

GENERALIZED NULL BERTRAND CURVES IN MINKOWSKI SPACE-TIME

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Abstract. ÇÖKEN and ÇİFTÇİ proved that a null Cartan curve in Minkowski space-time $\mathbb{E}_1^4$ is a null Bertrand curve if and only if k_2 is nonzero constant and k_3 is zero. That is, the null curve with non-zero curvature k_3 is not a Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$.

So, in this paper we defined a new type of Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$ for a null curve with non-zero curvature k_3 by using the similar idea of generalized Bertrand curve given by MATSUDA and YOROZU and we called it a null $(1, 3)$ -Bertrand curve. Also, we proved that if a null curve with non-zero curvatures in Minkowski space-time $\mathbb{E}_1^4$ is a null $(1, 3)$ -Bertrand curve then it is a null helix. We give an example of such curves.

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1. Introduction

In the classical differential geometry of curves in Euclidean space, SAINT VENANT (see [15]) proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by BERTRAND in 1850 in a paper (see [3]) in which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. In other

words, if we denote first and second curvatures of a given curve by k_1 and k_2 , respectively then we have $\lambda k_1 + \mu k_2 = 1$ where $\lambda, \mu \in \mathbb{R}$. Since Bertrand published his paper, curve pairs of this type have been called *Conjugate Bertrand Curves* or more commonly *Bertrand Curves* (see [11]). In 1888, BIOCHE [4] gave a new theorem to obtain Bertrand curves by using the given two curves C_1 and C_2 in Euclidean 3-space. Later, BURKE [6] gave a theorem related with *Bioche's theorem on Bertrand curves*.

The following properties of Bertrand curves are well known: If two curves have the same principal normals: (i) corresponding points are a fixed distance apart; (ii) the tangents at corresponding points are at a fixed angle. These well known properties of Bertrand curves in Euclidean 3-space were extended by PEARS in [14] to Riemannian n -space and found general results for Bertrand curves. When we apply these general results to Euclidean n -space, it is easily found that either k_2 or k_3 is zero. In other words, Bertrand curves in $\mathbb{E}^n (n > 3)$ are degenerate curves. This result was restated in [12] by MATSUDA and YOROZU. They proved that *there is no special Bertrand curve in $\mathbb{E}^n (n > 3)$* and they defined new type which is called $(1, 3)$ -type Bertrand curve in 4-dimensional Euclidean space.

In differential geometry of curves in Minkowski space, there are three different kinds of curves called spacelike, timelike and null (lightlike) depending on their causal characters. Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by a similar approach to that in positive definite Riemannian geometry. However, since the induced metric of a null curve is degenerate, this case is much more complicated and also different from a non-degenerate case. The presence of null curves often causes important and interesting differences as will be the case in the present study.

In Minkowski 3-space (also in a Lorentzian manifold), spacelike and timelike Bertrand curves and their characterizations were studied in [2, 7, 9, 10]. Null Bertrand curves in Minkowski 3-space were studied by BALGETİR, BEKTAŞ and INOBUCHI in [1] and they proved the following theorem for a null Cartan curve to be a Bertrand curve:

Theorem A. Let α be a Cartan framed null curve. Then it is a Bertrand curve if and only if α is a null geodesic or a Cartan framed null curve with constant second curvature k_2 .

Null Bertrand curves were studied in a Lorentzian manifold by JIN [10].

ÇOKEN and ÇİFTÇİ [5] proved the following theorem for Bertrand curves

in Minkowski space-time

Theorem B. A Cartan curve in Minkowski space-time $\mathbb{E}_1^4$ is a null Bertrand curve if and only if the curvature k_2 is a non-zero constant and k_3 is zero.

That is, the null curve with non-zero curvature k_3 is not a Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$. Also, GÖÇMEN and KELEŞ [8] obtained some new results for Cartan framed null Bertrand curves in $\mathbb{R}_1^4$.

In this paper, we define a new type of Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$ for a null curve with non-zero curvature k_3 by using the similar idea of generalized Bertrand curve given by MATSUDA and YOROZU [12] and we called it a null (1,3)-Bertrand curve. Also, we prove that if a null curve with non-zero curvatures in Minkowski space-time $\mathbb{E}_1^4$ is a null (1,3)-Bertrand curve then it is a null helix. Also, some properties of null (1,3)-Bertrand curves in Minkowski space-time are given. We complete the paper with an example of such curves.

2. Preliminaries

The Lorentzian 4 -space $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + \sum_{i=2}^4 dx_i^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4 \setminus \{0\}$ is *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike vector. The norm of a vector v is given by $\|v\|_L = \sqrt{|g(v, v)|}$ and two vectors v and w are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve α in $\mathbb{E}_1^4$ can locally be *spacelike*, *timelike* or *null (lightlike)* if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. Recall that a spacelike curve in $\mathbb{E}_1^4$ is called *pseudo-null curve* or *partially-null curve* if its principal normal vector is null and its first binormal vector is null, respectively. A spacelike or a timelike curve α has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([13]). A null curve α is parametrized by arclength function s if $g(\alpha''(s), \alpha''(s)) = 1$. In particular, a pseudo-null or a partially-null curve α has unit speed if $g(\alpha'(s), \alpha'(s)) = 1$.

Let $\{T, N_1, N_2, N_3\}$ be the moving Frenet frame along a null curve in $\mathbb{E}_1^4$. Then the Frenet frame field of the curve α satisfies the following Frenet equations:

$$(2.1) \quad \begin{aligned} T' &= k_1 N_1, \\ N_1' &= k_2 T - k_1 N_2, \\ N_2' &= -k_2 N_1 + k_3 N_3, \\ N_3' &= -k_3 T \end{aligned}$$

where the first curvature $k_1(s) = 0$ if α is a straight line or $k_1(s) = 1$ in all other cases [16]. Such curve has two non zero curvatures $k_2(s)$ and $k_3(s)$. Moreover, this moving Frenet frame $\{T, N_1, N_2, N_3\}$ satisfies the following conditions:

$$\begin{aligned} g(T, T) &= g(N_2, N_2) = 0, \quad g(N_1, N_1) = g(N_3, N_3) = 1, \\ g(T, N_1) &= g(T, N_3) = g(N_1, N_3) = g(N_1, N_2) = g(N_2, N_3) = 0, \quad g(T, N_2) = 1. \end{aligned}$$

In this study we consider the curve α is not a straight line, that is, the first curvature of the curve α is equal to one.

3. Null (1, 3)-bertrand curves in minkowski space-time

From [5], we know that a null curve with non-zero curvature k_3 is not a Bertrand curve in Minkowski space-time $\mathbb{E}_1^4$. In this section, we give the definition of null (1, 3)- Bertrand curve for a null curve with non-zero curvature k_3 in $\mathbb{E}_1^4$. Also we give some characterizations of such curves.

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ and $\beta : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be null curves with curvatures $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and $\bar{k}_1(\varphi(s))$, $\bar{k}_2(\varphi(s))$, $\bar{k}_3(\varphi(s))$, respectively, where $\varphi : I \rightarrow \bar{I}$, $\bar{s} = \varphi(s)$ is a regular C^∞ -function such that each point $\alpha(s)$ of the curve α corresponds to the point $\beta(\bar{s}) = \beta(\varphi(s))$ of the curve β for all $s \in I$. If the Frenet (1, 3)-normal plane at each point $\alpha(s)$ of the curve α coincides with the Frenet (1, 3)-normal plane at corresponding point $\beta(\bar{s}) = \beta(\varphi(s))$ of the curve β for all $s \in I$ then α is called a null (1, 3)-Bertrand curve in $\mathbb{E}_1^4$ and β is called a null (1, 3)-Bertrand mate of the curve α .

Theorem 3.1. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null curve with curvature functions $k_1(s) = 1$, $k_2(s)$ and $k_3(s) \neq 0$. Then α is a null (1, 3)-Bertrand

curve if and only if there exist constant real numbers λ , δ , γ and $\mu \neq 0$ satisfying

$$\begin{aligned} (3.1-a) \quad & \lambda \neq 0, \\ (3.1-b) \quad & 1 + \lambda k_2(s) - \mu k_3(s) = 0, \\ (3.1-c) \quad & (k_2(s))^2 + (k_3(s))^2 = \frac{\lambda^2}{\delta^4}, \\ (3.1-d) \quad & -\frac{k_2(s)}{k_3(s)} = \gamma, \end{aligned}$$

for all $s \in I$.

Proof. We assume that α is a null $(1, 3)$ -Bertrand curve parametrized by arc-length s and β is the null $(1, 3)$ -Bertrand mate of the curve α with arc-length $\bar{s}$. Then we can write the curve β as

$$(3.2) \quad \beta(\bar{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda(s)N_1(s) + \mu(s)N_3(s)$$

for all $s \in I$ where $\lambda(s)$ and $\mu(s)$ are C^∞ -functions on I . Differentiating (3.2) with respect to s and by using the Frenet equations, we have

$$(3.3) \quad \begin{aligned} \overline{T}(\varphi(s))\varphi'(s) = & [1 + \lambda(s)k_2(s) - \mu(s)k_3(s)]T(s) + \lambda'(s)N_1(s) \\ & - \lambda(s)k_1(s)N_2(s) + \mu'(s)N_3(s) \end{aligned}$$

for all $s \in I$. Since the plane spanned by $N_1(s)$ and $N_3(s)$ coincides with the plane spanned by $\overline{N}_1(\varphi(s))$ and $\overline{N}_3(\varphi(s))$, we can write

$$(3.4) \quad \overline{N}_1(\varphi(s)) = \cos \theta(s)N_1(s) + \sin \theta(s)N_3(s),$$

$$(3.5) \quad \overline{N}_3(\varphi(s)) = -\sin \theta(s)N_1(s) + \cos \theta(s)N_3(s).$$

And then by using (3.4) and (3.5), we have

$$\begin{aligned} g(\overline{N}_1(\varphi(s)), \overline{T}(\varphi(s))\varphi'(s)) &= \lambda'(s)\cos \theta(s) + \mu'(s)\sin \theta(s) = 0, \\ g(\overline{N}_3(\varphi(s)), \overline{T}(\varphi(s))\varphi'(s)) &= -\lambda'(s)\sin \theta(s) + \mu'(s)\cos \theta(s) = 0. \end{aligned}$$

Thus we get

$$\lambda'(s) = 0, \mu'(s) = 0.$$

That is, λ and μ are constant functions on I . So, we can rewrite (3.2) and (3.3) for all $s \in I$ as follows:

$$(3.6) \quad \beta(\bar{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda N_1(s) + \mu N_3(s)$$

and

$$(3.7) \quad \overline{T}(\varphi(s))\varphi'(s) = [1 + \lambda k_2(s) - \mu k_3(s)]T(s) - \lambda k_1(s)N_2(s).$$

If we denote

$$(3.8) \quad \zeta(s) = \frac{1 + \lambda k_2(s) - \mu k_3(s)}{\varphi'(s)}, \quad \delta(s) = -\frac{\lambda k_1(s)}{\varphi'(s)}$$

for all $s \in I$. We can easily obtain that

$$(3.9) \quad \overline{T}(\varphi(s)) = \zeta(s)T(s) + \delta(s)N_2(s),$$

where $\zeta(s)$ and $\delta(s)$ are C^∞ -functions on I . Since $\overline{T}(\varphi(s))$, $T(s)$ and $N_2(s)$ are lightlike vectors, we get

$$(3.10) \quad \zeta(s)\delta(s) = 0.$$

That is, $1 + \lambda k_2(s) - \mu k_3(s) = 0$ or $\lambda = 0$. We assume that $\lambda = 0$ and $1 + \lambda k_2(s) - \mu k_3(s) \neq 0$. In that case, we can write $\overline{T}(\varphi(s)) = \zeta(s)T(s)$ and if we differentiate the last equation with respect to s , we get $\frac{d\overline{T}(\varphi(s))}{ds}\varphi'(s) = \zeta'(s)T(s) + \zeta(s)T'(s)$. By using the Frenet equations of α and β null curves, we have $\overline{k_1}(\varphi(s))\overline{N_1}(\varphi(s))\varphi'(s) = \zeta'(s)T(s) + \zeta(s)k_1(s)N_1(s)$. From (3.4), it holds $\zeta'(s) = 0$. That is, $\zeta(s)$ is non-zero constant function on I . So we get $\overline{k_1}(\varphi(s))\overline{N_1}(\varphi(s))\varphi'(s) = \zeta k_1(s)N_1(s)$, where $\varphi'(s) = |\zeta|$. Since the null curves α and β are not straight lines, the principal curvature functions of the null curves α and β are equal to one. That is, $\overline{k_1}(\varphi(s)) = 1$ and $k_1(s) = 1$. So we have $\overline{N_1}(\varphi(s)) = \pm N_1(s)$, for all $s \in I$. This implies that α is a null Bertrand curve. But by Theorem B, this fact is a contradiction. Thus we must consider only the case of $\lambda \neq 0$ and $1 + \lambda k_2(s) - \mu k_3(s) = 0$. Then we obtain the relations (3.1-a) and (3.1-b). Hence we can write

$$(3.11) \quad \overline{T}(\varphi(s)) = \delta(s)N_2(s).$$

Differentiating (3.11) and by using the Frenet equations, we obtain

$$(3.12) \quad \begin{aligned} \overline{k_1}(\varphi(s))\overline{N_1}(\varphi(s))\varphi'(s) &= -\delta(s)k_2(s)N_1(s) + \delta'(s)N_2(s) \\ &+ \delta(s)k_3(s)N_3(s). \end{aligned}$$

Since $\overline{N_1}(\varphi(s))$ is expressed by linear combination of $N_1(s)$ and $N_3(s)$, it holds that $\delta'(s) = 0$. That is, $\delta(s)$ is a non-zero constant function. Also from (3.8), we can write

$$(3.13) \quad \varphi'(s) = -\frac{\lambda}{\delta}.$$

Since $\lambda \neq 0$, it follows $\varphi'(s) \neq 0$. Hence there exists a regular map $\varphi : I \rightarrow \bar{I}$ defined by $\bar{s} = \varphi(s) = -\frac{\lambda}{\delta}s + \eta$, where η is a real constant. We can rewrite (3.12) as $\overline{k_1}(\varphi(s))\overline{N_1}(\varphi(s))\varphi'(s) = -\delta k_2(s)N_1(s) + \delta k_3(s)N_3(s)$. and we can easily see that

$$(3.14) \quad (\varphi'(s)\overline{k_1}(\varphi(s)))^2 = \delta^2 [(k_2(s))^2 + (k_3(s))^2].$$

By substituting (3.13) into (3.14) and using $\overline{k_1}(\varphi(s)) = 1$, we obtain the relation (3.1-c). From (3.12), we have

$$\overline{N_1}(\varphi(s)) = -\frac{\delta k_2(s)}{\varphi'(s)}N_1(s) + \frac{\delta k_3(s)}{\varphi'(s)}N_3(s),$$

where

$$(3.15) \quad \cos \theta(s) = -\frac{\delta k_2(s)}{\varphi'(s)}, \quad \sin \theta(s) = \frac{\delta k_3(s)}{\varphi'(s)}.$$

Differentiating (3.4) with respect to s and using the Frenet equations, we obtain

$$(3.16) \quad \begin{aligned} & \varphi'(s)\overline{k_2}(\varphi(s))\overline{T}(\varphi(s)) - \varphi'(s)\overline{k_1}(\varphi(s))\overline{N_2}(\varphi(s)) \\ &= (\cos \theta(s))'N_1(s) + (\sin \theta(s))'N_3(s) \\ &+ (\cos \theta(s)k_2(s) - \sin \theta(s)k_3(s))T(s) - \cos \theta(s)k_1(s)N_2(s) \end{aligned}$$

for all $s \in I$. From the above fact, it holds

$$(3.17) \quad (\cos \theta(s))' = 0, \quad (\sin \theta(s))' = 0.$$

That is, θ is a constant function on L with value θ_0 . Let $\gamma = (\cos \theta_0)(\sin \theta_0)^{-1}$ be a constant number. So from (3.15), we get $\frac{k_2(s)}{k_3(s)} = -\gamma$. Thus we obtain the relation (3.1-d).

Conversely, we assume that $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ is a null curve with curvature functions $k_1(s) = 1$, $k_2(s)$ and $k_3(s) \neq 0$ satisfying the relation (3.1-a), (3.1-b), (3.1-c) and (3.1-d) for constant numbers λ , δ , γ and μ . Then we define a null curve $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ such as

$$(3.18) \quad \beta(s) = \alpha(s) + \lambda N_1(s) + \mu N_3(s)$$

for all $s \in I$. Differentiating (3.18) with respect to s and by using the Frenet equations, we have

$$\frac{d\beta(s)}{ds} = (1 + \lambda k_2(s) - \mu k_3(s))T(s) + (-\lambda k_1(s))N_2(s).$$

By using (3.1-b), we obtain

$$\frac{d\beta(s)}{ds} = -\lambda k_1(s)N_2(s)$$

for all $s \in I$. From (3.1-a), it follows $\frac{d\beta(s)}{ds} \neq 0$. We consider that the curve α is not a straight line, that is, the first curvature of the curve α is equal to one. So we can write

$$(3.19) \quad \beta'(s) = -\lambda N_2(s).$$

Differentiating (3.19) with respect to s , we get

$$\beta''(s) = \lambda(k_2(s)N_1(s) - k_3(s)N_3(s)).$$

There exists a regular map $\varphi : I \rightarrow \bar{I}$ defined by

$$\bar{s} = \varphi(s) = \int_0^s g(\beta''(t), \beta''(t))^{\frac{1}{4}} dt, \quad (\forall s \in I)$$

where $\bar{s}$ denotes the pseudo-arc length parameter of the curve β . From (3.1-c), we obtain

$$(3.20) \quad \varphi'(s) = \epsilon \frac{\lambda}{\delta}$$

for all $s \in I$, where $\varphi : I \rightarrow \bar{I}$ is a regular C^∞ -function and

$$\epsilon = \begin{cases} 1, & \frac{\lambda}{\delta} > 0 \\ -1, & \frac{\lambda}{\delta} < 0 \end{cases}.$$

Thus the curve β is rewritten as follows:

$$(3.21) \quad \beta(\bar{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda N_1(s) + \mu N_3(s).$$

Differentiating (3.21) with respect to s , we obtain

$$\varphi'(s) \frac{d\beta(\bar{s})}{d\bar{s}} \Big|_{\bar{s}=\varphi(s)} = (1 + \lambda k_2(s) - \mu k_3(s)) T(s) - \lambda N_2(s).$$

From (3.1-b)

$$(3.22) \quad \bar{T}(\varphi(s)) = -\frac{\lambda}{\varphi'(s)} N_2(s),$$

where $\bar{T}(\varphi(s)) = \frac{d\beta(\bar{s})}{d\bar{s}}$. By substituting (3.20) into (3.22), we get

$$(3.23) \quad \bar{T}(\varphi(s)) = -\epsilon\delta N_2(s)$$

for all $s \in I$. Differentiating (3.23) with respect to s and by using the Frenet equations, we have

$$\frac{d\bar{T}(\varphi(s))}{d\bar{s}} = \frac{\epsilon\delta k_2(s)}{\varphi'(s)} N_1(s) - \frac{\epsilon\delta k_3(s)}{\varphi'(s)} N_3(s)$$

and

$$(3.24) \quad \left\| \frac{d\bar{T}(\varphi(s))}{d\bar{s}} \right\| = \frac{\delta^2 ((k_2(s))^2 + (k_3(s))^2)}{(\varphi'(s))^2}.$$

Since the curve α is a null (1, 3)-Bertrand curve, the curvatures of α satisfy (3.1-c). If we substitute (3.1-c) into (3.24), we get

$$(3.25) \quad \bar{k}_1(\varphi(s)) = \left\| \frac{d\bar{T}(\varphi(s))}{d\bar{s}} \right\| = 1.$$

Then we can define a unit vector field $\bar{N}_1(\varphi(s))$ along the curve β by

$$\begin{aligned} \bar{N}_1(\varphi(s)) &= \frac{1}{\bar{k}_1(\varphi(s))} \frac{d\bar{T}(\varphi(s))}{d\bar{s}} \\ &= \frac{\epsilon\delta k_2(s)}{\varphi'(s)} N_1(s) - \frac{\epsilon\delta k_3(s)}{\varphi'(s)} N_3(s). \end{aligned}$$

Since $\bar{N}_1(\varphi(s))$ is expressed by linear combination of $N_1(s)$ and $N_3(s)$, we can put $\bar{N}_1(\varphi(s)) = \cos \tau(s) N_1(s) + \sin \tau(s) N_3(s)$, where

$$(3.26) \quad \cos \tau(s) = \frac{\epsilon\delta k_2(s)}{\varphi'(s)} \quad \sin \tau(s) = -\frac{\epsilon\delta k_3(s)}{\varphi'(s)}$$

for all $s \in I$ and $\tau(s)$ is a C^∞ -function. (3.1-c) and (3.1-d) imply that the curvatures of the curve α $k_2(s)$ and $k_3(s)$ are constants. On the other hand, $\varphi'(s) = \frac{\lambda}{\delta} = \text{constant}$. Thus $\cos \tau(s)$ and $\sin \tau(s)$ are constants, that is, $\tau(s) = \tau_0 = \text{constant}$. We can rewrite as

$$(3.27) \quad \bar{N}_1(\varphi(s)) = \cos \tau_0 N_1(s) + \sin \tau_0 N_3(s).$$

Differentiating (3.27) with respect to s and by using the Frenet equations, we have

$$\frac{d\overline{N}_1(\varphi(s))}{d\overline{s}}\varphi'(s) = (k_2 \cos \tau_0 - k_3 \sin \tau_0) T(s) - \cos \tau_0 N_2(s).$$

By using (3.26) in the above equality, we get

$$\frac{d\overline{N}_1(\varphi(s))}{d\overline{s}} = \frac{\epsilon}{\delta} T(s) - \frac{\epsilon \delta k_2}{(\varphi'(s))^2} N_2(s)$$

for all $s \in I$. From the Frenet equations

$$(3.28) \quad \overline{k}_2(\varphi(s)) = -\frac{1}{2}g\left(\frac{d\overline{N}_1(\varphi(s))}{d\overline{s}}, \frac{d\overline{N}_1(\varphi(s))}{d\overline{s}}\right) = \frac{k_2}{(\varphi'(s))^2}.$$

Thus we can define a unit vector field $\overline{N}_2(\varphi(s))$ along the curve β by

$$\overline{N}_2(\varphi(s)) = \overline{k}_2(\varphi(s))T(\varphi(s)) - \frac{d\overline{N}_1(\varphi(s))}{d\overline{s}},$$

that is,

$$\overline{N}_2(\varphi(s)) = -\frac{\epsilon}{\delta} T(s)$$

Next we can define a unit vector $\overline{N}_3(\varphi(s))$ along the curve β by $\overline{N}_3(\varphi(s)) = -\sin \tau_0 N_1(s) + \cos \tau_0 N_3(s)$, that is,

$$\overline{N}_3(\varphi(s)) = \frac{\epsilon \delta k_3(s)}{\varphi'(s)} N_1(s) + \frac{\epsilon \delta k_2(s)}{\varphi'(s)} N_3(s).$$

Thus we obtain

$$(3.29) \quad \overline{k}_3(\varphi(s)) = -g\left(\frac{d\overline{N}_3(\overline{s})}{d\overline{s}}, \overline{N}_2(\overline{s})\right) = -\frac{k_3(s)}{(\varphi'(s))^2}.$$

Notice that $g(\overline{T}, \overline{T}) = g(\overline{N}_2, \overline{N}_2) = 0$, $g(\overline{N}_1, \overline{N}_1) = g(\overline{N}_3, \overline{N}_3) = 1$ and $g(\overline{T}, \overline{N}_1) = g(\overline{T}, \overline{N}_3) = g(\overline{N}_1, \overline{N}_3) = g(\overline{N}_1, \overline{N}_2) = g(\overline{N}_2, \overline{N}_3) = 0$, $g(\overline{T}, \overline{N}_2) = 1$ for all $s \in I$, where $\{\overline{T}, \overline{N}_1, \overline{N}_2, \overline{N}_3\}$ is moving Frenet frame along null curve β in E_1^4 . And it is trivial that the Frenet (1,3)-normal plane at each point $\alpha(s)$ of the curve α coincides with the Frenet (1,3)-normal plane at corresponding point $\beta(\overline{s})$ of the curve β . Hence α is a null (1,3)-Bertrand curve in $\mathbb{E}_1^4$. This completes the proof. $\square$

Corollary 3.1. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand curve with curvature functions $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and $\beta : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand mate of the curve α with curvature functions $\bar{k}_1(\bar{s})$, $\bar{k}_2(\bar{s})$, $\bar{k}_3(\bar{s})$ where s and $\bar{s}$ denote the arc-length parameter of the curves α and β , respectively. Then the relations between these curvature functions are*

$$\bar{k}_1(\varphi(s)) = 1, \bar{k}_2(\varphi(s)) = \frac{k_2(s)}{(\varphi'(s))^2}, \bar{k}_3(\varphi(s)) = -\frac{k_3(s)}{(\varphi'(s))^2},$$

where $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$ is a regular C^∞ -function for all $s \in I$.

Proof. It is obvious from the Theorem (3.1). $\square$

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand curve with curvatures $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and β be a null $(1, 3)$ -Bertrand mate of the curve α and $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$ is a regular C^∞ -function such that each point $\alpha(s)$ of the curve α corresponds to the point $\beta(\bar{s}) = \beta(\varphi(s))$ of the curve β for all $s \in I$. Then the distance between the points $\alpha(s)$ and $\beta(\bar{s})$ is constant for all $s \in I$.*

Proof. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand curve with curvatures $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and β be a null $(1, 3)$ -Bertrand mate of the curve α . We assume that β is distinct from α . Let the pairs of $\alpha(s)$ and $\beta(\bar{s}) = \beta(\varphi(s))$ (where $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$ is a regular C^∞ -function) be corresponding points of the curves α and β . Then we can write $\beta(\bar{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda N_1(s) + \mu N_3(s)$, where λ and μ are non-zero constants. Thus we can write $\beta(\bar{s}) - \alpha(s) = \lambda N_1(s) + \mu N_3(s)$ and $\|\beta(\bar{s}) - \alpha(s)\| = \sqrt{\lambda^2 + \mu^2}$. So, $d(\alpha(s), \beta(\bar{s})) = \text{constant}$. This completes the proof. $\square$

Corollary 3.3. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand curve with curvature functions $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and β be a null $(1, 3)$ -Bertrand mate of the curve α with curvature functions $\bar{k}_1(\varphi(s))$, $\bar{k}_2(\varphi(s))$, $\bar{k}_3(\varphi(s))$. Then the curvatures of the curves α and β are constants.*

Proof. We assume that $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ is a null $(1, 3)$ -Bertrand curve with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ and β is a null $(1, 3)$ -Bertrand mate of the curve α . In that case, the relations (3.1-c) and (3.1-d) imply that $k_2(s)$, $k_3(s)$ are constant. From the Corollary 3.1, it can be easily seen that the curvatures of the curve β are constant curvatures. $\square$

Corollary 3.4. *If $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ is a null $(1, 3)$ -Bertrand curve with non-zero curvatures and β is a null $(1, 3)$ -Bertrand mate of α then α and β are null helices.*

Proof. We assume that $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ is a null $(1, 3)$ -Bertrand curve with curvatures $k_1(s) = 1$, $k_2(s)$, $k_3(s) \neq 0$ and β is a null $(1, 3)$ -Bertrand mate of the curve α . Then both the curvatures $k_2(s)$, $k_3(s)$ belong to the curve α and the curvatures $\bar{k}_2(\varphi(s))$, $\bar{k}_3(\varphi(s))$ belong to the curve β are nonzero constants. Hence they are null helices. $\square$

Corollary 3.5. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$ be a null $(1, 3)$ -Bertrand curve and β be a null $(1, 3)$ -Bertrand mate of α . Then the curvatures $k_2(s)$, $k_3(s)$ belong to the curve α and the curvatures $\bar{k}_2(\varphi(s))$, $\bar{k}_3(\varphi(s))$ belong to the curve β satisfy $\bar{k}_2 k_3 + k_2 \bar{k}_3 = 0$.*

Proof. It is obvious from (3.28) and (3.29). $\square$

Example 3.1. Let α be a null curve in E_1^4 given by

$$\alpha(s) = \frac{1}{\sqrt{2}} (\sinh(s), \cosh(s), \sin(s), \cos(s)).$$

The Frenet frame of the curve α is given by

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}} (\cosh(s), \sinh(s), \cos(s), -\sin(s)), \\ N_1(s) &= \frac{1}{\sqrt{2}} (\sinh(s), \cosh(s), -\sin(s), -\cos(s)), \\ N_2(s) &= \frac{1}{\sqrt{2}} (-\cosh(s), -\sinh(s), \cos(s), -\sin(s)), \\ N_3(s) &= \frac{1}{\sqrt{2}} (\sinh(s), \cosh(s), \sin(s), \cos(s)). \end{aligned}$$

Then we get the curvatures of α as follows $k_1(s) = 1$, $k_2(s) = 0$, $k_3(s) = -1$. By using the Theorem B, the curve α is not a null Bertrand curve. But if we take constants λ , δ , γ and μ as $\lambda = 1$, $\delta = 1$, $\gamma = 0$, $\mu = -1$, then it is trivial that the relations (3.1-a), (3.1-b), (3.1-c) and (3.1-d) hold. Therefore, the curve α is a null $(1, 3)$ -Bertrand curve in $\mathbb{E}_1^4$. In this case, the null $(1, 3)$ -Bertrand mate of the curve α is given by

$$\beta(s) = \frac{1}{\sqrt{2}} (\sinh(s), \cosh(s), -\sin(s), -\cos(s)).$$

By using (3.25), (3.28), (3.29), we obtain the curvatures of the curve β as $\bar{k}_1 = 1$, $\bar{k}_2 = 0$ and $\bar{k}_3 = 1$.

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