] \text{ and a.s. in } C([0,T];\mathbb{R}^m), \\ &Z^{\varepsilon_n} \to Z, \text{ in } \Lambda_{m\times k}^2(0,T) \text{ and a.s. in } L^2\left(0,T;\mathbb{R}^{m\times k}\right), \\ &\nabla\varphi_\varepsilon(Y^\varepsilon) \rightharpoonup U, \text{ weakly in } \Lambda_m^2(0,T), \\ &J_{\varepsilon_n}(Y^{\varepsilon_n}) \to Y, \text{ in } \Lambda_m^2(0,T) \text{ and a.s. in } L^2(0,T;\mathbb{R}^m). \end{aligned}$$

Passing to limit in (16) we conclude that

$$Y_t + \int_t^T U_s ds = \eta + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \text{ a.s.}$$

Since $\nabla\varphi_\varepsilon(Y_s^\varepsilon) \in \partial\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right)$ then for all $A \in \mathcal{F}$, $0 \le s \le t \le T$ and $v \in S_m^2\left[0,T\right]$,

$$\mathbb{E}\int_s^t \mathbf{1}_A \left\langle \nabla\varphi_\varepsilon(Y_r^\varepsilon), v_r - Y_r^\varepsilon\right\rangle dr + \mathbb{E}\int_s^t \mathbf{1}_A\varphi(J_\varepsilon\left(Y_r^\varepsilon\right))dr \le \mathbb{E}\int_s^t \mathbf{1}_A\varphi(v_r)dr.$$

Passing to $\liminf$ for $\varepsilon = \varepsilon_n \searrow 0$ in the above inequality we obtain that $U_s \in \partial\varphi\left(Y_s\right)$. Hence $(Y,Z,U) \in S_m^p\left[0,T\right] \times \Lambda_{m\times k}^p\left(0,T\right) \times \Lambda_m^2\left(0,T\right)$ and $\left(Y,Z,K\right)$, with $K_t = \int_0^t U_s ds$, is the solution of BSVI (1).

Step 5. *Remarks in case* (A_5).

Passing to $\liminf$ for $\varepsilon = \varepsilon_n \searrow 0$ in (23) and (24) it follows, using assumptions (A_5), that the solution also satisfies

$$\begin{aligned} &(a) \quad \left|Y_t\right| \le R_0, \text{ a.s. for all } t \in \left[0,T\right], \\ &(b) \quad \mathbb{E}\varphi(Y_t) + \frac{1}{2}\mathbb{E}\int_t^T \left|U_s\right|^2 ds \\ &\qquad \le \mathbb{E}\big(\varphi(\eta) + \int_0^T \left|F_{R_0}^{\#}\left(s\right)\right|^2 ds + L^2\int_0^T \left|Z_s\right|^2 ds\big). \end{aligned}$$

The proof is completed now. $\square$

Remark 8. The existence Theorem 7 is well adapted to the Hilbert spaces since we do not impose an assumption of type $\mathrm{Int}\left(\mathrm{Dom}\left(\varphi\right)\right) \neq \emptyset$, which is very restrictive for the infinite dimensional spaces. In the context of the Hilbert spaces Theorem 7 holds in the same form and one can give, as examples, partial differential backward stochastic variational inequalities (see [14]).

4. BSVI - a general existence result

We replace now assumptions (A_5) with $\mathrm{Int}\left(\mathrm{Dom}\left(\varphi\right)\right) \neq \emptyset$.

Theorem 9 (Existence)**.** *Let* $p \ge 2$ *and assumptions* $(A_1 - A_3)$ *be satisfied with* $s \to \mu_s = \mu\left(s\right)$ *and* $s \to \ell_s = \ell\left(s\right)$ *deterministic processes. We suppose moreover that* $\mathrm{Int}\left(\mathrm{Dom}\left(\varphi\right)\right) \neq \emptyset$ *and for all* $\rho \ge 0$

$$\mathbb{E}\left|\eta\right|^p + \mathbb{E}\left(\int_0^T F_\rho^{\#}(s)ds\right)^p < \infty.$$

Then there exists a unique triple $(Y,Z,K) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \times S_m^p(0,T)$, $\mathbb{E} \updownarrow K \updownarrow_T^{p/2} < \infty$, *such that for all* $t \in [0,T]$:

$$\begin{cases} Y_t + K_T - K_t = \eta + \int_t^T F(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s, \ a.s., \\ dK_t \in \partial\varphi(Y_t)\, dt, \ a.s., \\ Y_T = \eta, \ a.s., \end{cases} \tag{25}$$

which means that BSVI (1) *has a unique solution, and moreover*

$$\mathbb{E}\,\|Y\|_T^p + \mathbb{E}\,\|K\|_T^p + \mathbb{E}\updownarrow K \updownarrow_T^{p/2} + \mathbb{E}\int_0^T |Z_t|^2\, dt < \infty.$$

Proof. The uniqueness was proved in Proposition 6.

Step 1. *Existence under supplementary assumption*

$$\begin{aligned} &\exists M > 0, \ u_0 \in \mathrm{Int}(\mathrm{Dom}(\partial\varphi)) \ \textit{such that} \\ &\mathbb{E}|\varphi(\eta)| + |\eta| + \int_0^T |F(s,u_0,0)| ds \le M, \ \text{a.s. } \omega \in \Omega. \end{aligned} \tag{26}$$

Let R_0 defined by (23) and denote $\zeta_t = \ell(t) + F^{\#}_{R_0}(t)$. By Theorem 7 there exists a unique $(Y^n, Z^n, U^n) \in S_m^p[0,T] \times \Lambda_{m\times k}^p(0,T) \times \Lambda_m^2(0,T)$ such that $U_s^n \in \partial\varphi(Y_s^n)$ and for all $t \in [0,T]$:

$$Y_t^n + \int_t^T U_s^n ds = \eta + \int_t^T F(s, Y_s^n, Z_s^n)\, \mathbf{1}_{\zeta_t \le n} ds - \int_t^T Z_s^n dB_s, \ \text{a.s.} \tag{27}$$

Moreover $\sup_{s\in[0,T]} |Y_s^n| \le R_0$, a.s. and

$$\mathbb{E}\left(\int_0^T |\varphi(Y_s^n)|\, ds\right)^{p/2} + \mathbb{E}\left(\int_0^T |Z_s^n|^2\, ds\right)^{p/2} \le C. \tag{28}$$

Let $q = p/2$, $n_q = 1 \wedge (q-1)$, $a = 2$ and $V_t^{2,q}$ given by (6).

Since

$$\begin{aligned} &\langle Y_t^n - Y_t^{n+l}, (F(t,Y_t^n,Z_t^n)\mathbf{1}_{\zeta_t\le n} - U_t^n - F(t,Y_t^{n+l},Z_t^{n+l})\mathbf{1}_{\zeta_t \le n+l} + U_t^{n+l})\rangle dt \\ &\le \langle Y_t^n - Y_t^{n+l}, F(t,Y_t^n,Z_t^n)\rangle(\mathbf{1}_{\zeta_t\le n} - \mathbf{1}_{\zeta_t\le n+l})dt \\ &+ |Y_t^n - Y_t^{n+l}|^2 dV_t^{2,q} + \frac{n_q}{4}|Z_t^n - Z_t^{n+l}|^2 dt, \end{aligned}$$

then by Proposition 11, from Appendix, (with $a=2$) there exists a constant depending only on p, such that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}|Y_s^n-Y_s^{n+l}|^{p/2}+\mathbb{E}\left(\int_0^T|Z_s^n-Z_s^{n+l}|^2ds\right)^{p/4}\\&\leq C_p\ e^{p\left\|V^{2,q}\right\|_T}\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}\left|F\left(s,Y_s^n,Z_s^n\right)\right|ds\right)^{p/2}.\end{aligned}$$

But

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}\left|F\left(s,Y_s^n,Z_s^n\right)\right|ds\right)^{p/2}\\&\leq\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}\left(F_{R_0}^{\#}\left(s\right)+\ell\left(s\right)\left|Z_s^n\right|\right)ds\right)^{p/2}\\&\leq C_p'\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}F_{R_0}^{\#}\left(s\right)ds\right)^{p/2}\\&\quad+C_p'\left[\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}\ell^2\left(s\right)ds\right)^{p/2}\right]^{1/2}\cdot\left[\mathbb{E}\left(\int_0^T\left|Z^n\left(s\right)\right|^2ds\right)^{p/2}\right]^{1/2}\\&\leq C_p'\left[\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}F_{R_0}^{\#}\left(s\right)ds\right)^{p}\right]^{1/2}\\&\quad+C_p'\ C^{1/2}\left[\mathbb{E}\left(\int_0^T\mathbf{1}_{\zeta_s\geq n}\ell^2\left(s\right)ds\right)^{p}\right]^{1/2}\rightarrow0,\ \text{as}\ n\rightarrow\infty.\end{aligned}$$

Hence there exists a pair $(Y,Z)\in S_m^{p/2}\left[0,T\right]\times\Lambda_{m\times k}^{p/2}\left(0,T\right)$ such that, as $n\rightarrow\infty$, $(Y^n,Z^n)\rightarrow(Y,Z)$ in $S_m^{p/2}\left[0,T\right]\times\Lambda_{m\times k}^{p/2}\left(0,T\right)$. In particular $Y_0^n\rightarrow Y_0$ in $\mathbb{R}^m$ and from equation (27) it follows that $K_\cdot^n=\int_0^\cdot U_s^nds\rightarrow K$, in $S_m^0[0,T]$. Now by (11) for $V_t=V_t^{2,p}$ we obtain

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\left|U_t^n\right|dt\right)^{p/2}=\mathbb{E}\updownarrow K^n\updownarrow_T^{p/2}\\&\leq Ce^{2p\|V\|_T}\left[1+T+\mathbb{E}\left|\eta\right|^p+\mathbb{E}\Big(\int_0^T\left|F\left(t,u_0,0\right)\right|dt\Big)^p\right]\end{aligned}$$

with $C=C\left(p,u_0,\hat{u}_0,r_0,\varphi\right)$.

Therefore

$$\mathbb{E}\updownarrow K\updownarrow_T^{p/2}\leq Ce^{2p\|V\|_T}\left[1+T+\mathbb{E}\left|\eta\right|^p+\mathbb{E}\Big(\int_0^T\left|F\left(s,u_0,0\right)\right|ds\Big)^p\right].$$

Passing to lim inf as $n \to \infty$, eventually on a subsequence, we deduce from (18) and (20) that $\sup_{s\in[0,T]} |Y_s| \le R_0,$ a.s. and

$$\mathbb{E}\left(\int_0^T |\varphi(Y_s)|\, ds\right)^{p/2} + \mathbb{E}\left(\int_0^T |Z_s|^2\, ds\right)^{p/2} \le C.$$

To show that (Y, Z, K) is solution of BSDE (25) it remains to show that $dK_t \in \partial\varphi(Y_t)(dt)$. Applying Corollary 13 we obtain $dK_t \in \partial\varphi(Y_t)(dt)$, since $dK_t^n = U_t^n dt \in \partial\varphi(Y_t^n)\, dt$.

Step 2. *Existence without supplementary assumption (26).*

Let $(u_0, \hat{u}_0) \in \partial\varphi$ such that $u_0 \in \mathrm{Int}(\mathrm{Dom}(\varphi))$ and $\overline{B}(u_0, r_0) \subset \mathrm{Dom}(\varphi)$. Recall that

$$\varphi^{\#}_{u_0,r_0} \overset{def}{=} \sup\{\varphi(u_0 + r_0 v) : |v| \le 1\} < \infty.$$

We introduce $\eta^n = \eta \mathbf{1}_{[0,n]}(|\eta| + |\varphi(\eta)|) + u_0 \mathbf{1}_{(n,\infty)}(|\eta| + |\varphi(\eta)|)$ and $F^n(t, y, z) = F(s, y, z) - F(s, u_0, 0)\, \mathbf{1}_{|F(s,u_0,0)|\ge n}$ Clearly $|\eta^n| + |\varphi(\eta_n)| + |F^n(t, u_0, 0)| \le 3n + |\varphi(u_0)|$. By *Step* 1, for each $n \in \mathbb{N}^*$ there exists a unique triple $(Y^n, Z^n, K^n) \in S_m^p[0,T] \times \Lambda^p_{m\times k}(0,T) \times S_m^{p/2}(0,T)$ solution of BSDE

$$Y_t^n + (K_T^n - K_t^n) = \eta^n + \int_t^T F^n(s, Y_s^n, Z_s^n)\, ds - \int_t^T Z_s^n dB_s, \text{ a.s.} \tag{29}$$

From Corollary 5 and Proposition 6 we infer that there exists a constant C_p such that

$$\begin{aligned}
&\mathbb{E} r_0^{p/2} \updownarrow K^n \updownarrow_T^{p/2} + \mathbb{E} \sup_{s\in[0,T]} |(Y_s^n - u_0)|^p + \mathbb{E}\Big(\int_0^T |\varphi(Y_s^n) - \varphi(u_0)|\, ds\Big)^{p/2} \\
&+ \mathbb{E}\Big(\int_0^T |Z_s^n|^2\, ds\Big)^{p/2} \le C_p e^{2p\|V\|_T}\Big[\Big[\varphi^{\#}_{u_0,r_0} - \varphi(u_0)\Big]^{p/2} T^{p/2} + |\hat{u}_0|^p T^p \\
&+ \mathbb{E}|\eta^n - u_0|^p + \mathbb{E}\Big(\int_0^T |F^n(s, u_0, 0)|\, ds\Big)^p\Big] \\
&\le C_p e^{2p\|V\|_T}\Big[\Big[\varphi^{\#}_{u_0,r_0} - \varphi(u_0)\Big]^{p/2} T^{p/2} + |\hat{u}_0|^p T^p + \mathbb{E}|\eta - u_0|^p + \\
&+ \mathbb{E}\Big(\int_0^T |F(s, u_0, 0)|\, ds\Big)^p\Big].
\end{aligned} \tag{30}$$

Remark that $p \ge 2$ is required only to obtain the estimate of $\mathbb{E} \updownarrow K^n \updownarrow_T^{p/2}$.

Since

$$\begin{aligned}&\langle Y_s^n - Y_s^{n+l}, F^n\left(s, Y_s^n, Z_s^n\right) - F^{n+l}(s, Y_s^{n+l}, Z_s^{n+l})\rangle \\ &\leq |Y_s^n - Y_s^{n+l}||F\left(s, u_0, 0\right)|\mathbf{1}_{|F(s,u_0,0)|\geq n} \\ &+ |Y_s^n - Y_s^{n+l}|^2 dV_t + \frac{1}{4}|Z_s^n - Z_s^{n+l}|^2 ds\end{aligned}$$

then by Proposition 11 we obtain

$$\begin{aligned}&\mathbb{E}\Big(\sup_{s\in[0,T]} |Y_s^n - Y_s^{n+l}|^p\Big) + \mathbb{E}\Big(\int_0^T |Z_s^n - Z_s^{n+l}|^2 ds\Big)^{p/2} \\ &\leq C_p e^{2p\|V\|_T}\left[\mathbb{E}\left(|\eta - u_0|^p\, \mathbf{1}_{|\eta|+|\varphi(\eta)|\geq n}\right)\right. \\ &\left.+\mathbb{E}\Big(\int_0^T |F\left(s, u_0, 0\right)|\, \mathbf{1}_{|F(s,u_0,0)|\geq n}\Big)^p\right].\end{aligned}$$

Hence there exists a pair $(Y, Z) \in S_m^p\,[0, T] \times \Lambda_{m\times k}^p\,(0, T)$ such that $(Y^n, Z^n) \to (Y, Z)$, as $n \to \infty$, in $S_m^p[0, T] \times \Lambda_{m\times k}^p(0, T)$. In particular $Y_0^n \to Y_0$ in $\mathbb{R}^m$. From equation (29) we have $K^n \to K$ in $S_m^0\,[0, T]$, and for all $t \in [0, T]$

$$Y_t + K_T - K_t = \eta + \int_t^T F\left(s, Y_s, Z_s\right) ds - \int_t^T Z_s dB_s, \text{ a.s.}$$

Letting $n \to \infty$ and applying Proposition 12 we can assert that estimate (30) holds without n. To complete the proof remark that from $dK_t^n \in \partial\varphi\left(Y_t^n\right) dt$ we can infer, using Corollary 13, that $dK_t \in \partial\varphi\left(Y_t\right) dt$.

Therefore (Y, Z, K) is solution of BSDE (25) in the sense of Definition 1. □

Remark 10. When μ and ℓ are stochastic processes we obtain, with similar proofs as in Theorems 7 and 9, the existence of a solution in the space

$$\mathbb{U}_{m,k}^p(0, T) \stackrel{def}{=} \Big\{(Y, Z)\in S_m^0[0, T]\times\Lambda_{m\times k}^0(0, T) : \|(Y, Z)\|_{a,p} <\infty,\ \forall a > 1\Big\},$$

where

$$\|(Y, Z)\|_{a,p}^p \stackrel{def}{=} \mathbb{E}\Big(\sup_{s\in[0,T]} e^{pV_s^{a,p}}|Y_s|^p\Big) + \mathbb{E}\Big(\int_0^T e^{2V_s^{a,p}}|Z_s|^2 ds\Big)^{p/2}.$$

5. Appendix

In this section we first present some useful and general estimates on $(Y,Z) \in S_m^0[0,T] \times \Lambda_{m\times k}^0(0,T)$ satisfying an identity of type

$$Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s,\ t \in [0,T],\ \mathbb{P}\text{-a.s.},$$

where $K \in S_m^0[0,T]$ and $K_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$ $\mathbb{P}$-a.s., $\omega \in \Omega$.

The following results and their proofs are given in the monograph of Pardoux, Răşcanu [15], Annex C.

Assume there exist

♢ D, R, N progressively measurable increasing continuous stochastic processes with $D_0 = R_0 = N_0 = 0$,

♢ V progressively measurable bounded-variation continuous stochastic process with $V_0 = 0$,

♢ $a, p > 1$,

such that, as signed measures on $[0,T]$,

$$dD_t + \langle Y_t, dK_t\rangle \le \left(\mathbf{1}_{p\ge 2} dR_t + |Y_t| dN_t + |Y_t|^2 dV_t\right) + \frac{n_p}{2a}|Z_t|^2 dt, \tag{31}$$

where $n_p = (p-1)\wedge 1$.

Let $\left\|Ye^V\right\|_{[t,T]} \overset{def}{=} \sup_{s\in[t,T]} \left|Y_s e^{V_s}\right|$ and $\left\|Ye^V\right\|_T \overset{def}{=} \left\|Ye^V\right\|_{[0,T]}$.

Proposition 11. *Assume* (31) *and*

$$\mathbb{E}\left\|Ye^V\right\|_T^p + \mathbb{E}\left(\int_0^T e^{2V_s}\mathbf{1}_{p\ge 2} dR_s\right)^{p/2} + \mathbb{E}\left(\int_0^T e^{V_s} dN_s\right)^p < \infty.$$

Then there exists a positive constant $C_{a,p}$, depending only of a, p, such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$:

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{s\in[t,T]} |e^{V_s}Y_s|^p + \Big(\int_t^T e^{2V_s} dD_s\Big)^{p/2} + \Big(\int_t^T e^{2V_s}|Z_s|^2 ds\Big)^{p/2}\Big] \\ &+ \mathbb{E}^{\mathcal{F}_t}\Big[\int_t^T e^{pV_s}|Y_s|^{p-2}\mathbf{1}_{Y_s\ne 0} dD_s + \int_t^T e^{pV_s}|Y_s|^{p-2}\mathbf{1}_{Y_s\ne 0}|Z_s|^2 ds\Big] \\ &\le C_{a,p}\,\mathbb{E}^{\mathcal{F}_t}\Big[\left|e^{V_T}Y_T\right|^p + \Big(\int_t^T e^{2V_s}\mathbf{1}_{p\ge 2} dR_s\Big)^{p/2} + \Big(\int_t^T e^{V_s} dN_s\Big)^p\Big]. \end{aligned} \tag{32}$$

In particular for all $t \in [0,T]$:

$$|Y_t|^p \le C_{a,p}\, \mathbb{E}^{\mathcal{F}_t}\left[\left(|Y_T|^p + \mathbf{1}_{p\ge 2} R_T^p + N_T^p\right) e^{p\left\|(V_\cdot - V_t)^+\right\|_{[t,T]}}\right], \ \mathbb{P}\text{-}a.s.$$

Moreover if there exists a constant $b \ge 0$ *such that for all* $t \in [0,T]$:

$$\left|e^{V_T - V_t} Y_T\right| + \left(\int_t^T e^{2(V_s - V_t)} \mathbf{1}_{p\ge 2} dR_s\right)^{1/2} + \int_t^T e^{(V_s - V_t)} dN_s \le b, \ a.s.$$

then for all $t \in [0,T]$:

$$|Y_t|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2(V_s - V_t)} |Z_s|^2 ds\right)^{p/2} \le b^p C_{a,p}, \ \ \mathbb{P}\text{-}a.s. \tag{33}$$

The following results provides a criterion for passing to the limit in Stieltjes integral (for the proofs we refer the reader to [15], Chapter I).

Proposition 12. *Let* Y, K, Y^n, K^n *be* $C([0,T];\mathbb{R}^m)$*-valued random variables,* $n \in \mathbb{N}$*. Assume*

(*i*) $\exists p > 0$ *such that* $\sup_{n\in\mathbb{N}^*} \mathbb{E} \updownarrow K^n \updownarrow_T^p < \infty$,

(*ii*) $\left(\|Y^n - Y\|_T + \|K^n - K\|_T\right) \xrightarrow{prob.} 0, \ as\ n \to \infty,$
i.e. $\forall \varepsilon > 0, \ \mathbb{P}\left\{\left(\|Y^n - Y\|_T + \|K^n - K\|_T\right) > \varepsilon\right\} \to 0, \ as\ n \to \infty.$

Then, for all $0 \le s \le t \le T$:

$$\int_s^t \langle Y_r^n, dK_r^n \rangle \xrightarrow{prob.} \int_s^t \langle Y_r, dK_r \rangle, \ as\ n \to \infty,$$

and moreover, $\mathbb{E} \updownarrow K \updownarrow_T^p \le \liminf_{n\to+\infty} \mathbb{E} \updownarrow K^n \updownarrow_T^p$.

Corollary 13. *Let the assumptions of Proposition* 12 *be satisfied. If* $A : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ *is a (multivalued) maximal monotone operator then the following implication holds*

$$dK_t^n \in A(Y_t^n)\, dt \ on\ [0,T], \ a.s. \ \Rightarrow \ dK_t \in A(Y_t)\, dt \ on\ [0,T], \ a.s.$$

In particular if $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ *is a proper convex l.s.c. function then*

$$dK_t^n \in \partial\varphi(Y_t^n)\, dt \ on\ [0,T], \ a.s. \ \Rightarrow \ dK_t \in \partial\varphi(Y_t)\, dt \ on\ [0,T], \ a.s.$$

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