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(FUZZY) ISOMORPHISM THEOREMS OF SOFT Γ-HYPERRINGS

BY

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Abstract. Soft set theory, introduced by Molodtsov, has been considered as an effective mathematical tool for modeling uncertainties. In this paper, we apply soft sets to Γ-hyperrings. The concept of soft Γ-hyperrings is first introduced. Then three isomorphism theorems of soft Γ-hyperrings are established. Finally, we derive three fuzzy isomorphism theorems of soft Γ-hyperrings.

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1. Introduction

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. Alternatively, mathematical theories, such as probability theory, fuzzy set theory, vague set theory, rough set theory and interval mathematics, have been proven to be useful mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties, as pointed out by Molodtsov in [25]. At present, works on the soft set theory are progressing rapidly. Ali et al. [2] proposed some new operations on soft sets. Chen et al. [5] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attribute reduction in rough set theory. In particular, fuzzy soft set theory has been investigated by some researchers, for examples, see [12, 22, 23]. Recently, the algebraic structures of soft sets have been studied increasingly, for example, see [1, 13].

On the other hand, the theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by MARTY in 1934 (see [24]) when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. Later on, many people have observed that the theory of hyperstructures also have many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Some review of the theory of hyperstructures can be found in [6, 7, 9, 14, 18, 26], respectively. A well known type of a hyperring, called the *Krasner hyperring* [17]. Krasner hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation (i.e., $a+b$ is a set). Then this concept has been studied by a variety of authors, see [8, 9]. In particular, the relationships between the fuzzy sets and hyperrings have been considered by many researchers, for example, see [19, 27, 28, 29, 30].

The concept of Γ-rings was introduced by BARNES [4]. After that, this concept was discussed further by some researchers. The notion of fuzzy ideals in a Γ-ring was introduced by JUN and LEE in [16]. They studied some preliminary properties of fuzzy ideals of Γ-rings. JUN [15] defined fuzzy prime ideals of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal. In particular, DUTTA and CHANDA [11], studied the structures of the set of fuzzy ideals of a Γ-ring. MA ET AL. [21] considered the characterizations of Γ-hemirings and Γ-rings, respectively. Recently, AMERI ET AL. [3] considered the concept of fuzzy hyperideals of Γ-H_V-rings. MA ET AL. [20] considered the (fuzzy) isomorphism theorems of Γ-hyperrings. In the same time, DAVVAZ ET AL. [10] considered the properties of Γ-hypernear-rings, and derived some related results.

In this paper, we will discuss soft Γ-hyperrings.

In Section 2, we recall some basic concepts of Γ-hyperrings.

In Section 3, we derive three isomorphism theorems of soft Γ-hyperrings. In particular, we establish three fuzzy isomorphism theorems of soft Γ-hyperrings in Section 4.

2. Preliminaries

A *hypergroupoid* is a non-empty set $\mathcal{H}$ together with a mapping $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$, where $\mathcal{P}^*(\mathcal{H})$ is the set of all the non-empty subsets of $\mathcal{H}$.

A *quasicanonical hypergroup* (not necessarily commutative) is an algebraic structure $(\mathcal{H}, +)$ satisfying the following conditions:

(i) for every $x, y, z \in \mathcal{H}, x + (y + z) = (x + y) + z$;

(ii) there exists a $0 \in \mathcal{H}$ such that $0 + x = x$, for all $x \in \mathcal{H}$;

(iii) for every $x \in \mathcal{H}$, there exists a unique element $x' \in \mathcal{H}$ such that $0 \in (x + x') \cap (x' + x)$. (we call the element $-x$ the opposite of x);

(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Quasicanonical hypergroups are also called *polygroups.*

We note that if $x \in \mathcal{H}$ and A, B are non-empty subsets in $\mathcal{H}$, then by $A + B$, $A + x$ and $x + B$ we mean that $A + B = \bigcup_{a \in A, b \in B} a + b, A + x = A + \{x\}$ and $x + B = \{x\} + B$, respectively. Also, for all $x, y \in \mathcal{H}$, we have $-(-x) = x, -0 = 0$, where 0 is unique and $-(x + y) = -y - x$.

A sub-hypergroup $A \subset \mathcal{H}$ is said to be *normal* if $x + A - x \subseteq A$ for all $x \in \mathcal{H}$. A normal sub-hypergroup A of $\mathcal{H}$ is called *left (right) hyperideal* of $\mathcal{H}$ if $xA \subseteq A$ ($Ax \subseteq A$ respectively) for all $x \in \mathcal{H}$. Moreover A is said to be a *hyperideal* of $\mathcal{H}$ if it is both a left and a right hyperideal of $\mathcal{H}$. A *canonical hypergroup* is a commutative quasicanonical hypergroup.

Definition 2.1 ([17])**.** A *hyperring* is an algebraic structure $(R, +, \cdot)$, which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $0 \cdot x = x \cdot 0 = 0$, for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation "+" that is, $z \cdot (x + y) = z \cdot x + z \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

Definition 2.2 ([3])**.** Let $(R, \oplus)$ and $(\Gamma, \oplus)$ be two canonical hypergroups. Then R is called a Γ-*hyperring*, if the following conditions are satisfied for all $x, y, z \in R$ and for all $\alpha, \beta \in \Gamma$,

(1) $x\alpha y \in R$;

(2) $(x \oplus y)\alpha z = x\alpha z \oplus y\alpha z, x(\alpha \oplus \beta)y = x\alpha y \oplus x\beta y, x\alpha(y \oplus z) = x\alpha y \oplus x\alpha z$;

(3) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

In the sequel, unless otherwise stated, $(R, \oplus, \Gamma)$ always denotes a Γ-hyperring.

A subset A in R is said to be a *left* (*right*) Γ-*hyperideal* of R if it satisfies the following conditions:

(1) $(A, \oplus)$ is a normal sub-hypergroup of $(R, \oplus)$;

(2) $x\alpha y \in A$ ($y\alpha x \in A$ respectively) for all $x \in R$, $y \in A$ and $\alpha \in \Gamma$.

A is said to be a Γ- *hyperideal* of R if it is both a left and a right Γ-hyperideal of R.

Definition 2.3 ([20]). A fuzzy set μ of a Γ-hyperring R is called a *fuzzy* Γ- *hyperideal* of R if the following conditions hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$, for all $x, y \in R$;

(2) $\mu(x) \leq \mu(-x)$, for all $x \in R$;

(3) $\max\{\mu(x), \mu(y)\} \leq \mu(x\alpha y)$, for all $x, y \in R$ and for all $\alpha \in \Gamma$;

(4) $\mu(x) \leq \inf_{z \in -y+x+y} \mu(z)$, for all $x, y \in R$.

Definition 2.4. Let R be a Γ-ring such that $x(-\alpha)y = -x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$. Denote $\overline{R} = \{\overline{x} = \{x, -x\} | x \in R\}$ and $\overline{\Gamma} = \{\overline{\alpha} = \{\alpha, -\alpha\} | \alpha \in \Gamma\}$. Define the hyperoperations on $\overline{R}$ and $\overline{\Gamma}$ as follows: $\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\}, \overline{\alpha} \oplus \overline{\beta} = \{\overline{\alpha+\beta}, \overline{\alpha-\beta}\}$ and $\overline{x}\,\overline{\alpha}\,\overline{y} = \overline{x\alpha y}$ for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. Then $(\overline{R}, \oplus, \overline{\Gamma})$ is a Γ-hyperring.

Example 2.5 ([20]). Let $(G, \cdot)$ be a group and $\Gamma = G$. Denote $G^0 = \Gamma^0 = G \cup \{0\}$ and define $x\alpha y = x \cdot \alpha \cdot y$ for all $x, y \in G$ and $\alpha \in \Gamma$. Then $(G^0, \oplus, \Gamma^0)$ is a Γ^0-hyperring with respect to the hyperoperation " $\oplus$ " on G^0 and Γ^0, defined by

$x \oplus 0 = 0 \oplus x = \{x\}$, for all $x \in G^0$,

$x \oplus x = G^0 \backslash \{x\}$, for all $x \in G^0 \backslash \{0\}$,

$x \oplus y = \{x, y\}$, for all $x, y \in G^0 \backslash \{0\}$ with $x \neq y$,

and

$\alpha \oplus 0 = 0 \oplus \alpha = \{\alpha\}$, for all $\alpha \in \Gamma^0$,

$\alpha \oplus \alpha = \Gamma^0 \backslash \{\alpha\}$, for all $\alpha \in \Gamma^0 \backslash \{0\}$,

$\alpha \oplus \beta = \{\alpha, \beta\}$, for all $\alpha, \beta \in \Gamma^0 \backslash \{0\}$ with $\alpha \neq \beta$, respectively.

Definition 2.6 ([20]). If R and R' are Γ-hyperrings, then a mapping $f : R \longrightarrow R'$ such that $f(x \oplus y) = f(x) \oplus f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$, for all $x, y \in R$ and $\alpha \in \Gamma$, is called a Γ-*hyperring homomorphism*.

Clearly, a Γ-hyperring homomorphism f is an isomorphism if f is injective and surjective. We write $R \cong R'$ if R is isomorphic to R'.

If N is a Γ-hyperideal of R, then we define the relation N^* by $x \equiv y (\operatorname{mod} N) \Longleftrightarrow (x - y) \cap N \neq \emptyset$.

This is a congruence relation on R.

Let N be a Γ-hyperideal of R. Then, for $x, y \in N$, the following are equivalent:

(1) $(x - y) \cap N \neq \emptyset$;

(2) $x - y \subseteq N$;

(3) $y \in x + N$.

The class $x + N$ is represented by x and we denote it with $N^*(x)$. Moreover, $N^*(x) = N^*(y)$ if and only if $x \equiv y(modN)$. We can define R/N as follows $R/N = \{N^*(x)|x \in R\}$.

Define a hyperoperation $\boxplus$ and an operation $\odot_\alpha$ on R/N by

$N^*(x) \boxplus N^*(y) = \{N^*(z)|z \in N^*(x) \oplus N^*(y)\}$;

$N^*(x) \odot_\alpha N^*(y) = N^*(x\alpha y)$, for all $N^*(x), N^*(y) \in R/N$.

Then, $(R/I, \boxplus, \odot_\alpha)$ is a Γ-hyperring, see [20].

3. Isomorphism theorems

In what follows, let R be a Γ-hyperring and A be a non-empty set. A set-valued function $F : A \to \mathcal{P}(R)$ can be defined as $F(x) = \{y \in R \mid (x, y) \in \rho\}$ for all $x \in A$, where ρ is an arbitrary binary relation between an element of A and an element of R, that is, ρ is a subset of $A \times R$. Then the pair (F, A) is a soft set over R.

For a soft set (F, A) over R, the set $\text{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the support of the soft set (F, A). A soft set (F, A) is non-null if $\text{Supp}(F, A) \neq \emptyset$.

Definition 3.1. Let (F, A) be a non-null soft set over R. Then (F, A) is called a *soft* Γ*-hyperring over* R if $F(x)$ is a Γ-hyperideal of R for all $x \in \text{Supp}(F, A)$.

Example 3.2. Let $R = \Gamma = \{0, 1, 2\}$ be two canonical hypergroups with hyperoperation $\oplus$ as follows:

$\oplus$	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

Define a mapping $R \times \Gamma \times R \to R$ by $a\gamma b = a \cdot \gamma \cdot b$ for all $a, b \in R$ and $\gamma \in \Gamma$, where "$\cdot$" is the following multiplication.

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then it can be easily verified that $(R, \oplus, \Gamma)$ is a Γ-hyperring.

Let (F, A) be a soft set over R, where $A = R$ and $F : A \to \mathcal{P}(R)$ is a set-valued function given by $F(x) = \{y \in R \mid x\rho y \Leftrightarrow x \in \{0\}\}$ for all $x \in A$. Then $F(0) = \{0, 1, 2\}$ and $F(1) = F(2) = \{0\}$ are all Γ-hyperideals of R. Thus (F, A) is a soft Γ-hyperring over R.

Definition 3.3. Let R_1 and R_2 be two Γ-hyperrings, (F, A) and (G, B) be soft Γ-hyperrings over R_1 and R_2, respectively, and $f : R_1 \to R_2$ and $g : A \to B$ be two functions. Then (f, g) is called a *soft* Γ-*hyperring homomorphism* if the following conditions hold:

(1) f is a Γ-hyperring homomorphism;

(2) g is a mapping;

(3) for all $x \in A$, $f(F(x)) = G(g(x))$.

If there is a soft Γ-hyperring homomorphism (f, g) between (F, A) and (G, B), we say that (F, A) is soft Γ-hyperring homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$. Furthermore, if f is a monomorphism (resp. epimorphism, isomorphism) and g is a injective (resp. surjective, bijective) mapping, then (f, g) is called a soft monomorphism (resp. epimorphism, isomorphism), and (F, A) is soft monomorphic (resp. epimorphic, isomorphic) to (G, B). We use $(F, A) \cong (G, B)$ to denote that (F, A) is soft Γ-hyperring isomorphic to (G, B).

The following proposition is obvious.

Proposition 3.4. *Let N be a Γ-hyperideal of R, and (F, A) be a soft Γ-hyperring over R, then (F, A) is soft Γ-hyperring epimorphic to $(F/N, A)$, where $(F/N)(x) = F(x)/N$ for all $x \in A$, and $N \subseteq F(x)$ for all $x \in$ Supp(F, A) (if $x \in A-$ Supp(F, A), we mean that $(F/N)(x) = \emptyset$).*

Next, we establish three isomorphism theorems of soft Γ-hyperrings.

Theorem 3.5 (First Isomorphism Theorem)**.** *Let R_1 and R_2 be two Γ-hyperrings, (F, A) and (G, B) be soft Γ-hyperrings over R_1 and R_2, respectively. If (f, g) is a soft Γ-hyperring epimorphism from (F, A) to (G, B)*

with kernel N such that N is a Γ-hyperideal of R_1 and $N \subseteq F(x)$ for all $x \in supp(F, A)$, then:

(1) $(F/N, A) \simeq (f(F), A)$;

(2) *if g is bijective, then* $(F/N, A) \simeq (G, B)$.

Proof. (1) It is clear that $(F/N, A)$ and $(f(F), A)$ are soft Γ-hyperrings over R_1/N and R_2, respectively.

Define $\overline{f} : R_1/N \to R_2$ by $\overline{f}(N^*[x]) = f(x)$, for all $x \in R_1$. If xN^*y, we have $(x - y) \cap N \neq \emptyset$, that is, there exists $z \in (x - y) \cap N$. Hence $f(z) = 0$ and $f(z) \in f(x) - f(y)$. It follows that $f(x) = f(y)$. So $\overline{f}$ is well-defined.

Since f is surjective, it is clear that $\overline{f}$ is surjective. To show that $\overline{f}$ is injective, assume that $f(x) = f(y)$, then we have $0 \in f(x - y)$. Thus, there exists $z \in x - y$ such that $z \in ker f$. It follows that $(x - y) \cap N \neq \emptyset$, which implies $N^*[x] = N^*[y]$. There $\overline{f}$ is injective. Furthermore, we have

$$\begin{aligned} \overline{f}(N^*[x] \boxplus N^*[y]) &= \overline{f}(\{N^*[z] \mid z \in N^*[x] \oplus N^*[y]\}) \\ &= \{f(z) \mid z \in N^*[x] \oplus N^*[y]\} = f(N^*[x]) \oplus f(N^*[y]) \\ &= f(x) \oplus f(y) = \overline{f}([N^*[x]) \oplus \overline{f}(N^*[y]), \end{aligned} \tag{1}$$

$$\begin{aligned} \overline{f}(N^*[x] \odot_\alpha N^*[y]) &= \overline{f}(N^*[x\alpha y]) = f(x\alpha y) \\ &= f(x)\alpha f(y) = \overline{f}(N^*[x])\alpha\overline{f}(N^*[y]). \end{aligned} \tag{2}$$

Thus, $\overline{f}$ is a Γ-hyperring isomorphism.

Define $\overline{g} : A \to A$ by $\overline{g}(x) = x$ for all $x \in A$, then $\overline{g}$ is a bijective mapping. Furthermore, $\overline{f}(F(x)/N) = f(F(x)) = f(F(g(x)))$ for all $x \in A$. Therefore, $(\overline{f}, \overline{g})$ is a soft Γ-hyperring isomorphism, and $(F/N, A) \cong (f(F), A)$.

(2) Since $\overline{f}$ is an isomorphism, g is bijective and for all $x \in A$, $\overline{f}(F(x)/N) = f(F(x)) = G(g(x))$. Hence, $(\overline{f}, g)$ is a soft Γ-hyperring isomorphism. So we have $(F/N, A) \cong (G, B)$. □

Theorem 3.6 (Second Isomorphism Theorem)**.** *Let N and K be two Γ-hyperideals of R. If (F, A) is a soft Γ-hyperring of K, then we have $(F/(N \cap K), A) \simeq ((N + F)/N, A)$, where $N \cap K \subseteq F(x)$ for all $x \in supp(F, A)$.*

Proof. It is clear that $(F/(N \cap K), A)$ and $((N + F)/N, A)$ are soft Γ-hyperrings over $(K/(N \cap K)$ and $(N + K)/N$, respectively.

Define $f : K \to (N + K)/N$ by $f(x) = N^*[x]$ for all $x \in K$. It is easy to check that f is a Γ-hyperring homomorphism. For any $N^*[x] \in (N + K)/N$, where $x \in N+K$, that is, there exist $a \in N$ and $b \in K$ such that $x \in a+b$, we have $N^*[x] = N+x = N+a+b = N+b = N^*[b] = f(b)$. Thus, f is a Γ-hyperring epimorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$. Then g is bijective.

We know $\{N^*[a] \mid a \in F(x)\} \subseteq (N + F(x))/N$. On the other hand, for any $N^*[b] \in (N + F(x))/N$, where $b \in N + F(x)$, which implies that there exist $n \in N$ and $k \in F(x)$ such that $b \in n + k$, we have $N^*[b] = N + b = N + n + k = N + k = N^*[k] \in \{N^*[a] \mid a \in F(x)\}$, which implies, $(N + F(x))/N \subseteq \{N^*[a] \mid a \in F(x)\}$, and so $\{N^*[a] \mid a \in F(x)\} = (N + F(x))/N$. For all $x \in A$, we have $f(F(x)) = \{N^*[a] \mid a \in F(x)\} = (N + F(x))/N = (N + F(g(x)))/N$.

Therefore, (f, g) is a soft Γ-hyperring epimorphism from (F, A) to $((N+F)/N, A)$.

Since $N \cap K$ is a Γ-hyperideal of K, we have $kerf = N \cap K$. In fact, for any $x \in K$, $x \in kerf \Leftrightarrow f(x) = N^*[0] = N \Leftrightarrow N^*[x] = N+x = N \Leftrightarrow x \in N$ (since $x \in K$)$\Leftrightarrow x \in N \cap K$. Hence $kerf = N \cap K$.

Therefore, it follows from Theorem 3.5 that $(F/(N \cap K), A) \cong ((N + F)/N, A)$. □

Theorem 3.7 (Third Isomorphism Theorem)**.** *Let N and K be two Γ-hyperrings of R such that $N \subseteq K$. If (F, A) is a soft Γ-hyperring over R, and $K \subseteq F(x)$ for all $x \in supp(F, A)$, then we have $((F/N)/(K/N), A) \simeq (F/K, A)$.*

Proof. Since K and N are Γ-hyperrings of R, and $N \subseteq K$, we know that K/N is a Γ-hyperring of R/N, and so $(R/N)/(K/N)$ is well-defined.

Furthermore, we can deduce easily that $(F/N, A)$, $(F/K, A)$ and $((F/N)/(K/N), A)$ are soft Γ-hyperrings over R/N, R/K and $(R/N)/(K/N)$, respectively.

Define $f : R/N \to R/K$ by $f(N^*[x]) = K^*[x]$. It is clear that f is a Γ-hyperring epimorphism.

We define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective. Furthermore, for all $x \in A$, $f(F(x)/N) = F(x)/K = F(g(x))/K$.

Consequently, (f, g) is a soft Γ-hyperring epimorphism from $(F/N, A)$ to $(F/K, A)$. To show that $kerf = K/N$. In fact, for any $N^*[x] \in R/N$, $N^*[x] \in kerf \Leftrightarrow f(N^*[x]) = K^*[0] = K \Leftrightarrow K^*[x] = K + x = K \Leftrightarrow x \in K \Leftrightarrow N^*[x] \in K/N$. Thus, we have $kerf = K/N$. Therefore, it follows from Theorem 3.5 that $((F/N)/(K/N), A) \simeq (F/K, A)$. □

4. Fuzzy isomorphism theorems

Let μ be a fuzzy Γ-hyperideal of R. Define the relation on R:

$$x \equiv y(mod\ \mu)$$

if and only if there exists $r \in x - y$ such that $\mu(r) = \mu(0)$, denoted by $x\mu^*y$. The relation μ^* is an equivalence relation. If $x\mu^*y$, then $\mu(x) = \mu(y)$.

Let $\mu^*[x]$ be the equivalence class containing the element $x \in R$, and R/μ be the set of all equivalence classes, i.e., $R/\mu = \{\mu^*[x] \mid x \in R\}$. Define the following two operations in R/μ:

$$\mu^*[x] \boxplus \mu^*[y] = \{\mu^*[z] \mid z \in \mu^*[x] + \mu^*[y]\}, \quad \mu^*[x] \odot_\alpha \mu^*[y] = \mu^*[x\alpha y].$$

Then $(R/\mu, \boxplus, \odot_\alpha)$ is a Γ-hyperring, see [20].

Let N be a Γ-hyperideal of R, and μ be a fuzzy Γ-hyperideal of R. If μ is restricted to N, then μ is a fuzzy Γ-hyperideal of N, and N/μ is a Γ-hyperideal of R/μ. Furthermore, if μ and ν are fuzzy Γ-hyperideals of R, then so is $\mu \cap \nu$, see [20].

If X and Y are two non-empty sets, $f : X \to Y$ is a mapping, and μ and ν are the fuzzy sets of X and Y, respectively, then the image $f(\mu)$ of μ is the fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu(x)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for all $y \in Y$. The inverse image $f^{-1}(\nu)$ of ν is the fuzzy subset of X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Let R_1 and R_2 be two Γ-hyperrings, and $f : R_1 \to R_2$ be a Γ-hyperring homomorphism. If μ and ν are fuzzy Γ-hyperideals of R_1 and R_2, respectively, then (1) $f(\mu)$ is a fuzzy Γ-hyperideal of R_2; (2) if f is an Γ-hyperring epimorphism, then $f^{-1}(\nu)$ is a fuzzy Γ-hyperideal of R_1. If μ and ν are fuzzy Γ-hyperideals of R_1 and R_2, respectively, then (1) if f is Γ-hyperring epimorphism, then $f(f^{-1}(\nu)) = \nu$; (2) if μ is a constant on $ker f$, then $f^{-1}(f(\mu)) = \mu$ (see [20]).

Let μ be a fuzzy Γ-hyperideal of R, then $R_\mu = \{x \in M \mid \mu(x) = \mu(0)\}$ is a Γ-hyperideal of R.

Theorem 4.1 (First Fuzzy Isomorphism Theorem)**.** *Let R_1 and R_2 be two Γ-hyperrings, and (F, A) and (G, B) be soft Γ-hyperrings over R_1 and*

R_2, respectively. If (f,g) is a soft Γ-hyperring epimorphism from (F,A) to (G,B) and μ is a fuzzy Γ-hyperideal of R_1 with $(R_1)_\mu \supseteq ker f$, then

(1) $(F/\mu, A) \simeq (f(F)/f(\mu), A)$, where $(F/\mu)(x) = F(x)/\mu$ for all $x \in A$;

(2) if g is bijective, then $(F/\mu, A) \simeq (G/f(\mu), B)$.

Proof. (1) Since (F,A) is soft Γ-hyperring over R_1, and μ is a fuzzy Γ-hyperideal of R_1, $(F/\mu, A)$ is a soft Γ-hyperring over R_1/μ. For all $x \in \mathrm{supp}(F,A)$, $f(F(x)) = G(g(x)) \neq \emptyset$ is a Γ-hyperideal of R_2. It follows that $(f(F)/f(\mu), A)$ is a soft Γ-hyperring over $R_2/f(\mu)$.

Define $\overline{f} : R_1/\mu \to R_2/f(\mu)$ by $\overline{f}(\mu^*[x]) = f(\mu)^*[f(x)]$, for all $x \in R_1$. If $\mu^*[x] = \mu^*[y]$, then $\mu(x) = \mu(y)$. Since $(R_1)_\mu \supseteq ker f$, μ is a constant on $ker f$. So we have $f^{-1}(f(\mu)) = \mu$.

It follows that $f^{-1}(f(\mu))(x)=f^{-1}(f(\mu))(y)$, i.e., $f(\mu)(f(x))=f(\mu)(f(y))$. Thus, $f(\mu)^*[(f(x))] = f(\mu)^*[(f(y))]$. So $\overline{f}$ is well-defined. Furthermore, we have

(i) $\overline{f}(\mu^*[x] \boxplus \mu^*[y]) = \overline{f}(\{\mu^*[z] \mid z \in \mu^*[x] \oplus \mu^*[y]\}) = \{f(\mu)^*[f(z)] \mid z \in \mu^*[x] \oplus \mu^*[y]\} = f(\mu)^*(f(\mu^*[x])) \oplus f(\mu)^*(f(\mu^*[y])) = \overline{f}(\mu^*[x]) \oplus \overline{f}(\mu^*[y])$;

(ii) $\overline{f}(\mu^*[x] \odot_\alpha \mu^*[y]) = \overline{f}(\mu^*[x\alpha y]) = f(\mu)^*(f(x\alpha y)) = f(\mu)^*(f(x)\alpha f(y)) = f(\mu)^*([f(x)])\alpha f(\mu)^*([f(y)]) = \overline{f}(\mu^*[x])\alpha\ \overline{f}(\mu^*[y])$.

Hence, $\overline{f}$ is a Γ-hyperring homomorphism. Clearly, $\overline{f}$ is a Γ-hyperring epimorphism. Now, we show that $\overline{f}$ is a Γ-hyperring monomorphism. Let $f(\mu)^*[f(x)] = f(\mu)^*[f(y)]$, then we have $f(\mu)(f(x)) = f(\mu)(f(y))$, i.e.,

$(f^{-1}(f(\mu)))(x) = (f^{-1}(f(\mu)))(y)$, and so $\mu(x) = \mu(y)$. Furthermore, we have $\mu^*[x] = \mu^*[y]$. Therefore, $\overline{f}$ is a Γ-hyperring isomorphism.

Define $\overline{g} : A \to A$ by $\overline{g}(x) = x$ for all $x \in A$, then $\overline{g}$ is a bijective mapping. Furthermore, for all $x \in A$, we have $\overline{f}(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = f(F(\overline{g}(x)))/f(\mu)$. Consequently, $(\overline{f}, \overline{g})$ is a soft Γ-hyperring isomorphism. So we have $(F/\mu, A) \cong (f(F)/f(\mu), A)$.

(2) Since $\overline{f}$ is a Γ-hyperring isomorphism, g is bijective and for all $x \in A$, $\overline{f}(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = G(g(x))/f(\mu)$. Hence, $(\overline{f}, g)$ is a soft Γ-hyperring isomorphism. Furthermore, we have $(F/\mu, A) \cong (G/f(\mu), B)$. $\square$

Corollary 4.2. *Let R_1 and R_2 be two Γ-hyperrings, and (F,A) and (G,B) be soft Γ-hyperrings over R_1 and R_2 respectively. If (f,g) is a soft Γ-hyperring epimorphism from (F,A) to (G,B) and ν is a fuzzy Γ-hyperideal of R_2, then we have:*

(1) $(F/f^{-1}(\nu), A) \cong (f(F)/\nu, A)$;

(2) if g is bijective, then $(F/f^{-1}(\nu), A) \cong (G/\nu, B)$.

Now, we give the Second Fuzzy and Third Fuzzy Isomorphism Theorems.

Theorem 4.3 (Second Fuzzy Isomorphism Theorem)**.** *Let (F, A) be a soft Γ-hyperring over R. If μ and ν are two fuzzy Γ-hyperideals with $\mu(0) = \nu(0)$, then we have $(F_\mu/(\mu \cap \nu), A) \simeq ((F_\mu + F_\nu)/\nu, A)$.*

Proof. We know ν and $\mu \cap \nu$ are two fuzzy Γ-hyperideals of $R_\mu + R_\nu$ and R_μ, respectively. Thus $(R_\mu + R_\nu)/\nu$ and $R_\mu/(\mu \cap \nu)$ are both Γ-hyperrings.

Since (F, A) is a soft Γ-hyperring over R, we can deduce that $(F_\mu/(\mu \cap \nu), A)$ and $((F_\mu + F_\nu)/\nu, A)$ are soft Γ-hyperrings over $R_\mu/(\mu \cap \nu)$ and $(R_\mu + R_\nu)/\nu$, respectively.

Define $f : R_\mu \to (R_\mu + R_\nu)/\nu$ by $f(x) = \nu^*[x]$, for all $x \in R_\mu$. It is easy to see that f is a Γ-hyperring epimorphism. We check that $ker f = \mu \cap \nu$.

$\ker f = \{x \in R_\mu | \ f(x) = \nu^*[0]\} = \{x \in R_\mu | \ \nu^*[x] = \nu^*[0]\} = \{x \in R_\mu | \ \nu(x) = \nu(0)\} = \{x \in R_\mu | \ \mu(x) = \mu(0) = \nu(0) = \nu(x)\} = \{x \in R_\mu | \ x \in R_\nu\} = \mu \cap \nu$.

This implies, f is a Γ-hyperring isomorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective.

To show that $F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu$. In fact, clearly, $F_\mu(x)/\nu \subseteq (F_\mu + F_\nu)(x)/\nu$. For all $\nu^*[a] \in (F_\mu + F_\nu)(x)/\nu$, where $a \in (F_\mu + F_\nu)(x)$, which implies that there exist $m \in F_\mu(x)$ and $n \in F_\nu(x)$ such that $a \in m+n$, there is $\alpha \in a - m \subseteq m + n - m \subseteq F_\nu(x)$, i.e., $\nu(\alpha) = \nu(0)$, and so we have $\nu^*[a] = \nu^*[m] \in F_\mu(x)/\nu$. Hence, for all $x \in A$, $f(F_\mu(x)/(\mu \cap \nu)) = F_\mu(x)/\nu = (F_\mu + F_\nu)(x)/\nu = (F_\mu + F_\nu)(g(x))/\nu$.

Therefore, (f, g) is a soft Γ-hyperring epimorphism and $(F_\mu/\mu \cap \nu, A) \cong ((F_\mu + F_\nu)/\nu, A)$. □

Theorem 4.4 (Third Fuzzy Isomorphism Theorem)**.** *Let (F, A) be a soft Γ-hyperring over R. If μ and ν are two fuzzy Γ-hyperideals with $\nu \leq \mu$, $\mu(0) = \nu(0)$ and $F_\mu(x) = R_\mu$ for all $x \in$Supp(F, A), then we have $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$.*

Proof. We know that R_μ/ν is a Γ-hyperideal of R/ν. Since (F, A) is a soft Γ-hyperring over R, it follows that $(F/\nu, A)$, $((F/\nu)/(F_\mu/\nu), A)$ and $(F/\mu, A)$ are soft Γ-hyperrings over R/ν, $(R/\nu)/(R_\mu/\nu)$ and R/μ, respectively.

Define $f : R/\nu \to R/\mu$ by $f(\nu^*[x]) = \mu^*[x]$, for all $x \in R$. If $\nu^*[x] = \nu^*[y]$, for all $x, y \in R$, then there exists $r \in x - y$, such that $\nu(r) = \nu(0)$.

Since $\nu \leq \mu$ and $\mu(0) = \nu(0)$, we have $\mu(r) \geq \nu(r) = \nu(0) = \mu(0)$, which implies that $\mu(r) = \mu(0)$, and so $\mu^*(x) = \mu^*(y)$. Hence, f is well-defined.

Moreover, we have

(i) $f(\nu^*[x] \boxplus \nu^*[y]) = f(\{\nu^*[z] \mid z \in \nu^*[x] \oplus \nu^*[y]\}) = \{\mu^*[z\ ]| \ z \in \nu^*[x] \oplus \nu^*[y]\} = \mu^*[\nu^*[x]] \boxplus \mu^*[\nu^*[y]] = \mu^*[x] \boxplus \mu^*[y] = f(\nu^*[x]) \boxplus f(\nu^*[y])$;

(ii) $f(\nu^*[x] \odot_\alpha \nu^*[y]) = f(\nu^*[x\alpha y]) = \mu^*[x\alpha y] = \mu^*[x] \odot_\alpha \mu^*[y] = f(\nu^*[x]) \odot_\alpha f(\nu^*[y])$.

Hence, f is a Γ-hyperring homomorphism. Clearly, f is a Γ-hyperring epimorphism. Next, we show that $kerf = R_\mu/\nu$. In fact, $\ker f = \{\nu^*[x] \in R/\nu \mid f(\nu^*[x]) = \mu^*[0]\} = \{\nu^*[x] \in R/\nu \mid \mu^*[x] = \mu^*[y]\} = \{\nu^*[x] \in R/\nu \mid \mu(x) = \mu(0)\} = \{\nu^*[x] \in R/\nu \mid x \in R_\mu\} = R_\mu/\nu$.

This implies f is a Γ-hyperring isomorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective. For all $x \in A$, $f(F(x)/\nu) = F(x)/\mu = F(g(x))/\mu$. Thus, (f, g) is a soft isomorphism from $(F/\nu, A)$ to $(F/\mu, A)$. Therefore, from Theorem 4.1 it follows that $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$. □

5. Conclusions

In this paper, we investigate three isomorphism theorems and three fuzzy isomorphism theorems in the context soft Γ-hyperrings.

In our future study of fuzzy structure of Γ-hyperrings, the following topics could be considered:

(1) To consider roughness of soft Γ-hyperrings;

(2) To establish three fuzzy isomorphism theorems of fuzzy soft Γ-hyperrings;

(3) To describe the fuzzy soft Γ-hyperrings and their applications.

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