ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LX, 2014, f.2
DOI: 10.2478/aicu-2013-0028

INTERPOLATION OPERATORS ON SOME TRIANGLES WITH CURVED SIDES

BY

TEODORA CĂTINAŞ, PETRU BLAGA and GHEORGHE COMAN

Abstract. This paper contains a survey regarding interpolation and Bernstein-type operators defined on triangles having one or all curved sides; we consider as well some of the product and Boolean sum operators. We study the interpolation properties, the orders of accuracy and the remainders of the generated approximation formulas.

Mathematics Subject Classification 2010: 41A05, 41A25, 41A80.

Key words: product and boolean sum operators, triangles and tetrahedrons with curved sides, interpolation operators, Bernstein-type operators, remainders.

1. Introduction

The aim of this survey is to present some interpolation and Bernstein-type operators for functions defined on triangles with one or all curved sides (see [11], [12], [14], [15]). They come as an extension of the corresponding operators for functions defined on triangles with all straight sides (see, e.g., [3]-[6], [8]-[10], [13], [23], [24], [26]-[29], [32]). The operators defined on domains with curved sides permit essential boundary conditions to be satisfied exactly. Such operators can be used in construction of surfaces which satisfy some given conditions (see, e.g., [17], [18]), in finite element method for differential equation problems (Lagrange operators for Dirichlet boundary conditions, Birkhoff operators for Neumann boundary conditions and Hermite operators for Robin boundary conditions) (see, e.g., [19], [25], [26], [35]) and in numerical integration of functions (see, e.g., [16]).

We study these operators especially from the theoretical point of view. The idea came from the paper of Barnhill and Gregory [4], where there is considered a triangle with one curved side and there are used Lagrange projectors on the straight sides, and Taylor projector on the curved side. Such operators were also studied in many other papers in connection with their applications in computer aided geometric design (see, e.g., [1], [2], [5]) and in finite element analysis (see, e.g., [1], [7], [20], [22], [23], [24], [35]).

We study three main aspects of the constructed operators: 1) the interpolation properties; 2) the orders of accuracy; 3) the remainders of the corresponding interpolation formulas.

The order of accuracy of an interpolation operator P is given by *the degree of exactness* ($\text{dex}(P)$), respectively by *the precision set* ($\text{pres}(P)$). Recall that $\text{dex}(P) = r$ if $Pf = f$, for $f \in \mathbb{P}_r$ and there exists $g \in \mathbb{P}_{r+1}$ such that $Pg \neq g$, where $\mathbb{P}_m$ denotes the space of the polynomials in 2 variables of global degree at most m. The precision set of an interpolation operator is the set of monomials for which the interpolant is exact ([4]).

The characteristics 1) and 2) can be verified by a straightforward computation. The remainders of the interpolation formulas will be studied using the Peano's Theorem for the functions from a Sard-type space (see, e.g., [31]). The Sard-type space, denoted by $B_{pq}(a,c)$, $(p,q \in \mathbb{N},\ p+q=m)$, is the space of the functions $f : D \to \mathbb{R}$, $D = [a,b] \times [c,d]$ satisfying $f^{(p,q)} \in C(D)$; $f^{(m-j,j)}(\cdot,c) \in C[a,b]$, $j < q$; $f^{(i,m-i)}(a,\cdot) \in C[c,d]$, $i < p$.

Given $h > 0$, denote by $\tilde{T}_h$ the triangle having the vertices $V_1 = (h,0)$, $V_2 = (0,h)$ and $V_3 = (0,0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$ (see Figure 1).

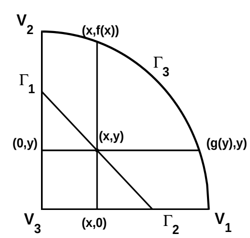


Figure 1: Triangle $\tilde{T}_h$.

There is no restriction in considering this standard triangle $\tilde{T}_h$, since any triangle with one curved side can be obtained from this standard triangle by an affine transformation which preserves the form and order of accuracy of the interpolant ([4]).

In Section 2 we study Lagrange, Hermite and Birkhoff interpolation operators, as well as their product and Boolean sum on $\tilde{T}_h$. In Section 3 we present some Bernstein-type operators together with their product and Boolean sum for the same triangle $\tilde{T}_h$. Section 4 is dedicated to Bernstein-type operators defined on a triangle with all curved sides, denoted by $\tilde{T}'_h$. This triangle has the vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$, and the three curved sides γ_1, γ_2 (along the coordinate axes) and γ_3 (opposite to the vertex V_3). We define γ_1 by $(x, f_1(x))$, with $f_1(0) = f_1(h) = 0$, $f_1(x) \leq 0$, for $x \in [0, h]$; γ_2 defined by $(g_2(y), y)$, with $g_2(0) = g_2(h) = 0$, $g_2(y) \leq 0$, for $y \in [0, h]$ and γ_3 defined by the one-to-one functions f_3 and g_3, where g_3 is the inverse of the function f_3, i.e., $y = f_3(x)$ and $x = g_3(y)$, with $x, y \in [0, h]$ and $f_3(0) = g_3(0) = h$, $h \in \mathbb{R}_+$, (see Figure 2). For example, f_1 and g_2 can be convex functions.

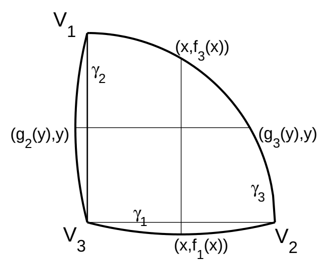


Figure 2. Triangle $\tilde{T}'_h$.

2. Interpolation operators on a triangle with one curved side

2.1. Lagrange-type operators

Suppose that F is a real-valued function defined on $\tilde{T}_h$. Let L_1, L_2 and L_3 be the interpolation operators defined by

$$
\begin{aligned}
(L_1F)(x,y) &= \frac{g(y)-x}{g(y)}F(0,y) + \frac{x}{g(y)}F(g(y),y),\\
(L_2F)(x,y) &= \frac{f(x)-y}{f(x)}F(x,0) + \frac{y}{f(x)}F(x,f(x)),\\
(L_3F)(x,y) &= \frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y).
\end{aligned} \tag{1}
$$

1) Each of the operators L_1, L_2 and L_3 interpolates the function F along two sides of the triangle $\tilde{T}_h$:

$$
\begin{aligned}
&(L_1F)(0,y) = F(0,y), \quad (L_1F)(g(y),y) = F(g(y),y), \quad y \in [0,h],\\
&(L_2F)(x,0) = F(x,0), \quad (L_2F)(x,f(x)) = F(x,f(x)), \quad x \in [0,h],\\
&(L_3F)(x+y,0) = F(x+y,0), \;\; (L_3F)(0,x+y) = F(0,x+y), \;\; x,y \in [0,h],
\end{aligned}
$$

properties illustrated in Figure 3. The bold sides and points indicate the interpolation domains.

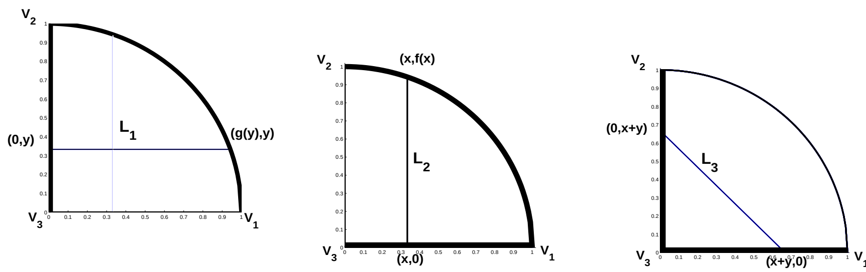


Figure 3. The interpolation domains for L_1, L_2 and L_3.

2) The orders of accuracy:

$$
\begin{aligned}
\text{(2)} \qquad &\operatorname{dex}(L_i) = 1, \quad i = 1,2,3, \quad \operatorname{pres}(L_1) = \{1, x, y^j, \quad j \in \mathbb{N}^*\},\\
&\operatorname{pres}(L_2) = \{1, x^i, y, \quad i \in \mathbb{N}^*\}, \quad \operatorname{pres}(L_3) = \{1, x, y\}.
\end{aligned}
$$

3) Regarding the remainders $R_i^L F$, $i = 1,2,3$, of the interpolation formulas $F = L_iF + R_i^LF$, $\quad i = 1,2,3$, we have:

Theorem 1 ([14])**.** *If* $F \in B_{11}(0,0)$ *then*

$$
\begin{aligned}
(R_1^LF)(x,y) = {} & \frac{x[x-g(y)]}{2} F^{(2,0)}(\xi,0)\\
& + \frac{xy[g(y)-x]}{g(y)} \left[F^{(1,1)}(\xi_1,\eta_1) - F^{(1,1)}(\xi_2,\eta_2)\right],
\end{aligned}
$$

with $\xi \in [0,h]$, $(\xi_1,\eta_1) \in [0,x] \times [0,y]$ *and* $(\xi_2,\eta_2) \in [x,g(y)] \times [0,y]$, *respectively*

$$
\text{(3)} \qquad \left|(R_1^LF)(x,y)\right| \le \frac{h^2}{8} \left[\|F^{(2,0)}(\cdot,0)\|_\infty + \|F^{(1,1)}\|_\infty\right],
$$

where $\|\cdot\|_\infty$ *denotes the Chebyshev norm.*

Proof. From (2) we have $\text{dex}(L_1) = 1$ and applying the Peano's Theorem we get

$$(R_1^L F)(x,y) = \int_0^h K_{20}(x,y,s)F^{(2,0)}(s,0)ds + \iint_{\tilde{T}_h} K_{11}(x,y,s,t)F^{(1,1)}(s,t)dsdt, \tag{4}$$

with the Peano's kernels given by

$$\begin{aligned} K_{20}(x,y,s) &= (x-s)_+ - \frac{x}{g(y)}(g(y)-s)_+, \\ K_{11}(x,y,s,t) &= (y-t)_+^0[(x-s)_+^0 - \frac{x}{g(y)}(g(y)-s)_+^0]. \end{aligned} \tag{5}$$

As,

$$\begin{array}{ll} K_{20}(x,y,s) \le 0, & s \in [0,h], \\ K_{11}(x,y,s,t) \ge 0, & (s,t) \in [0,x] \times [0,y], \\ K_{11}(x,y,s,t) \le 0, & (s,t) \in [x,g(y)] \times [0,y], \\ K_{11}(x,y,s,t) = 0, & (s,t) \in D_1 \times D_2, \end{array} \tag{6}$$

with D_1 and D_2 illustrated in Figure 4, by the Mean Value Theorem we obtain

$$\begin{aligned} (R_1^L F)(x,y) = {} & \varphi_{20}(x,y)F^{(2,0)}(\xi,0) + \varphi_{11}^1(x,y)F^{(1,1)}(\xi_1,\eta_1) \\ & + \varphi_{11}^2(x,y)F^{(1,1)}(\xi_2,\eta_2), \end{aligned}$$

with $\xi \in [0,h]$, $(\xi_1,\eta_1) \in [0,x] \times [0,y]$, $(\xi_2,\eta_2) \in [x,g(y)] \times [0,y]$, and

$$\begin{aligned} \varphi_{20}(x,y) &= \int_0^h K_{20}(x,y,s)ds = \frac{x(x-g(y))}{2}, \\ \varphi_{11}^1(x,y) &= \int_0^y \int_0^x K_{11}(x,y,s,t)dsdt = \frac{xy[g(y)-x]}{g(y)}, \\ \varphi_{11}^2(x,y) &= \int_0^y \int_x^{g(y)} K_{11}(x,y,s,t)dsdt = -\frac{xy[g(y)-x]}{g(y)}. \end{aligned} \tag{7}$$

As $|\varphi_{20}(x,y)| \le \frac{h^2}{8}$, $\left|\varphi_{11}^1(x,y)\right| \le \frac{h^2}{16}$, $\left|\varphi_{11}^2(x,y)\right| \le \frac{h^2}{16}$, relation (3) follows.□

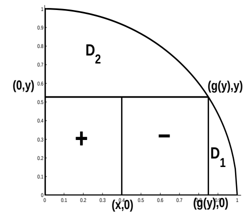


Figure 4: The sign of the kernel K_{11}.

Remark 2. Analogous formulas can be obtained for the remainders $R_2^L F$ and $R_3^L F$.

Let P_{ij}^L be the product of the operators L_i and L_j, i.e., $P_{ij} = L_i L_j$, $i, j = 1, 2, 3$, $i \neq j$. We have

$$(P_{12}^L F)(x, y) = \frac{h-y}{h}\frac{g(y)-x}{g(y)}F(0,0) + \frac{y}{h}\frac{g(y)-x}{g(y)}F(0,h) + \frac{x}{g(y)}F(g(y),y),$$

$$(P_{13}^L F)(x, y) = \frac{g(y)-x}{g(y)}F(0,y) + \frac{x}{g(y)[y+g(y)]}[g(y)F(y+g(y),0) + yF(0,y+g(y))],$$

$$(P_{23}^L F)(x, y) = \frac{f(x)-y}{f(x)}F(x,0) + \frac{y}{f(x)[x+f(x)]}[xF(x+f(x),0) + f(x)F(0,x+f(x))].$$

1) The interpolation properties: $P_{12}^L F = F$, on $\Gamma_3 \cup V_3$; $P_{13}^L F = F$, on $\Gamma_1 \cup V_1$; $P_{23}^L F = F$, on $\Gamma_2 \cup V_2$.

Remark 3. The operator P_{ij}^L has the same interpolation properties as the operator P_{ji}^L, $i, j = 1, 2, 3$, $i \neq j$. These properties are illustrated in Figure 5.

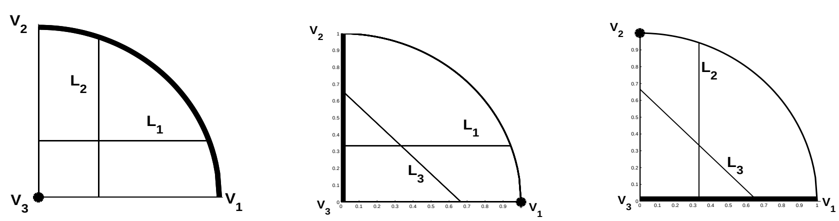


Figure 5. The interpolation domains for P_{12}^L, P_{13}^L and P_{23}^L.

2) The orders of accuracy: $\text{dex}(P_{ij}^L) = 1$, $\text{pres}(P_{ij}^L) = \{1, x, y\}$, $\quad i, j = 1, 2, 3,\ i \neq j$.

3) For the remainders $R_{ij}^P F$, of the interpolation formulas $F = P_{ij}^L F + R_{ij}^{LP} F$, $\quad i, j = 1, 2, 3,\ i \neq j$, we have:

Theorem 4 ([14])**.** *If $F \in B_{11}(0,0)$ then*

$$(R_{12}^{LP} F)(x,y) = \frac{x[x - g(y)]}{2} F^{(2,0)}(\xi, 0) + \frac{y(y-h)[g(y) - x]}{2g(y)} F^{(0,2)}(0, \eta) + \frac{xy[g(y) - x]}{g(y)} [F^{(1,1)}(\xi_1, \eta_1) - F^{(1,1)}(\xi_2, \eta_2)],$$

with $\xi, \eta \in [0, h]$, $(\xi_1, \eta_1) \in [0, x] \times [0, y]$ and $(\xi_2, \eta_2) \in [x, g(y)] \times [0, y]$, respectively

$$\left|(R_{12}^{LP} F)(x,y)\right| \leq \frac{h^2}{8} \left[\|F^{(2,0)}(\cdot, 0)\|_\infty + \|F^{(0,2)}(0, \cdot)\|_\infty + \|F^{(1,1)}\|_\infty\right]. \tag{8}$$

Proof. By $\text{dex}(P_{12}^L) = 1$, applying Peano's Theorem we get that

$$(R_{12}^{LP} F)(x,y) = \int_0^h K_{20}(x, y, s) F^{(2,0)}(s, 0) ds + \int_0^h K_{02}(x, y, t) F^{(0,2)}(0, t) dt + \iint_{\tilde{T}_h} K_{11}(x, y, s, t) F^{(1,1)}(s, t) ds dt, \tag{9}$$

with the Peano's kernels

$$K_{20}(x, y, s) = (x - s)_+ - \frac{x}{g(y)} [g(y) - s]_+,$$
$$K_{02}(x, y, t) = \frac{g(y) - x}{g(y)} \left[(y - t)_+ - \frac{y(h - t)}{h}\right],$$
$$K_{11}(x, y, s, t) = (y - t)_+^0 \{(x - s)_+^0 - \frac{x}{g(y)} [g(y) - s]_+^0\}.$$

We notice that the Peano's kernels K_{20} and K_{11} are the same as the kernels given in (5). Therefore, their sign is given in (6) and we have $K_{02}(x, y, t) \leq 0$, for $t \in [0, h]$. By the Mean Value Theorem we obtain

$$(R_{12}^{LP} F)(x,y) = \varphi_{20}(x,y) F^{(2,0)}(\xi, 0) + \varphi_{02}(x,y) F^{(0,2)}(0, \eta) + \varphi_{11}^1(x,y) F^{(1,1)}(\xi_1, \eta_1) + \varphi_{11}^2(x,y) F^{(1,1)}(\xi_2, \eta_2),$$

for $\xi, \eta \in [0,h]$, $(\xi_1,\eta_1) \in [0,x] \times [0,y]$, $(\xi_2,\eta_2) \in [x,g(y)] \times [0,y]$, with φ_{20}, φ_{11}^1, φ_{11}^2 given in (7) and

$$\varphi_{02}(x,y) = \int_0^h K_{02}(x,y,t)dt = \frac{y(y-h))[g(y)-x]}{2g(y)}.$$

We have $|\varphi_{20}(x,y)| \le \frac{h^2}{8}$, $|\varphi_{02}(x,y)| \le \frac{h^2}{8}$, $\left|\varphi_{11}^1(x,y)\right| \le \frac{h^2}{16}$, $\left|\varphi_{11}^2(x,y)\right| \le \frac{h^2}{16}$, so the relation (8) follows. □

Remark 5. Analogous formulas can be obtained for the remainders $R_{23}^{LP}F$ and $R_{13}^{LP}F$.

Let S_{ij}^L be the Boolean sum of the operators L_i and L_j, i.e., $S_{ij}^L = L_i \oplus L_j = L_i + L_j - L_iL_j$, $i,j = 1,2,3$, $i<j$ (see, e.g., [21]).

We have

$$\begin{aligned}\left(S_{12}^L F\right)(x,y) &= \frac{g(y)-x}{g(y)}F(0,y) + \frac{f(x)-y}{f(x)}F(x,0) + \frac{y}{f(x)}F(x,f(x)) \\ &\quad - \frac{g(y)-x}{g(y)}\left[\frac{h-y}{h}F(0,0) + \frac{y}{h}F(0,h)\right], \\ \left(S_{13}^L F\right)(x,y) &= \frac{x}{g(y)}F(g\left(y\right),y) + \frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y) - \\ &\quad - \frac{x}{g(y)}\left[\frac{g(y)}{y+g(y)}F(y+g(y),0) + \frac{y}{y+g(y)}F(0,y+g(y))\right],\end{aligned}$$

$$\begin{aligned}\left(S_{23}^L F\right)(x,y) &= \frac{y}{f(x)}F(x,f\left(x\right)) + \frac{x}{x+y}F(x+y,0) + \frac{y}{x+y}F(0,x+y) - \\ &\quad - \frac{y}{f(x)}\left[\frac{y}{x+f(x)}F(x+f(x),0) + \frac{f(x)}{x+f(x)}F(0,x+f(x))\right].\end{aligned}$$

1) The interpolation properties: $S_{ij}^L F = F$, $i,j = 1,2,3$, $i<j$, on $\partial \tilde{T}_h$.

2) The orders of accuracy:

$$\begin{aligned} &\mathrm{dex}(S_{12}^L) = 1, \quad \mathrm{dex}(S_{13}^L) = \mathrm{dex}(S_{23}^L) = 2, \\ &\mathrm{pres}(S_{12}^L) = \{1, y, xy, x^k, \quad k \in \mathbb{N}^*\}, \\ &\mathrm{pres}(S_{13}^L) = \{1, x, y, x^2, y^2, xy^k, \quad k \in \mathbb{N}^*\}, \\ &\mathrm{pres}(S_{23}^L) = \{1, x, y, x^2, y^2, x^k y, \quad k \in \mathbb{N}^*\}. \end{aligned} \tag{10}$$

3) For the remainders $R_{ij}^{LS}F$, of the interpolation formulas $F = S_{ij}^L F + R_{ij}^{LS}F$, $i,j = 1,2,3$, $i<j$, we have:

Theorem 6 ([14]). *If $F \in B_{11}(0,0)$ then*

$$(R_{12}^{LS}F)(x,y) = \int_0^h K_{02}(x,y,t)F^{(0,2)}(0,t)dt + \iint_{\tilde{T}_h} K_{11}(x,y,s,t)F^{(1,1)}(s,t)dsdt, \tag{11}$$

with the Peano's kernels

$$K_{02}(x,y,t) = \frac{x}{g(y)}(y-t)_+ - \frac{x}{f(x)}[f(x)-t]_+ + \frac{[g(y)-x]y}{hg(y)}(h-t),$$
$$K_{11}(x,y,s,t) = (x-s)_+^0\{(y-t)_+^0 - \frac{y}{f(x)}[f(x)-t]_+^0\}.$$

Furthermore,

$$\left|(R_{12}^{LS}F)(x,y)\right| \leq \|F^{(2,0)}(\cdot,0)\|_\infty \int_0^h |K_{02}(x,y,t)|dt + \|F^{(1,1)}\|_\infty \iint_{\tilde{T}_h} |K_{11}(x,y,s,t)|dsdt.$$

Proof. The proof follows directly by Peano's Theorem, taking into account that $\operatorname{dex}(S_{12}^L) = 1$. □

Theorem 7 ([14]). *If $F \in B_{12}(0,0)$ then*

$$(R_{13}^{LS}F)(x,y) = \int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds + \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds + \int_0^h K_{03}(x,y,t)F^{(0,3)}(0,t)dt + \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dsdt, \tag{12}$$

with the Peano's kernels

$$K_{30}(x,y,s) = \frac{(x-s)_+^2}{2} - \frac{x[g(y)-s]_+^2}{2g(y)} - \frac{x(x+y-s)_+^2}{2(x+y)} + \frac{x[y+g(y)-s]_+^2}{2(y+g(y))},$$
$$K_{21}(x,y,s) = y(x-s)_+ - \frac{xy[g(y)-s]_+}{g(y)},$$
$$K_{03}(x,y,t) = \frac{[g(y)-x](y-t)_+^2}{2g(y)} - \frac{y(x+y-t)_+^2}{2(x+y)} + \frac{xy(y+g(y)-t)_+^2}{g^2(y)[y+g(y)]},$$
$$K_{12}(x,y,s,t) = (y-t)_+\{(x-s)_+^0 - \frac{x}{g(y)}[g(y)-s]_+^0\}.$$

Furthermore,

$$
\begin{aligned}
\left|(R_{13}^{LS}F)(x,y)\right| \leq & \|F^{(3,0)}(\cdot,0)\|_\infty \int_0^h |K_{30}(x,y,s)|ds \\
& + \|F^{(2,1)}(\cdot,0)\|_\infty \int_0^h |K_{21}(x,y,s)|ds \\
& + \|F^{(0,3)}(0,\cdot)\|_\infty \int_0^h |K_{03}(x,y,t)|dt \\
& + \|F^{(1,2)}\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|dsdt.
\end{aligned} \tag{13}
$$

Proof. From (10) it follows that $\mathrm{dex}(S_{13}^L) = 2$ and applying Peano's Theorem we get (12) and the inequality (13). □

Remark 8. An analogous formula can be obtained for $R_{23}^{LS}F$.

2.2. Hermite-type operators

Suppose that the real valued function F is defined on the triangle $\tilde{T}_h$ and it possesses the partial derivatives $F^{(1,0)}$ and $F^{(0,1)}$ on the side Γ_3. We consider the operators H_1 and H_2 defined by

$$
\begin{aligned}
(H_1F)(x,y) = & \frac{[x-g(y)]^2}{g^2(y)}F(0,y) + \frac{x[2g(y)-x]}{g^2(y)}F(g(y),y) \\
& + \frac{x[x-g(y)]}{g(y)}F^{(1,0)}(g(y),y), \\
(H_2F)(x,y) = & \frac{[y-f(x)]^2}{f^2(x)}F(x,0) + \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x)) \\
& + \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)).
\end{aligned} \tag{14}
$$

1) The interpolation properties:

$$
\begin{aligned}
& H_1F = F, \text{ on } \Gamma_1 \cup \Gamma_3; \quad (H_1F)^{(1,0)} = F^{(1,0)}, \text{ on } \Gamma_3 \\
& H_2F = F, \text{ on } \Gamma_2 \cup \Gamma_3; \quad (H_2F)^{(0,1)} = F^{(0,1)}, \text{ on } \Gamma_3.
\end{aligned}
$$

These interpolation properties are illustrated in Figure 6.

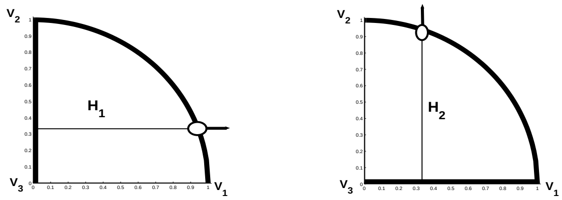


Figure 6. The interpolation domains for H_1 and H_2.

2) The orders of accuracy:

$$\begin{aligned} &\mathrm{dex}(H_1) = \mathrm{dex}(H_2) = 2, \\ &\mathrm{pres}(H_1) = \{1, x, y, x^2, y^2, xy^n, n \in \mathbb{N}^*\}, \\ &\mathrm{pres}(H_2) = \{1, x, y, x^2, y^2, x^n y, n \in \mathbb{N}^*\}. \end{aligned} \tag{15}$$

3) The interpolation formulas are $F = H_i F + R_i^H F$, $i = 1, 2$, where $R_i^H F$, $i = 1, 2$ are the remainder terms, for which we have:

Theorem 9 ([14])**.** *If* $F \in B_{12}(0,0)$ *then the following inequality holds*

$$\begin{aligned} \left|(R_1^H F)(x,y)\right| \leq &\frac{x[g(y)-x]^2}{6}\|F^{(3,0)}(\cdot,0)\|_\infty + \frac{xy[g(y)-x]^2}{2g(y)-x}\|F^{(2,1)}(\cdot,0)\|_\infty \\ &+ \frac{xy^2[g(y)-x][3g(y)-2x]}{2g^2(y)}\|F^{(1,2)}(\cdot,\cdot)\|_\infty, \end{aligned} \tag{16}$$

and further,

$$\begin{aligned} \left|(R_1^H F)(x,y)\right| \leq &\frac{2h^3}{81}\|F^{(3,0)}(\cdot,0)\|_\infty + \frac{xy[g(y)-x]^2}{2g(y)-x}\|F^{(2,1)}(\cdot,0)\|_\infty \\ &+ \frac{xy^2[g(y)-x][3g(y)-2x]}{2g^2(y)}\|F^{(1,2)}(\cdot,\cdot)\|_\infty. \end{aligned} \tag{17}$$

Proof. By (15) it follows that $\mathrm{dex}(H_1) = 2$, and therefore by Peano's Theorem we get

$$\begin{aligned} (R_1^H F)(x,y) = &\int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds + \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds \\ &+ \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dsdt, \end{aligned} \tag{18}$$

with

$$\begin{aligned}
K_{30}(x,y,s) &= \frac{(x-s)_+^2}{2} - \frac{x[2g(y)-x]}{g^2(y)}\frac{(g(y)-s)_+^2}{2}\\
&- \frac{x[x-g(y)]}{g(y)}(g(y)-s)_+,\\
K_{21}(x,y,s) &= y(x-s)_+ - \frac{xy[2g(y)-x]}{g^2(y)}(g(y)-s)_+\\
&- \frac{xy[x-g(y)]}{g(y)}(g(y)-s)_+^0,\\
K_{12}(x,y,s,t) &= (y-t)_+[(x-s)_+^0\\
&- \frac{x[2g(y)-x]}{g^2(y)}(g(y)-s)_+^0].
\end{aligned}$$

We have

$$K_{30}(x,y,s) = \begin{cases} \dfrac{s^2[g(y)-x]^2}{2g^2(y)} \geq 0, & s \in [0,x) \\ \dfrac{x[g(y)-s]}{2g^2(y)}[(s-x)g(y)+s[g(y)-x] \geq 0, & s \in [x,g(y)) \\ 0, & s \in [g(y),h], \end{cases}$$

$$K_{21}(x,y,s) = \begin{cases} -\dfrac{1}{g^2(y)}sy[g(y)-x]^2 \leq 0, & s \in [0,x) \\ -\dfrac{xy}{g^2(y)}[g^2(y)-s(2g(y)-x)] \leq 0, & s \in [x, \frac{g^2(y)}{2g(y)-x}) \\ -\dfrac{xy}{g^2(y)}[g^2(y)-s(2g(y)-x)] \geq 0, & s \in [\frac{g^2(y)}{2g(y)-x}, g(y)) \\ 0, & s \in [g(y),h], \end{cases}$$

and

$$K_{12}(x,y,s,t) = (y-t)_+ \begin{cases} \dfrac{[g(y)-x]^2}{g^2(y)} \geq 0, & (s,t) \in [0,x) \times [0,y) \\ \dfrac{x[x-2g(y)]}{g^2(y)} \leq 0, & (s,t) \in [x,g(y)) \times [0,y) \\ 0, & (s,t) \in D_1 \cup D_2, \end{cases}$$

with domains D_1, D_2 and the sign of K_{12} as in Figure 4. We obtain that

$$
\begin{aligned}
&\left|(R_1^H F)(x,y)\right| \\
&\le \|F^{(3,0)}(\cdot,0)\|_\infty \int_0^h K_{30}(x,y,s)ds + \|F^{(2,1)}(\cdot,0)\|_\infty \int_0^h |K_{21}(x,y,s)|\,ds \\
&+ \|F^{(1,2)}(\cdot,\cdot)\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|\,dsdt,
\end{aligned}
$$

whence, after some computation we get (16), and further we obtain (17). □

Remark 10. An analogous formula can be obtained for the remainder $R_2^H F$.

The product of the operators H_1 and H_2 is given by

$$
\begin{aligned}
(P_{12}^H F)(x,y) =& \frac{[x-g(y)]^2}{g^2(y)} \left[\frac{(y-h)^2}{h^2} F(0,0) \right. \\
&\left. + \frac{y(2h-y)}{h^2} F(0,h) + \frac{y(y-h)}{h} F^{(0,1)}(0,h) \right] \\
&+ \frac{x[2g(y)-x]}{g^2(y)} F(g(y),y) + \frac{x[x-g(y)]}{g(y)} F^{(1,0)}(g(y),y).
\end{aligned}
$$

1) The interpolation properties:

$$
\begin{aligned}
&P_{12}^H F = F, \qquad \text{on } V_3 \cup \Gamma_3, \\
&(P_{12}^H F)^{(1,0)} = F^{(1,0)}, \quad (P_{12}^H F)^{(0,1)} = F^{(0,1)}, \quad \text{on } \Gamma_3.
\end{aligned}
$$

The interpolation properties are illustrated in Figure 7.

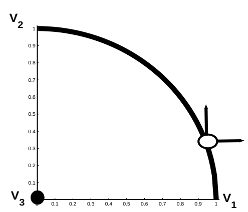


Figure 7. The interpolation domain for $P_{12}^H F$.

2) The orders of accuracy:

$$
\text{(19)} \qquad \operatorname{dex}(P_{12}^H) = 2, \quad \operatorname{pres}(P_{12}^H) = \{1, x, y, x^2, xy, y^2, x^2y, xy^2\}.
$$

For the remainder of the corresponding interpolation formula, $F = P_{12}^H F + R_{12}^{HP} F$, we have:

Theorem 11 ([14]). *If $F \in B_{12}(0,0)$ then the following inequality holds*

$$
\begin{aligned}
\left|(R_{12}^{HP}F)(x,y)\right| \leq\ & \frac{x[g(y)-x]^2}{6}\|F^{(3,0)}(\cdot,0)\|_\infty \\
& + \frac{xy[g(y)-x]^2}{2g(y)-x}\|F^{(2,1)}(\cdot,0)\|_\infty \\
& + \frac{y[g(y)-x]^2(h-y)}{6g^2(y)}\|F^{(0,3)}(0,\cdot)\|_\infty \\
& + \frac{xy[g(y)-x][3g(y)-2x]}{g^2(y)}\|F^{(1,2)}(\cdot,\cdot)\|_\infty.
\end{aligned} \tag{20}
$$

Proof. By (19) it follows that $\text{dex}(P_{12}^H) = 2$ and applying Peano's Theorem we get

$$
\begin{aligned}
(R_{12}^{HP}F)(x,y) =& \int_0^h K_{30}(x,y,s)F^{(3,0)}(s,0)ds + \int_0^h K_{21}(x,y,s)F^{(2,1)}(s,0)ds \\
&+ \int_0^h K_{03}(x,y,t)F^{(0,3)}(0,t)dt + \iint_{\tilde{T}_h} K_{12}(x,y,s,t)F^{(1,2)}(s,t)dsdt,
\end{aligned}
$$

with

$$
\begin{aligned}
K_{30}(x,y,s) =& \frac{(x-s)_+^2}{2} - \frac{x[2g(y)-x]}{g^2(y)}\frac{(g(y)-s)_+^2}{2} - \frac{x[x-g(y)]}{g(y)}(g(y)-s)_+, \\
K_{21}(x,y,s) =& y(x-s)_+ - \frac{xy[2g(y)-x]}{g^2(y)}(g(y)-s)_+ - \frac{xy[x-g(y)]}{g(y)}, \\
K_{03}(x,y,t) =& \frac{(y-t)_+^2}{2} - \frac{x[2g(y)-x]}{g^2(y)}\frac{(y-t)_+^2}{2} \\
& - \frac{[x-g(y)]^2}{g^2(y)}\left[\frac{y(2h-y)}{h^2}\frac{(h-t)^2}{2} + \frac{y(y-h)(h-t)}{h}\right] \\
K_{12}(x,y,s,t) =& (y-t)_+\left[(x-s)_+^0 - \frac{x[2g(y)-x]}{g^2(y)}\right].
\end{aligned}
$$

We have

$$
K_{30}(x,y,s) = \begin{cases}
\dfrac{s^2[g(y)-x]^2}{2g^2(y)} \geq 0, & s \in [0,x) \\
-\dfrac{x[g(y)-s]}{2g^2(y)}[(x-s)g(y)+(x-g(y))s] \geq 0, & s \in [x, g(y)) \\
0, & s \in [g(y), h],
\end{cases}
$$

$$K_{21}(x,y,s)=\begin{cases} -\dfrac{sy}{g^2(y)}[g(y)-x]^2 \le 0, & s\in[0,x)\\ -\dfrac{xy}{g^2(y)}[g(y)(g(y)-s)+s(x-g(y))]\le 0, & s\in[x,\frac{g^2(y)}{2g(y)-x})\\ -\dfrac{xy}{g^2(y)}[g(y)(g(y)-s)+s(x-g(y))]\ge 0, & s\in[\frac{g^2(y)}{2g(y)-x},g(y))\\ 0, & s\in[g(y),h], \end{cases}$$

$$K_{03}(x,y,t)=\begin{cases} \dfrac{[g(y)-x]^2}{2g^2(y)}t^2(h-y)^2\ge 0, & t\in[0,y)\\ -\dfrac{[x-g(y)]^2(h-t)y}{2h^2g^2(y)}[h(y-t)+t(y-h)]\ge 0, & t\in[y,h] \end{cases}$$

and

$$K_{12}(x,y,s,t)=\begin{cases} (y-t)\dfrac{[g(y)-x]^2}{g^2(y)}\ge 0, & (s,t)\in[0,x)\times[0,y)\\ (y-t)\dfrac{x[x-2g(y)]}{g^2(y)}\le 0, & (s,t)\in[x,g(y))\times[0,y)\\ 0, & (s,t)\in([0,h]\times[y,h])\cap\tilde{T}_h. \end{cases}$$

We obtain that

$$\begin{aligned} &\left|(R_{12}^{HP}F)(x,y)\right|\le\|F^{(3,0)}(\cdot,0)\|_\infty\int_0^h K_{30}(x,y,s)ds\\ &+\|F^{(2,1)}(\cdot,0)\|_\infty\int_0^h|K_{21}(x,y,s)|\,ds+\|F^{(0,3)}(0,\cdot)\|_\infty\int_0^h K_{03}(x,y,t)dt\\ &+\|F^{(1,2)}(\cdot,\cdot)\|_\infty\iint_{\tilde{T}_h}|K_{12}(x,y,s,t)|\,dsdt, \end{aligned}$$

whence, after some computation, we get (20). □

The Boolean sum of the operators H_1 and H_2 is given by

$$\begin{aligned} (S_{12}^HF)(x,y)&=\frac{[x-g(y)]^2}{g^2(y)}F(0,y)+\frac{[y-f(x)]^2}{f^2(x)}F(x,0)\\ &+\frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))+\frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x))\\ &-\frac{[x-g(y)]^2}{g^2(y)}\left[\frac{(y-h)^2}{h^2}F(0,0)+\frac{y(2h-y)}{h^2}F(0,h)+\frac{y(y-h)}{h}F^{(0,1)}(0,h)\right]. \end{aligned}$$

1) The interpolation properties:

$$\begin{aligned} &S_{12}^HF=F, \quad \text{on } \partial\tilde{T}_h\\ &(S_{12}^HF)^{(1,0)}=F^{(1,0)}, \quad (S_{12}^HF)^{(0,1)}=F^{(0,1)}, \quad \text{on } \Gamma_3. \end{aligned}$$

2) The orders of accuracy: $\mathrm{dex}(S_{12}^H) = 2$, $\mathrm{pres}(S_{12}^H) = \{1, x, y, x^2, y^2, x^n y, n \in \mathbb{N}^*\}$.

For the remainder of the interpolation formula, $F = S_{12}^H F + R_{12}^{HS} F$, we have:

Theorem 12 ([14])**.** *If $F \in B_{12}(0,0)$ then the following inequality holds*

$$\begin{aligned}\left|(R_{12}^{HS}F)(x,y)\right| \leq & \|F^{(0,3)}(0,\cdot)\|_\infty \int_0^h |K_{03}(x,y,t)| dt \\ & + \|F^{(1,2)}(\cdot,\cdot)\|_\infty \iint_{\tilde{T}_h} |K_{12}(x,y,s,t)|\, dsdt.\end{aligned}$$

Proof. By (15) it follows that $\mathrm{dex}(S_{12}^H) = 2$ and therefore by Peano's Theorem we get

$$\begin{aligned}(R_{12}^{HS}F)(x,y) = & \int_0^h K_{30}(x,y,s) F^{(3,0)}(s,0) ds + \int_0^h K_{21}(x,y,s) F^{(2,1)}(s,0) ds \\ & + \int_0^h K_{03}(x,y,t) F^{(0,3)}(0,t) dt + \iint_{\tilde{T}_h} K_{12}(x,y,s,t) F^{(1,2)}(s,t) dsdt,\end{aligned}$$

with

$$\begin{aligned}K_{30}(x,y,s) =& K_{21}(x,y,s) = 0, \\ K_{03}(x,y,t) =& \frac{(y-t)_+^2}{2} \frac{g^2(y) - (x-g(y))^2}{g^2(y)} \\ & - \frac{y(2f(x)-y)}{f^2(x)} \frac{(f(x)-t)_+^2}{2} - \frac{y(y-f(x))(f(x)-t)_+}{f(x)} \\ & + \frac{(x-g(y))^2}{g^2(y)} \left[\frac{y(2h-y)(h-t)^2}{2h^2} + \frac{y(y-h)(h-t)}{h} \right], \\ K_{12}(x,y,s,t) =& (x-s)_+^0 [(y-t)_+ - \frac{y[2f(x)-y]}{f^2(x)} (f(x)-t)_+ \\ & - \frac{y(y-f(x))}{f(x)} (f(x)-t)_+^0].\end{aligned}$$

We have

$$K_{03}(x,y,t)=\begin{cases}\frac{(y-t)^2}{2}\frac{g^2(y)-(x-g(y))^2}{g^2(y)}-\frac{y(2f(x)-y)}{f^2(x)}\frac{(f(x)-t)^2}{2}\\ +\frac{(x-g(y))^2}{g^2(y)}\left[\frac{y(2h-y)(h-t)^2}{2h^2}+\frac{y(y-h)(h-t)}{h}\right]\leq 0,\\ \hfill t\in[0,y),\\ -\frac{y(2f(x)-y)}{f^2(x)}\frac{(f(x)-t)^2}{2}\\ +\frac{(x-g(y))^2}{g^2(y)}\left[\frac{y(2h-y)(h-t)^2}{2h^2}+\frac{y(y-h)(h-t)}{h}\right],\\ \hfill t\in[y,f(x))\\ \frac{(x-g(y))^2}{g^2(y)}\left[\frac{y(2h-y)(h-t)^2}{2h^2}+\frac{y(y-h)(h-t)}{h}\right]\leq 0,\\ \hfill t\in[f(x),h],\end{cases}$$

and

$$K_{12}(x,y,s,t)$$

$$=(y-t)_+\cdot\begin{cases}(y-t)-\frac{y[2f(x)-y]}{f^2(x)}(f(x)-t)-\frac{y(y-f(x))}{f(x)}\leq 0,\\ \hfill (s,t)\in[0,x)\times[0,y),\\ -\frac{y[2f(x)-y]}{f^2(x)}(f(x)-t)-\frac{y(y-f(x))}{f(x)}\leq 0,\\ \hfill (s,t)\in[0,x)\times[y,\frac{f^2(x)}{2f(x)-y}),\\ -\frac{y[2f(x)-y]}{f^2(x)}(f(x)-t)-\frac{y(y-f(x))}{f(x)}\geq 0,\\ \hfill (s,t)\in[0,x)\times[\frac{f^2(x)}{2f(x)-y},f(x))\\ 0,\\ (s,t)\in([x,h]\times[0,f(x)]\cup[0,x]\times[f(x),h])\cap\tilde{T}_h.\end{cases}$$

We obtain that

$$\begin{aligned}\left|(R_{12}^{HS}F)(x,y)\right|\leq&\|F^{(0,3)}(0,\cdot)\|_\infty\int_0^h|K_{03}(x,y,t)|dt\\ &+\|F^{(1,2)}(\cdot,\cdot)\|_\infty\iint_{\tilde{T}_h}|K_{12}(x,y,s,t)|\,dsdt.\end{aligned}$$

□

2.3. Birkhoff-type operators

In this section we give some examples of operators which interpolate the given function $F : \tilde{T}_h \to \mathbb{R}$ on a side of the triangle and its first partial derivatives on another side of $\tilde{T}_h$, respectively.

First, we suppose that the function $F : \tilde{T}_h \to \mathbb{R}$ has the partial derivatives $F^{(1,0)}$ and $F^{(0,1)}$ on the side Γ_3. We consider the Birkhoff-type operators B_1 and B_2 defined respectively by

$$(B_1F)(x,y) = F(0,y) + xF^{(1,0)}(g(y),y),$$
$$(B_2F)(x,y) = F(x,0) + yF^{(0,1)}(x,f(x)).$$

1) The interpolation properties:

$$B_1F = F \text{ on } \Gamma_1 \text{ and } (B_1F)^{(1,0)} = F^{(1,0)} \text{ on } \Gamma_3,$$
$$B_2F = F \text{ on } \Gamma_2 \text{ and } (B_2F)^{(0,1)} = F^{(0,1)} \text{ on } \Gamma_3.$$

These interpolation properties are illustrated in Figure 8.

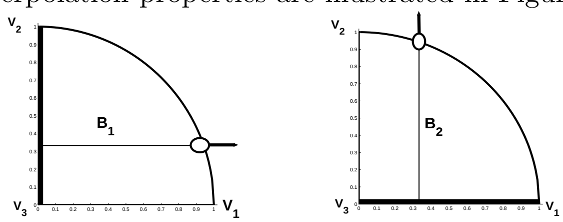


Figure 8. The interpolation domains for B_1 and B_2.

2) The orders of accuracy:

(21) $$\operatorname{dex}(B_1) = \operatorname{dex}(B_2) = 1, \quad \operatorname{pres}(B_1) = \left\{1, x, y^j, j \in \mathbb{N}^*\right\},$$
$$\operatorname{pres}(B_2) = \left\{1, y, x^i, i \in \mathbb{N}^*\right\}.$$

3) For the remainders of the interpolation formulas $F = B_1F + R_1^BF$ and $F = B_2F + R_2^BF$ we have:

Theorem 13 ([14]). *If $F \in B_{11}(0,0)$ then*

$$(R_1^BF)(x,y) = \frac{x[x-2g(y)]}{2}F^{(2,0)}(\xi,0) + xyF^{(1,1)}(\xi_1,\eta),$$

with $\xi \in [0,h]$, $(\xi_1,\eta) \in \tilde{T}_h$, respectively

(22) $$\left|(R_1^BF)(x,y)\right| \leq \frac{h^2}{2}\|F^{(2,0)}(\cdot,0)\|_\infty + \frac{h^2}{4}\|F^{(1,1)}(\cdot,\cdot)\|_\infty.$$

Proof. By (21) it follows that $\operatorname{dex}(B_1) = 1$, and applying Peano's Theorem we obtain

$$\left(R_1^B F\right)(x,y) = \int_0^h K_{20}(x,y,s)\, F^{(2,0)}(s,0)\, ds + \iint_{\tilde{T}_h} K_{11}(x,y,s,t)\, F^{(1,1)}(s,t)\, dsdt$$

with

$$\begin{aligned} K_{20}(x,y,s) &= (x-s)_+ - x\,[g(y)-s]_+^0\,, \\ K_{11}(x,y,s,t) &= (x-s)_+^0\,(y-t)_+^0\,. \end{aligned}$$

We have

$$\begin{array}{ll} K_{20}(x,y,s) \le 0, & s \in [0,h] \\ K_{11}(x,y,s,t) > 0, & (s,t) \in [0,x] \times [0,y] \\ K_{11}(x,y,s,t) = 0, & (s,t) \in (([x,h] \times [0,y]) \cup ([0,x] \times [y,h])) \cap \tilde{T}_h, \end{array}$$

and by the Mean Value Theorem we obtain

$$(R_1^B F)(x,y) = \varphi_{20}(x,y) F^{(2,0)}(\xi,0) + \varphi_{11}(x,y) F^{(1,1)}(\xi_1,\eta),$$

with $\xi \in [0,h]$, $(\xi_1,\eta) \in \tilde{T}_h$, and

$$\begin{aligned} \varphi_{20}(x,y) &= \int_0^h K_{20}(x,y,s)ds = \frac{x[x-2g(y)]}{2}, \\ \varphi_{11}(x,y) &= \int_0^y \int_0^x K_{11}(x,y,s,t)dsdt = xy. \end{aligned}$$

As, $|\varphi_{20}(x,y)| \le \frac{h^2}{2}$, $|\varphi_{11}(x,y)| \le \frac{h^2}{4}$, the relation (22) follows. □

Remark 14. Analogous formula can be obtained for the remainder $R_2^B F$.

Next, we suppose that the function $F : \tilde{T}_h \to \mathbb{R}$ admits the partial derivatives $F^{(1,0)}$ on Γ_1 and $F^{(0,1)}$ on Γ_2.

We consider the Birkhoff-type operators B_3 and B_4 defined by

$$\begin{aligned} (B_3 F)(x,y) &= F(g(y),y) + [x-g(y)] F^{(1,0)}(0,y)\,, \\ (B_4 F)(x,y) &= F(x,f(x)) + [y-f(x)] F^{(0,1)}(x,0)\,. \end{aligned}$$

1) The interpolation properties:

$$B_3F = F \text{ on } \Gamma_3 \text{ and } (B_3F)^{(1,0)} = F^{(1,0)} \text{ on } \Gamma_1,$$
$$B_4F = F \text{ on } \Gamma_3 \text{ and } (B_2F)^{(0,1)} = F^{(0,1)} \text{ on } \Gamma_2.$$

These properties are illustrated in Figure 9.

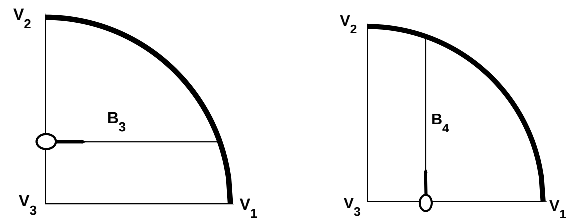


Figure 9. The interpolation domains for B_3 and B_4.

2) The orders of accuracy:

$$\operatorname{dex}(B_3) = \operatorname{dex}(B_4) = 1, \quad \operatorname{pres}(B_3) = \left\{1, x, y^j, j \in \mathbb{N}^*\right\},$$
$$\operatorname{pres}(B_4) = \left\{1, y, x^i, i \in \mathbb{N}^*\right\}.$$

3) For the remainders of the interpolation formulas $F = B_3F + R_3^BF$ and $F = B_4F + R_4^BF$, we have:

Theorem 15 ([14]). *If $F \in B_{11}(0,0)$ then*

$$(R_3^BF)(x,y) = \frac{x^2 - g^2(y)}{2} F^{(2,0)}(\xi, 0) + y[x - g(y)]F^{(1,1)}(\xi_1, \eta), \tag{23}$$

with $\xi \in [0, g(y)]$, $(\xi_1, \eta) \in \tilde{T}_h$, respectively

$$\left|(R_3^BF)(x,y)\right| \leq \frac{h^2}{2}\|F^{(2,0)}(\cdot, 0)\|_\infty + \frac{h^2}{4}\|F^{(1,1)}(\cdot,\cdot)\|_\infty. \tag{24}$$

Proof. As $\operatorname{dex}(B_1) = 1$, by Peano's Theorem, we obtain

$$\left(R_3^BF\right)(x,y) = \int_0^{g(y)} K_{20}(x,y,s)\, F^{(2,0)}(s,0)\, ds + \int_0^{g(y)}\int_0^h K_{11}(x,y,s,t)\, F^{(1,1)}(s,t)\, dsdt$$

where

$$K_{20}(x,y,s) = (x-s)_+ - [g(y)-s], K_{11}(x,y,s,t) = [(x-s)^0_+ - 1](y-t)^0_+ .$$

Taking into account that

$$\begin{array}{ll} K_{20}(x,y,s) \leq 0, & s \in [0, g(y)] \\ K_{11}(x,y,s,t) = -1, & (s,t) \in D = [0, g(y)-x] \times [0,y] \\ K_{11}(x,y,s,t) = 0, & (s,t) \in \tilde{T}_h \setminus D, \end{array}$$

and

$$\int_0^{g(y)} K_{20}(x,y,s)ds = \frac{x^2 - g^2(y)}{2},$$
$$\int_0^{g(y)} \int_0^h K_{11}(x,y,s,t)dsdt = y[x - g(y)],$$

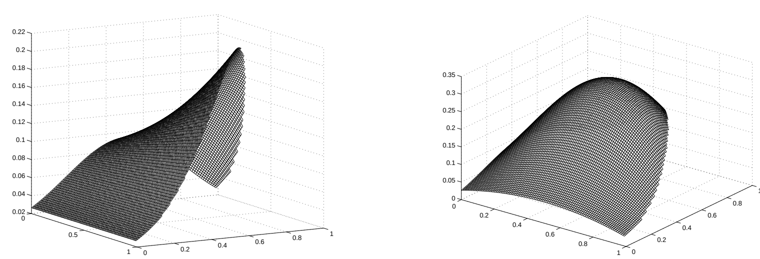
the relation (23) follows. Further, as $\left|\frac{x^2-g^2(y)}{2}\right| \leq \frac{h^2}{2}$, $|y[x-g(y)]| \leq \frac{h^2}{4}$, the inequality (24) follows. □

Remark 16. Analogous formula can be obtained for the remainder $R_4^B F$.

Example 17. We consider the following test function (see, e.g., [30]):

$$F_1(x,y) = \exp[-\frac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3. \quad \text{(Gentle)} \tag{25}$$

We take the triangle with one curved side $\tilde{T}_1$, $(h=1)$, with $f : [0,1] \to [0,1]$, $f(x) = \sqrt{1-x^2}$. In Figure 10 we plot the graphs of L_1F_1 and H_1F_1.



L_1F_1 H_1F_1

Figure 10. Graphs of L_1F_1 and H_1F_1.

Example 18. Table 1 contains some maximum approximation errors for F_1, defined on $\tilde{T}_1$.

Maximum error			
L_1F_1	0.2097	$\mathcal{B}_m^x F_1$	0.0525
$P_{13}F_1$	0.2943	$\mathcal{B}_n^y F_1$	0.0452
$S_{12}F_1$	0.1718	$P_{mn}F_1$	0.0858
H_1F_1	0.0758	$Q_{nm}F_1$	0.0857
B_1F_1	0.6235	$S_{mn}F_1$	0.0095
		$T_{nm}F_1$	0.0095

Table 1. Maximum approximation errors for F_1.

3. Bernstein-type operators on a triangle with one curved side

Since the Bernstein-type operators interpolate a given function at the endpoints of the interval, these operators can also be used as interpolation operators both on triangles with straight sides (see, e.g., [9], [33], [34]) and with curved sides.

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(0, y)$, $(g(y), y)$, respectively, $(x,\ 0)$, $(x, f(x))$ be the points where the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}_h$, intersect the sides Γ_i, $i = 1, 2, 3$, (see Figure 1). One considers the Bernstein-type operators $\mathcal{B}_m^x$ and $\mathcal{B}_n^y$ defined by

$$(\mathcal{B}_m^x F)(x, y) = \sum_{i=0}^{m} p_{m,i}(x, y)\, F\left(i\frac{g(y)}{m}, y\right),$$

$$(\mathcal{B}_n^y F)(x, y) = \sum_{j=0}^{n} q_{n,j}(x, y)\, F\left(x, j\frac{f(x)}{n}\right)$$

with

$$p_{m,i}(x, y) = \tbinom{m}{i}\Big(\frac{x}{g(y)}\Big)^i\Big(1 - \frac{x}{g(y)}\Big)^{m-i}, \quad 0 \leqslant x + y \leqslant g(y),$$

$$q_{n,j}(x, y) = \tbinom{n}{j}\Big(\frac{y}{f(x)}\Big)^j\Big(1 - \frac{y}{f(x)}\Big)^{n-j}, \quad 0 \leq x + y \leq f(x),$$

where $\Delta_m^x = \left\{ i\frac{g(y)}{m} \mid i = \overline{0, m} \right\}$ and $\Delta_n^y = \left\{ j\frac{f(x)}{n} \mid j = \overline{0, n} \right\}$ are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$.

Theorem 19 ([11]). *If F is a real-valued function defined on $\widetilde{T}_h$ then:*

(*i*) $\mathcal{B}_m^x F = F$ *on* $\Gamma_2 \cup \Gamma_3$, *and* $\mathcal{B}_n^y F = F$ *on* $\Gamma_1 \cup \Gamma_3$,

(*ii*) $(\mathcal{B}_m^x e_{ij})(x,y) = x^i y^j, \quad i = 0,1;\ (\mathcal{B}_m^x e_{2j})(x,y) = \left[x^2 + \frac{x(g(y)-x)}{m}\right] y^j$, $j \in \mathbb{N}$,

(*iii*) $(\mathcal{B}_n^y e_{ij})(x,y) = x^i y^j, \quad j = 0,1;\ (\mathcal{B}_n^y e_{i2})(x,y) = x^i \left[y^2 + \frac{y(f(x)-y)}{n}\right]$, $i \in \mathbb{N}$.

Proof. The interpolation properties (i) follow from the relations:

$$p_{m,i}(0,y) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases} \qquad p_{m,i}(g(y),y) = \begin{cases} 0, & \text{for } i < m, \\ 1, & \text{for } i = m, \end{cases}$$

respectively by

$$q_{n,j}(x,0) = \begin{cases} 1, & \text{for } j = 0, \\ 0, & \text{for } j > 0, \end{cases} \qquad q_{n,j}(x,f(x)) = \begin{cases} 0, & \text{for } j < n, \\ 1, & \text{for } j = n. \end{cases}$$

Regarding the properties (ii), we have

$$(\mathcal{B}_m^x e_{ij})(x,y) = y^j (\mathcal{B}_m^x e_{i0})(x,y), \qquad j \in \mathbb{N}$$

$$(\mathcal{B}_m^x e_{00})(x,y) = \left(\frac{x}{g(y)} + 1 - \frac{x}{g(y)}\right)^m = 1,$$

$$\begin{aligned}\mathcal{B}_m^x e_{10}(x,y) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{g(y)}\right)^i \left(1 - \frac{x}{g(y)}\right)^{m-i} i\frac{g(y)}{m} \\ &= x \sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{x}{g(y)}\right)^i \left(1 - \frac{x}{g(y)}\right)^{m-1-i} = x,\end{aligned}$$

$$\begin{aligned}\mathcal{B}_m^x e_{20}(x,y) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{g(y)}\right)^i \left(1 - \frac{x}{g(y)}\right)^{m-i} i^2 \left(\frac{g(y)}{m}\right)^2 \\ &= \left(\frac{g(y)}{m}\right)^2 \sum_{i=0}^{m} \binom{m}{i} i(i-1) \left(\frac{x}{g(y)}\right)^i \left(1 - \frac{x}{g(y)}\right)^{m-i} + x\frac{g(y)}{m} \\ &= \frac{m-1}{m} x^2 + x\frac{g(y)}{m} = x^2 + \frac{x[g(y)-x]}{m}.\end{aligned}$$

Properties (iii) are proved in the same way. □

Now, we consider the approximation formula $F = \mathcal{B}_m^x F + R_m^x F$.

Theorem 20 ([11])**.** *If $F(\cdot, y) \in C[0, g(y)]$ then*

$$\left| (R_m^x F)(x,y) \right| \leq \left(1 + \frac{h}{2\delta\sqrt{m}}\right) \omega(F(\cdot, y); \delta), \quad y \in [0,h],$$

where $\omega(F(\cdot, y); \delta)$ is the modulus of continuity of the function F with regard to the variable x. Moreover, if $\delta = 1/\sqrt{m}$ then

$$\left| (R_m^x F)(x,y) \right| \leq \left(1 + \frac{h}{2}\right) \omega(F(\cdot, y); \frac{1}{\sqrt{m}}), \quad y \in [0,h]. \tag{26}$$

Proof. From the property $(\mathcal{B}_m^x e_{00})(x,y) = 1$, it follows that

$$\left| (R_m^x F)(x,y) \right| \leq \sum_{i=0}^{m} p_{m,i}(x,y) \left| F(x,y) - F(i\frac{g(y)}{m}, y) \right|.$$

Using the inequality

$$\left| F(x,y) - F(i\frac{g(y)}{m}, y) \right| \leq \left(\frac{1}{\delta} \left| x - i\frac{g(y)}{m} \right| + 1 \right) \omega(F(\cdot, y); \delta)$$

one obtains

$$\begin{aligned} |(R_m^x F)(x,y)| &\leq \sum_{i=0}^{m} p_{m,i}(x,y) \left(\frac{1}{\delta} \left| x - i\frac{g(y)}{m} \right| + 1 \right) \omega(F(\cdot, y); \delta) \\ &\leq \left[1 + \frac{1}{\delta} \Big(\sum_{i=0}^{m} p_{m,i}(x,y) \big(x - i\frac{g(y)}{m} \big)^2 \Big)^{1/2} \right] \omega(F(\cdot, y); \delta) \\ &= \left[1 + \frac{1}{\delta} \sqrt{\frac{x(g(y) - x)}{m}} \right] \omega(F(\cdot, y); \delta). \end{aligned}$$

Since $\max_{0 \leq x \leq g(y)} \left[x(g(y) - x) \right] = \frac{g^2(y)}{4}$ and $\max_{0 \leq y \leq h} g^2(y) = h^2$, it follows that $\max_{\widetilde{T}_h} \left[x(g(y) - x) \right] = \frac{h^2}{4}$, hence

$$\left| (R_m^x F)(x,y) \right| \leq \left(1 + \frac{h}{2\delta\sqrt{m}}\right) \omega(F(\cdot, y); \delta).$$

Now, for $\delta = 1/\sqrt{m}$, one obtains (26). □

Theorem 21 ([11]). *If* $F(\cdot,y)\in C^2[0,h]$ *then*

$$(R_m^x F)(x,y)=\frac{x[x-g(y)]}{2m}F^{(2,0)}(\xi,y),\qquad \text{for } \xi\in[0,g(y)]$$

and

$$\left|(R_m^x F)(x,y)\right|\le\frac{h^2}{8m}M_{20}F,\qquad (x,y)\in\widetilde{T}_h,$$

where $M_{ij}F=\max_{\widetilde{T}_h}\left|F^{(i,j)}(x,y)\right|$.

Proof. Taking into account that $\mathrm{dex}(\mathcal{B}_m^x)=1$, by Peano's Theorem, it follows

$$(R_m^x F)(x,y)=\int_0^{g(y)}K_{20}(x,y;s)F^{(2,0)}(s,y)ds,$$

where

$$K_{20}(x,y;s)=(x-s)_+-\sum_{i=0}^{m}p_{m,i}(x,y)\big(i\frac{g(y)}{m}-s\big)_+.$$

For a given $\nu\in\{1,...,m\}$ one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m},\nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu)=(x-s)_+-\sum_{i=\nu}^{m}p_{m,i}(x,y)\left(i\frac{g(y)}{m}-s\right),$$

whence,

$$K_{20}^{\nu}(x,y;s)=\begin{cases}x-s-\sum\limits_{i=\nu}^{m}p_{m,i}(x,y)(i\dfrac{g(y)}{m}-s), & s<x\\ -\sum\limits_{i=\nu}^{m}p_{m,i}(x,y)(i\dfrac{g(y)}{m}-s), & s\ge x.\end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s)\le 0$, for $s\ge x$. For $s<x$ we have

$$K_{20}^{\nu}(x,y;s)=x-s-\sum_{i=0}^{m}p_{m,i}(x,y)\left(i\frac{g(y)}{m}-s\right)+\sum_{i=0}^{\nu-1}p_{m,i}(x,y)\left(i\frac{g(y)}{m}-s\right).$$

As,

$$\sum_{i=0}^{m}p_{m,i}(x,y)\left(i\frac{g(y)}{m}-s\right)=x-s,$$

it follows that

$$K_{20}^{\nu}(x,y;s)=\sum_{i=0}^{\nu-1}p_{m,i}(x,y)\left(i\frac{g(y)}{m}-s\right)\leq 0.$$

So, $K_{20}^{\nu}(x,y;\cdot)\leq 0$ for any $\nu\in\{1,\ldots,m\}$, i.e., $K_{20}(x,y;s)\leq 0$, for $s\in[0,g(y)]$.

By the Mean Value Theorem, one obtains

$$(R_m^x F)(x,y)=F^{(2,0)}(\xi,y)\int_0^{g(y)}K_{20}(x,y;s)ds,\quad 0\leq\xi\leq g(y).$$

Since,

$$\int_0^{g(y)}K_{20}(x,y;s)ds=\frac{x(x-g(y))}{2m}$$

and $\max_{0\leq x\leq g(y)}\frac{|x(x-g(y))|}{2m}=\frac{g^2(y)}{8m}\leq\frac{h^2}{8m}$, $y\in[0,h]$ the conclusion follows.□

Remark 22. Analogous results are obtained for the remainder of the formula $F=\mathcal{B}_n^y F+R_n^y F$.

Let $P_{mn}=\mathcal{B}_m^x\mathcal{B}_n^y$, respectively, $Q_{nm}=\mathcal{B}_n^y\mathcal{B}_m^x$ be the products of the operators $\mathcal{B}_m^x$ and $\mathcal{B}_n^y$, i.e.,

$$(P_{mn}F)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(x,y)\,q_{n,j}\left(\frac{i}{m}g(y),y\right)F\Big(\frac{i}{m}g(y),\frac{j}{n}f\big(\frac{i}{m}g(y)\big)\Big)$$

$$(Q_{nm}F)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}\left(x,\frac{j}{n}f(x)\right)q_{n,j}(x,y)\,F\Big(\frac{i}{m}g\big(\frac{j}{n}f(x)\big),\frac{j}{n}f(x)\Big).$$

Remark 23. The nodes of the operator P_{mn}, respectively, Q_{nm} are given in Figure 11.

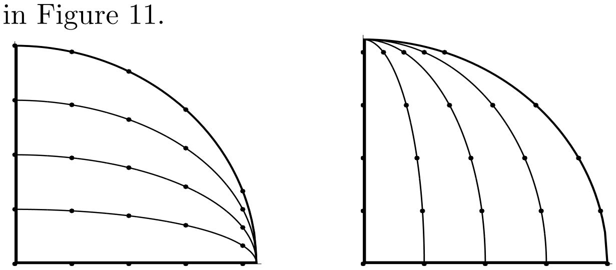


Figure 11. The nodes for P_{mn} and Q_{nm}, for $m=n=4$.

Theorem 24 ([11]). *If F is a real-valued function defined on $\widetilde{T}_h$ then:*

(*i*) $(P_{mn}F)(V_3) = F(V_3),\ P_{mn}F = F,$ *on* Γ_3

(*ii*) $(Q_{nm}F)(V_3) = F(V_3),\ Q_{nm}F = F,$ *on* Γ_3.

Proof. The proof follows from the properties

$$\begin{aligned}&(P_{mn}F)(x,0) = (\mathcal{B}_m^x F)(x,0), \quad (P_{mn}F)(0,y) = (\mathcal{B}_n^y F)(0,y),\\ &(P_{mn}F)(x,f(x)) = F(x,f(x)), \quad x,y \in [0,h]\end{aligned}$$

$$\begin{aligned}&(Q_{nm}F)(x,0) = (\mathcal{B}_m^x F)(x,0), \quad (Q_{nm}F)(0,y) = (\mathcal{B}_n^y F)(0,y),\\ &(Q_{nm}F)(g(y),y) = F(g(y),y), \quad x,y \in [0,h],\end{aligned}$$

which can be verified by a straightforward computation. □

Remark 25. The product operators P_{mn} and Q_{nm} interpolate the function F at the vertex $(0,0)$ and on the side $y = f(x)$ (or $x = g(y)$).

Let us consider now the approximation formula $F = P_{mn}F + R_{mn}^P F$, where R_{mn}^P is the corresponding remainder operator.

Theorem 26 ([11]). *If $F \in C(\widetilde{T}_h)$ then*

$$\left|\left(R_{mn}^P F\right)(x,y)\right| \leq (1+h)\omega\big(F; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\big), \quad (x,y) \in \widetilde{T}_h$$

Proof. We have

$$\begin{aligned}\left|(R_{mn}^P F)(x,y)\right| \leq \Bigg[&\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y) q_{n,j}\left(\frac{i}{m}g(y),y\right)\left|x - \frac{i}{m}g(y)\right|\\ &+ \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y) q_{n,j}\left(\frac{i}{m}g(y),y\right)\left|y - \frac{j}{n}f\left(\frac{i}{m}g(y)\right)\right|\\ &+ \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y) q_{n,j}\left(\frac{i}{m}g(y),y\right)\Bigg]\omega(F;\delta_1,\delta_2).\end{aligned}$$

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\frac{i}{m}g(y),y\right)\left|x-\frac{i}{m}g(y)\right| \le \sqrt{\frac{x(g(y)-x)}{m}},$$
$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\frac{i}{m}g(y),y\right)\left|y-\frac{j}{n}f\big(\frac{i}{m}g(y)\big)\right| \le \sqrt{\frac{y(f(x)-y)}{n}},$$
$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\frac{i}{m}g(y),y\right) = 1,$$

it follows that

$$\left|(R_{mn}^{P}F)(x,y)\right| \le \left(1+\frac{1}{\delta_1}\sqrt{\frac{x(g(y)-x)}{m}}+\frac{1}{\delta_2}\sqrt{\frac{y(f(x)-y)}{n}}\right)\omega(F;\delta_1,\delta_2).$$

But $x\big(g(y)-x\big) \le \frac{h^2}{4}$ and $y\big(f(x)-y\big) \le \frac{h^2}{4}$, whence,

$$\left|(R_{mn}^{P}F)(x,y)\right| \le \left(1+\frac{1}{\delta_1}\frac{h}{2\sqrt{m}}+\frac{1}{\delta_2}\frac{h}{2\sqrt{n}}\right)\omega(F;\delta_1,\delta_2)$$

and

$$\left|(R_{mn}^{P}F)(x,y)\right| \le (1+h)\,\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

□

Next we consider the Boolean sums of the operators $\mathcal{B}_m^x$ and $\mathcal{B}_n^y$, i.e.,

$$S_{mn} := \mathcal{B}_m^x \oplus \mathcal{B}_n^y = \mathcal{B}_m^x + \mathcal{B}_n^y - \mathcal{B}_m^x\mathcal{B}_n^y,$$
$$T_{nm} := \mathcal{B}_n^y \oplus \mathcal{B}_m^x = \mathcal{B}_n^y + \mathcal{B}_m^x - \mathcal{B}_n^y\mathcal{B}_m^x.$$

Theorem 27 ([11]). *If F is a real-valued function defined on $\widetilde{T}_h$ then*

$$S_{mn}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h} \qquad \text{and} \qquad T_{nm}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h}.$$

Proof. As,

$$(P_{mn}F)(x,0) = (\mathcal{B}_m^xF)(x,0), \qquad (P_{mn}F)(0,y) = (\mathcal{B}_n^yF)(0,y),$$
$$(\mathcal{B}_m^xF)(x,h-x) = (\mathcal{B}_n^yF)(x,h-x) = (P_{mn}F)(x,h-x) = F(x,h-x)$$

the conclusion follows. □

For the remainder of the Boolean sum approximation formula, $F = S_{mn}F + R_{mn}^{S}F$, we have the following result:

Theorem 28 ([11]). *If $F \in C(\widetilde{T}_h)$ then*

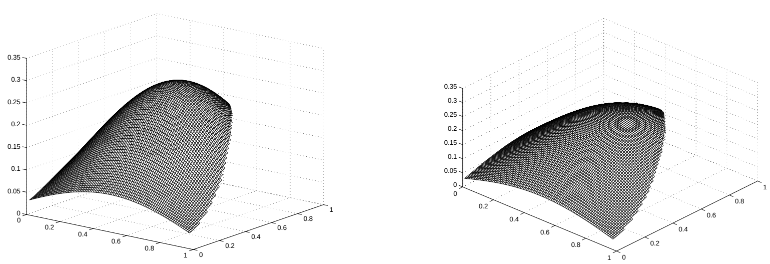
$$\left|(R^S_{mn}F)(x,y)\right| \leq (1+\frac{h}{2})\omega\big(F(\cdot,y);\frac{1}{\sqrt{m}}\big)+(1+\frac{h}{2})\omega\big(F(x,\cdot);\frac{1}{\sqrt{n}}\big)$$
$$+(1+h)\omega\big(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\big), \qquad (x,y)\in \widetilde{T}_h.$$

Proof. The identity $F - S_{mn}F = F - \mathcal{B}^x_m F + F - \mathcal{B}^y_n F - (F - P_{mn}F)$ implies that

$$\left|(R^S_{mn}F)(x,y)\right| \leq \left|(R^x_m F)(x,y)\right| + \left|(R^y_n F(x,y)\right| + \left|(R^P_{mn}F)(x,y)\right|$$

and the conclusion follows. □

Example 29. Consider the test function and the triangle from Example 17. In Figure 12 we plot the graphs of $\mathcal{B}^x_m F_1$ and $P_{mn}F_1$, for $m = 5$, $n = 6$.



$\mathcal{B}^x_m F_1$ $\quad$ $P_{mn}F_1$

Figure 12. Graphs of $\mathcal{B}^x_m F_1$ and $P_{mn}F_1$.

Example 30. Table 2 contains some maximum approximation errors for F_1.

	$\mathcal{B}^x_m F_1$	$\mathcal{B}^y_n F_1$	$P_{mn}F_1$	$Q_{nm}F_1$	$S_{mn}F_1$	$T_{nm}F_1$
Max error	0.0525	0.0452	0.0858	0.0857	0.0095	0.0095

Table 2. The maximum approximation error for F_1.

4. Bernstein-type operators on a triangle with all curved sides

Let F be a real-valued function defined on $\tilde{T}'_h$ and $(g_2(y), y)$, $(g_3(y), y)$, respectively, $(x,\ f_1(x))$, $(x, f_3(x))$ be the points where the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}'_h$, intersect the sides γ_1, γ_2 and γ_3. We consider the uniform partitions of the intervals $[g_2(y), g_3(y)]$ and $[f_1(x), f_3(x)]$, $x, y \in [0, h]$, $\Delta^x_m = \{g_2(y) + i\frac{g_3(y)-g_2(y)}{m} \mid i = \overline{0, m}\}$, respectively, $\Delta^y_n = \{f_1(x) + j\frac{f_3(x)-f_1(x)}{n} \mid j = \overline{0, n}\}$ and the Bernstein-type operators $\mathcal{B}^x_m$ and $\mathcal{B}^y_n$ defined by

$$(\mathcal{B}^x_m F)(x, y) = \sum_{i=0}^{m} p_{m,i}(x, y)\, F\left(g_2(y) + i\frac{g_3(y) - g_2(y)}{m}, y\right), \tag{27}$$

$$(\mathcal{B}^y_n F)(x, y) = \sum_{j=0}^{n} q_{n,j}(x, y)\, F\left(x, f_1(x) + j\frac{f_3(x) - f_1(x)}{n}\right), \tag{28}$$

with

$$p_{m,i}(x, y) = \binom{m}{i} \left[\frac{x - g_2(y)}{g_3(y) - g_2(y)}\right]^i \left[1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)}\right]^{m-i},$$

$$q_{n,j}(x, y) = \binom{n}{j} \left[\frac{y - f_1(x)}{f_3(x) - f_1(x)}\right]^j \left[1 - \frac{y - f_1(x)}{f_3(x) - f_1(x)}\right]^{n-j}.$$

Remark 31. In Figures 13 and 14 we plot the points

$$\big(g_2(y) + i\frac{g_3(y) - g_2(y)}{m}, y\big), i = \overline{0, m}$$

and respectively, $\big(x, f_1(x) + j\frac{f_3(x)-f_1(x)}{n}\big)$, $j = \overline{0, n}$, for $x, y \in [0, h]$.

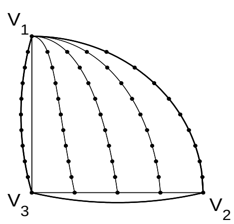


Figure 13. Points of Δ^x_m, for $m = 4$.

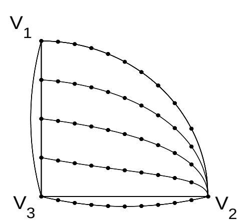


Figure 14. Points of Δ^n_y, for $n = 4$.

Theorem 32 ([12]). *If F is a real-valued function defined on $\tilde{T}'_h$ then:*

(*i*) $\mathcal{B}_m^x F = F$ *on* $\gamma_2 \cup \gamma_3$, *and* $\mathcal{B}_n^y F = F$ *on* $\gamma_1 \cup \gamma_3$,

(*ii*) $(\mathcal{B}_m^x e_{i0})(x,y) = x^i,\ i = 0,1;\ (\mathcal{B}_m^x e_{20})(x,y) = x^2 + \frac{[x-g_2(y)][g_3(y)-x]}{m}$;
$(\mathcal{B}_m^x e_{ij})(x,y) = y^j (\mathcal{B}_m^x e_{i0})(x,y),\ i = 0,1,2;\ j \in \mathbb{N}$;

(*iii*) $(\mathcal{B}_n^y e_{0j})(x,y) = y^j,\ j = 0,1;\ (\mathcal{B}_n^y e_{02})(x,y) = y^2 + \frac{[y-f_1(x)][f_3(x)-y]}{n}$;
$(\mathcal{B}_n^y e_{ij})(x,y) = x^i (\mathcal{B}_n^y e_{0j})(x,y),\ j = 0,1,2;\ i \in \mathbb{N}$.

Proof. The interpolation properties (i) follow by the relations:

$$p_{m,i}(g_2(y),y) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases} \quad p_{m,i}(g_3(y),y) = \begin{cases} 0, & \text{for } i < m, \\ 1, & \text{for } i = m, \end{cases}$$

respectively by

$$q_{n,j}(x,f_1(x)) = \begin{cases} 1, & \text{for } j = 0, \\ 0, & \text{for } j > 0, \end{cases} \quad q_{n,j}(x,f_3(x)) = \begin{cases} 0, & \text{for } j < n, \\ 1, & \text{for } j = n. \end{cases}$$

Regarding the properties (ii), we have

$$(\mathcal{B}_m^x e_{ij})(x,y) = y^j (\mathcal{B}_m^x e_{i0})(x,y), \qquad j \in \mathbb{N}$$

$$(\mathcal{B}_m^x e_{00})(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y) = 1,$$

$$\begin{aligned}
\mathcal{B}_m^x e_{10}(x,y) &= \sum_{i=0}^{m} p_{m,i}(x,y) \left[g_2(y) + i \frac{g_3(y)-g_2(y)}{m} \right] \\
&= g_2(y) + [g_3(y) - g_2(y)] \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \left[\frac{x-g_2(y)}{g_3(y)-g_2(y)} \right]^i \\
&\cdot \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)} \right]^{m-i} \frac{i}{m} \\
&= g_2(y) + [x - g_2(y)] \sum_{i=0}^{m-1} \binom{m-1}{i} \left[\frac{x-g_2(y)}{g_3(y)-g_2(y)} \right]^i \\
&\cdot \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)} \right]^{m-i-1} = x,
\end{aligned}$$

$$\mathcal{B}_m^x e_{20}(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y)\left[g_2(y)+i\frac{g_3(y)-g_2(y)}{m}\right]^2$$
$$=g_2^2(y)+2g_2(y)[x-g_2(y)]+\frac{[g_3(y)-g_2(y)][x-g_2(y)]}{m}$$
$$+\frac{(m-1)[g_3(y)-g_2(y)]^2}{m}\sum_{i=2}^{m}\frac{(m-2)!}{(i-2)!(m-i)!}$$
$$\cdot\left[\frac{x-g_2(y)}{g_3(y)-g_2(y)}\right]^i\left[1-\frac{x-g_2(y)}{g_3(y)-g_2(y)}\right]^{m-i}$$
$$=\frac{m-1}{m}[x-g_2(y)]^2+\frac{(2m-1)g_2(y)+g_3(y)}{m}$$
$$\cdot[x-g_2(y)]+g_2^2(y)=x^2+\frac{[x-g_2(y)][g_3(y)-x]}{m}.$$

Properties (iii) are proved in the same way. □

Remark 33. The interpolation properties of $\mathcal{B}_m^x F$ and $\mathcal{B}_n^y F$ are illustrated in Figures 15 and 16.

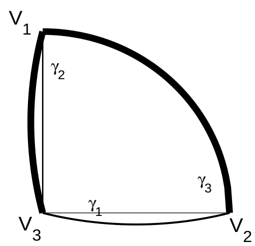


Figure 15. Interpolation set of $B_m^x F$

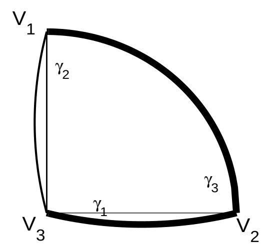


Figure 16. Interpolation set of $B_n^y F$.

Now, we consider the approximation formula $F = \mathcal{B}_m^x F + R_m^x F$.

Theorem 34 ([12])**.** *If* $F(\cdot,y)\in C[g_2(y),g_3(y)]$, $y\in[0,h]$, *then*

$$\left|\,(R_m^x F)(x,y)\,\right| \leq \left(1+\frac{g_3(y)-g_2(y)}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y);\delta), \quad y\in[0,h],$$

and if $M=\max_{0\leq y\leq h}|g_3(y)-g_2(y)|$ *then we have*

$$\left|\,(R_m^x F)(x,y)\,\right| \leq \left(1+\frac{M}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y);\delta), \quad y\in[0,h].$$

Moreover, if $\delta = 1/\sqrt{m}$ *then*

$$\left|\,(R_m^x F)\,(x,y)\right| \le \left(1+\frac{M}{2}\right)\omega(F(\cdot,y);\frac{1}{\sqrt{m}}),\quad y\in[0,h]. \tag{29}$$

Proof. As $(\mathcal{B}_m^x e_{00})(x,y)=1$, it follows that

$$\begin{aligned}\left|\,(R_m^x F)\,(x,y)\right| &\le \sum_{i=0}^m p_{m,i}(x,y)\left|F(x,y)-F\big(g_2(y)+i\frac{g_3(y)-g_2(y)}{m},y\big)\right|\\ &\le \sum_{i=0}^m p_{m,i}(x,y)\left(\frac{1}{\delta}\left|x-[g_2(y)+i\frac{g_3(y)-g_2(y)}{m}]\right|+1\right)\omega(F(\cdot,y);\delta)\\ &\le \Big\{1+\frac{1}{\delta}\Big[\sum_{i=0}^m p_{m,i}(x,y)\big(x-(g_2(y)+i\frac{g_3(y)-g_2(y)}{m})\big)^2\Big]^{1/2}\Big\}\omega(F(\cdot,y);\delta)\\ &\le \Big[1+\frac{1}{\delta}\sqrt{\frac{[x-g_2(y)][g_3(y)-x]}{m}}\Big]\omega(F(\cdot,y);\delta).\end{aligned}$$

Since $\max_{g_2(y)\le x\le g_3(y)}[x-g_2(y)][g_3(y)-x]=\frac{[g_3(y)-g_2(y)]^2}{4}$, one obtains

$$\left|(R_m^x F)(x,y)\right| \le \left[1+\frac{g_3(y)-g_2(y)}{2\delta\sqrt{m}}\right]\omega(F(\cdot,y);\delta)$$

and the proof follows. □

Remark 35. Analogous results can be obtained for the remainder of the formula $F=\mathcal{B}_n^y F + R_n^y F$.

Let $P_{mn}=\mathcal{B}_m^x\mathcal{B}_n^y$, respectively, $Q_{nm}=\mathcal{B}_n^y\mathcal{B}_m^x$ be the products of the operators $\mathcal{B}_m^x$ and $\mathcal{B}_n^y$, i.e.,

$$\begin{aligned}(P_{mn}F)\,(x,y) &= \sum_{i=0}^m\sum_{j=0}^n p_{m,i}\,(x,y)\,q_{n,j}\,(x_i,y)\,F\big(x_i,f_1(x_i)+j\frac{f_3(x_i)-f_1(x_i)}{n}\big),\\ (Q_{nm}F)\,(x,y) &= \sum_{i=0}^m\sum_{j=0}^n p_{m,i}\,(x,y_j)\,q_{n,j}\,(x,y)\,F\big(g_2(y_j)+i\frac{g_3(y_j)-g_2(y_j)}{m},y_j\big),\end{aligned}$$

with $x_i=g_2(y)+i\frac{g_3(y)-g_2(y)}{m}$, $y_j=f_1(x)+j\frac{f_3(x)-f_1(x)}{n}$.

Remark 36. The nodes of the operators P_{mn}, respectively Q_{nm} are given in Figures 17 and 18, where f_1, g_2 and f_3 are arcs of the circles (C1) $(x-\frac{1}{2})^2+(y+\frac{\sqrt{15}}{2})^2=4$, (C2) $(x+\frac{\sqrt{15}}{2})^2+(y-\frac{1}{2})^2=4$, respectively, (C3) $x^2+y^2=1$.

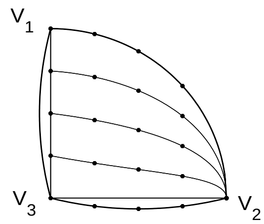


Figure 17. The nodes of P_{mn}, for $m = n = 4$.

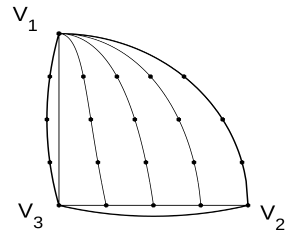


Figure 18. The nodes of Q_{nm}, for $m = n = 4$.

Theorem 37 ([12])**.** *If F is a real-valued function defined on $\tilde{T}_h'$ then:*

(*i*) $(P_{mn}F)(V_3) = F(V_3)$, $P_{mn}F = F$, *on* γ_3

(*ii*) $(Q_{nm}F)(V_3) = F(V_3)$, $Q_{nm}F = F$, *on* γ_3.

Proof. The proof follows from the properties:

$$\begin{aligned}
&(P_{mn}F)(x, f_1(x)) = (\mathcal{B}_m^x F)(x, f_1(x)),\\
&(P_{mn}F)(g_2(y), y) = (\mathcal{B}_n^y F)(g_2(y), y),\\
&(P_{mn}F)(x, f_3(x)) = F(x, f_3(x)), \qquad x, y \in [0, h]\\
&(Q_{nm}F)(x, f_1(x)) = (\mathcal{B}_m^x F)(x, f_1(x)),\\
&(Q_{nm}F)(g_2(y), y) = (\mathcal{B}_n^y F)(g_2(y), y),\\
&(Q_{nm}F)(g_3(y), y) = F(g_3(y), y), \qquad x, y \in [0, h],
\end{aligned}$$

which can be verified by a straightforward computation. □

The interpolation properties of $P_{mn}F$ and $Q_{nm}F$ are illustrated in Figure 19.

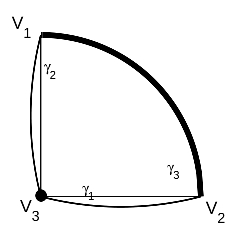


Figure 19. Interpolation set for $P_{mn}F$ and $Q_{nm}F$

For the remainder of the product approximation formula $F = P_{mn}F + R^P_{mn}F$ we have:

Theorem 38 ([12]). *If $F \in C(\tilde{T}'_h)$ then*

$$\left| \left(R^P_{mn}F\right)(x,y)\right| \leq \left(1 + \frac{M}{2\delta_1\sqrt{m}} + \frac{N}{2\delta_2\sqrt{n}}\right)\omega\left(F;\delta_1,\delta_2\right), \quad (x,y)\in \tilde{T}_h,$$

respectively

$$\left| \left(R^P_{mn}F\right)(x,y)\right| \leq \left(1 + \frac{M}{2} + \frac{N}{2}\right)\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right), \quad (x,y)\in \tilde{T}_h,$$

where $M = \max_{0\leq y\leq h}|g_3(y)-g_2(y)|$ and $N = \max_{0\leq x\leq h}|f_3(x)-f_2(x)|$.

Proof. We have

$$\begin{aligned}
\left|(R^P_{mn}F)(x,y)\right| &\leq \Big[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(x_i,y\right)|x-x_i| \\
&+ \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(x_i,y\right)\left|y-(f_1(x_i)+j\frac{f_3(x_i)-f_1(x_i)}{n})\right| \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(x_i,y\right)\Big]\omega(F;\delta_1,\delta_2) \\
&\leq \Big[\frac{1}{\delta_1}\sum_{j=0}^{n} q_{n,j}\left(x_i,y\right)\sqrt{\frac{[x-g_2(y)][g_3(y)-x]}{m}} \\
&+ \frac{1}{\delta_2}\sum_{i=0}^{m} p_{m,i}\left(x,y\right)\sqrt{\frac{[y-f_1(x)][f_3(x)-y]}{n}} + 1\Big]\omega(F;\delta_1,\delta_2),
\end{aligned}$$

and further,

$$\begin{aligned}
\left|(R^P_{mn}F)(x,y)\right| &\leq \Big[\frac{1}{\delta_1}\sqrt{\frac{[x-g_2(y)][g_3(y)-x]}{m}} \\
&+ \frac{1}{\delta_2}\sqrt{\frac{[x-f_1(x)][f_3(x)-y]}{n}} + 1\Big]\omega(F;\delta_1,\delta_2) \\
&\leq \Big[1 + \frac{1}{\delta_1}\frac{g_3(y)-g_2(y)}{2\sqrt{m}} + \frac{1}{\delta_2}\frac{f_3(x)-f_1(x)}{2\sqrt{n}} + 1\Big]\omega(F;\delta_1,\delta_2) \\
&\leq \Big[1 + \frac{1}{\delta_1}\frac{M}{2\sqrt{m}} + \frac{1}{\delta_2}\frac{N}{2\sqrt{n}} + 1\Big]\omega(F;\delta_1,\delta_2),
\end{aligned}$$

and the proof follows. □

For the remainder $R^Q_{nm}F = F - Q_{nm}F$ we can also obtain

$$\left| \left(R^Q_{nm}F\right)(x,y)\right| \le \left(1+\frac{M}{2}+\frac{N}{2}\right)\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right), \qquad (x,y)\in \tilde{T}_h.$$

We consider the Boolean sums of the operators $\mathcal{B}^x_m$ and $\mathcal{B}^y_n$, i.e.,

$$\begin{aligned} S_{mn} &:= \mathcal{B}^x_m \oplus \mathcal{B}^y_n = \mathcal{B}^x_m + \mathcal{B}^y_n - \mathcal{B}^x_m\mathcal{B}^y_n,\\ T_{nm} &:= \mathcal{B}^y_n \oplus \mathcal{B}^x_m = \mathcal{B}^y_n + \mathcal{B}^x_m - \mathcal{B}^y_n\mathcal{B}^x_m. \end{aligned}$$

Theorem 39 ([12]). *If F is a real-valued function defined on $\tilde{T}'_h$ then*

$$S_{mn}F\big|_{\partial\tilde{T}'_h} = F\big|_{\partial\tilde{T}'_h} \text{ and } T_{nm}F\big|_{\partial\tilde{T}'_h} = F\big|_{\partial\tilde{T}'_h}.$$

Proof. As

$$\begin{aligned} (P_{mn}F)(x,f_1(x)) &= (\mathcal{B}^x_mF)(x,f_1(x)),\\ (P_{mn}F)(g_2(y),y) &= (\mathcal{B}^y_nF)(g_2(y),y),\\ (P_{mn}F)(x,f_3(x)) &= F(x,f_3(x)), \end{aligned}$$

the proof follows. □

For the remainder of the Boolean sum approximation formula, $F = S_{mn}F + R^S_{mn}F$, we have the following result:

Theorem 40 ([12]). *If $F \in C(\tilde{T}'_h)$ then*

$$\begin{aligned} \left|(R^S_{mn}F)(x,y)\right| \le &(1+\frac{M}{2})\omega\big(F(\cdot,y);\frac{1}{\sqrt{m}}\big) + (1+\frac{N}{2})\omega\big(F(x,\cdot);\frac{1}{\sqrt{n}}\big)\\ &+ (1+\frac{M}{2}+\frac{N}{2})\omega\big(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\big), \qquad (x,y)\in\widetilde{T}_h. \end{aligned}$$

Proof. The identity $F - S_{mn}F = F - \mathcal{B}^x_mF + F - \mathcal{B}^y_nF - (F - P_{mn}F)$ implies that $\left|(R^S_{mn}F)(x,y)\right| \le \left|(R^x_mF)(x,y)\right| + \left|(R^y_nF(x,y)\right| + \left|(R^P_{mn}F)(x,y)\right|$ and the proof follows. □

An analogous inequality can be obtained for the error $R^T_{nm}F = F - T_{nm}F$.

Example 41. Consider the test function from Example 17. In Figure 20 we plot the graphs of $\mathcal{B}^x_mF_1$ and $S_{mn}F_1$, with $h=1, m=5$, $n=6$, and $f_1,g_2,f_3:[0,1]\to[0,1]$, $f_1(x) = -\frac{\sqrt{15}}{2} - \sqrt{4-(x-0.5)^2}$, $g_2(y) = -\frac{\sqrt{15}}{2} - \sqrt{4-(y-0.5)^2}$ and $f_3(x) = \sqrt{1-x^2}$.

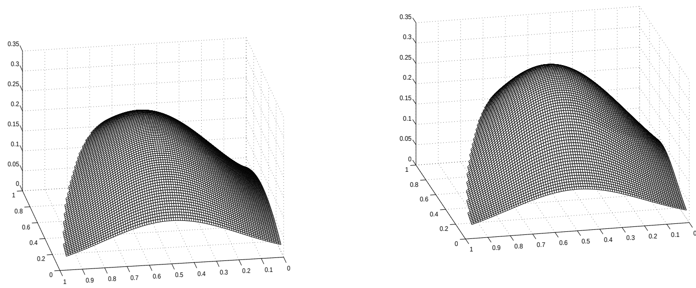


Figure 20. Graphs of $\mathcal{B}_m^x F_1$ and $S_{mn}F_1$.

Example 42. Table 3 contains the maximum approximation errors.

	$\mathcal{B}_m^x F_1$	$\mathcal{B}_n^y F_1$	$P_{mn}F_1$	$Q_{nm}F_1$	$S_{mn}F_1$	$T_{nm}F_1$
Max error	0.0847	0.0684	0.1269	0.1266	0.0239	0.0240

Table 3. The maximum approximation error for F_1.

REFERENCES

1. Barnhill, R.E. – *Blending function interpolation: a survey and some new results,* Numerishe Methoden der Approximationstheorie, Band 3, pp. 43–89. Internat. Ser. Numer. Math., Vol. 30, Birkhäuser, Basel, 1976.
2. Barnhill, R.E. – *Representation and approximation of surfaces*, Mathematical software, III, pp. 69–120. Publ. Math. Res. Center Univ. Wisconsin, No. 39, Academic Press, New York, 1977.
3. Barnhill, R.E.; Birkhoff, G.; Gordon, W.J. – *Smooth interpolation in triangles*, J. Approximation Theory, 8 (1973), 114–128.
4. Barnhill, R.E.; Gregory, J.A. – *Polynomial interpolation to boundary data on triangles,* Math. Comp., 29 (1975), 726–735.
5. Barnhill, R.E.; Gregory, J.A. – *Sard kernels theorems on triangular domains with applications to finite element error bounds*, Numer. Math., 25 (1976), 215–229.
6. Barnhill, R.E.; Mansfield, L. – *Error bounds for smooth interpolation in triangles*, J. Approximation Theory, 11 (1974), 306–318.

7. Bernardi, C. – *Optimal finite-element interpolation on curved domains,* SIAM J. Numer. Anal., 26 (1989), 1212–1240.

8. Birkhoff, G. – *Interpolation to boundary data in triangles*, J. Math. Anal. Appl., 42 (1973), 474–484.

9. Blaga, P.; Coman, G. – *Bernstein-type operators on triangles*, Rev. Anal. Numér. Théor. Approx., 38 (2009), 11–23 (2010).

10. Blaga, P.; Cătinaş, T.; Coman, G. – *Bernstein-type operators on tetrahedrons,* Stud. Univ. Babeş-Bolyai, Math., 54 (2009), 3–19.

11. Blaga, P.; Cătinaş, T.; Coman, G. – *Bernstein-type operators on a triangle with one curved side*, Mediterr. J. Math., 9 (2012), 833–845.

12. Blaga, P.; Cătinaş, T.; Coman, G. – *Bernstein-type operators on a triangle with all curved sides,* Appl. Math. Comput., 218 (2011), 3072–3082.

13. Böhmer, K.; Coman, G. – *Blending interpolation schemes on triangle with error bounds*, 14–37, Lecture Notes in Math., 571, Springer, Berlin, 1977.

14. Coman, G.; Cătinaş, T. – *Interpolation operators on a triangle with one curved side*, BIT, 50 (2010), 243–267.

15. Coman, G.; Cătinaş, T. – *Interpolation operators on a tetrahedron with three curved edges,* Calcolo, 47 (2010), 113–128.

16. Coman, G.; Cătinaş, T.; Birou, M.; Oprişan, A.; Oşan, C.; Pop, I.; Somogyi, I.; Todea, I. – *Interpolation operators*, Ed. Casa Cărţii de Ştiinţă, Cluj-Napoca, 2004.

17. Coman, G.; Gânscă, I. – *Some practical application of blending approximation II,* Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1986.

18. Coman, G.; Gânscă, I.; Ţâmbulea, L. – *Some new roof-surfaces generated by blending interpolation technique*, Studia Univ. Babeş-Bolyai Math., 36 (1991), 119–130.

19. Cavendish, J.C.; Gordon, W.J.; Hall, C.A. – *Ritz-Galerkin approximations in blending function spaces*, Numer. Math., 26 (1976), 155–178.

20. Dautray, R.; Lions, J.L. – *Analyse mathematique et calcul numerique,* Vol. 6, Methodes integrales et numeriques, Mason, Paris, 1988, 798–834.

21. Gordon, W.J. – *Distributive lattices and the approximation of multivariate functions*, Proc. Sympos. Univ. of Wisconsin, Madison, Wis., 1969, 223–277, Academic Press, New York.

22. Gordon, W.J.; Hall, C.A. – *Transfinite element methods: blending-function interpolation over arbitrary curved element domains,* Numer. Math., 21 (1973/74), 109–129.

23. Gordon, W.J.; Wixom, J.A. – *Pseudo-harmonic interpolation on convex domains*, SIAM J. Numer. Anal., 11 (1974), 909–933.

24. Gregory, J.A. – *A blending function interpolant for triangles*, Multivariate Approximation, 279–287, Academic Press, London-New York, 1978.

25. Marshall, J.A.; Mitchell, A.R. – *An exact boundary tehnique for improved accuracy in the finite element method,* J. Inst. Math. Appl., 12 (1973), 355–362.

26. Marshall, J.A.; Mitchell, A.R. – *Blending interpolants in the finite element method,* Internat. J. Numer. Methods Engrg., 12 (1978), 77–83.

27. Nielson, G.M. – *The side-vertex method for interpolation in triangles*, J. Approx. Theory, 25 (1979), 318–336.

28. Nielson, G.M. – *Minimum norm interpolation in triangles,* SIAM J. Numer. Anal., 17 (1980), 44–62.

29. Nielson, G.M.; Thomas, D.H.; Wixom, J.A. – *Interpolation in triangles*, Bull. Austral. Math. Soc., 20 (1979), 115–130.

30. Renka, R.J.; Cline, A.K. – *A triangle-based C^1 interpolation method,* Rocky Mountain J. Math., 14 (1984), 223–237.

31. Sard, A. – *Linear Approximation*, American Mathematical Society, Providence, R.I., 1963.

32. Schumaker, L.L. – *Fitting surfaces to scattered data*, Approximation theory, II, 203–268, Academic Press, New York, 1976.

33. Stancu, D.D. – *A method for obtaining polynomials of Bernstein type of two variables*, Amer. Math. Monthly, 70 (1963), 260–264.

34. Stancu, D.D. – *Approximation of bivariate functions by means of some Bernstein-type operators*, Multivariate approximation (Sympos., Univ. Durham, Durham, 1977), Academic Press, London-New York, 1978, 189–208.

35. Zlamal, M. – *Curved elements in the finite element method I,* SIAM J. Numer. Anal., 10 (1973), 229–240.

Received: 16.I.2012
Accepted: 29.II.2012

Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
1, M. Kogălniceanu St., RO-400084 Cluj-Napoca,
ROMANIA
tcatinas@math.ubbcluj.ro
blaga@math.ubbcluj.ro
ghcoman@math.ubbcluj.ro