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UNIQUENESS OF ENTIRE FUNCTIONS SHARING ONE SET*

BY

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Abstract. In this paper, we prove a uniqueness theorem for entire functions sharing values on a finite set. The result extends and improves some theorems obtained earlier by Fang, Zhang-Lin and Zhnag-Xiong.

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1. Introduction and main results

In this paper, we will use the standard notations of Nevanlinna's value distribution theory as in [3].

Let f be a nonconstant meromorphic function in the whole complex plane $\mathbf{C}$, we set $E(a, f) = \{z|f(z) - a = 0$, counting multiplicties$\}$, and $E(S, f) = \bigcup_{a\in S} E(a, f)$, where S denotes a set of complex numbers. Let p be a positive integer. Set

$$E_p(S, f) = \bigcup_{a\in S}\{z|f(z) - a = 0, \exists\, i, 0 < i \le p,\ \text{s.t. } f^{(i)}(z) \neq 0\},$$

where each zero of $f(z) - a$ with multiplicity m is counted m times when $m \le p$ in $E(S, f)$.

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Let f and g be two nonconstant entire functions, n, m, l, t and k be positive integers, we set

$$F = [f^n(f^l - 1)^t]^{(k)}, G = [g^n(g^l - 1)^t]^{(k)}, \tag{1.1}$$

$$H_m = \frac{(F^m)''}{(F^m)'} - 2\frac{(F^m)'}{F^m - 1} - \frac{(G^m)''}{(G^m)'} + 2\frac{(G^m)'}{G^m - 1}, \tag{1.2}$$

and $S_m = \{1, \omega, \omega^2, \cdots, \omega^{m-1}\}$, where $\omega = e^{\frac{2\pi}{m}i}$.

Fang [1] proved the following result.

Theorem A ([1]). *Let f and g be two nonconstant entire functions, and let n, k be tow positive integer with $n > 2k + 8$. If $[f^n(z)(f(z) - 1)]^{(k)}$ and $[g^n(z)(g(z) - 1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Zhang and Lin [7] improved Theorem A and obtained the following results.

Theorem B ([7]). *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > 2k + m_1 + 4$, and a, b be constants such that $|a| + |b| \neq 0$. If $[f^n(z)(af^m(z) + b)]^{(k)}$ and $[g^n(z)(ag^m(z) + b)]^{(k)}$ share 1 CM, then:*

(i) when $ab \neq 0$, $f(z) \equiv g(z)$;

(ii) when $ab = 0$, either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m_1} = 1$, or $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k a^2(c_1c_2)^{n+m_1}\{(n+m_1)c\}^{2t} = 1, \text{or} (-1)^k b^2(c_1c_2)^{n+m_1}\{(n+m_1)c\}^{2t} = 1,$$

when $a = 0$, $m_1 = 0$, when $a \neq 0$, $m_1 = m$.

Theorem C ([7]). *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > 2k + m + 4$. If $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$.*

Zhang and Xiong [8] improved Theorem B and Theorem B and obtained the following results.

Theorem D ([8]). *Let f and g be two transcendental entire functions, n, m, t, l, p be positive integers. If $E_1(S_m, [f^n(f^l - 1)^t]^{(p)}) = E_1(S_m, [g^n(g^l - 1)^t]^{(p)})$ and $n > \frac{6}{m} + 3tl + 4p$, then $f(z) \equiv bg(z)$, where $b^l = 1$.*

In this article, we prove

Theorem 1. *Let f and g be two transcendental entire functions, n, m, t, l, k and $p(\geq 2)$ be positive integers. If $E_p(S_m, [f^n(f^l-1)^t]^{(k)}) = E_p(S_m, [g^n(g^l-1)^t]^{(k)})$ and $n > \max\{\frac{p+1}{p-1}[\frac{4}{m}+2k+tl+\frac{2tl}{p+1}], \frac{3t}{2}\}$, then $f(z) \equiv bg(z)$, where $b^l = 1$.*

Remark 1. Under the condition of Theorem 1, let $p \to \infty$ and $m = 1$, one can check that the result of theorem 1 is still valid if $E(1, [f^n(f^l - 1)^t]^{(k)}) = E(1, [g^n(g^l-1)^t]^{(k)})$ and $n > \max\{4+2k+tl, \frac{3t}{2}\}$. Note that as p goes to ∞ our Theorem 1 includes Theorem A , Theorem B and Theorem C as special cases. We also note that our Theorem 1 together with Theorem D gives the complete solution to the uniqueness problem of entire functions sharing a set of values.

2. Lemmas

To prove the theorem, we need the following lemmas.

Lemma 1 ([4]). *Let $f(z)$ be a nonconstant meromorphic functions and let $R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$ be an irreducible rational function in f with constant coefficient $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then $T(r, R(f)) = dT(r, f) + S(r, f)$, where $d =max\{n, m\}$.*

Lemma 2 ([6]). *Let $f(z)$ be a nonconstant meromorphic function,k be positive integer, if $f^{(k)} \not\equiv 0$, then $N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$.*

Lemma 3 ([6, Second Fundamental Theorem]). *Let $f(z)$ be a nonconstant meromorphic function, $a_1, \cdots, a_n (n \geq 3)$ be complex numbers such that when $k \neq j$, $a_k \neq a_j$, then*

$$(n-2)T(r,f) \leq N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) + \cdots + N\left(r, \frac{1}{f-a_n}\right) - N_1(r) + S(r,f),$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$.

By second fundamental theorem, we have

$$(n-2)T(r,f) \leq \overline{N}(r, \frac{1}{f-a_1}) + \overline{N}(r, \frac{1}{f-a_2}) + \cdots + \overline{N}(r, \frac{1}{f-a_n}) - N_0(r, \frac{1}{f'}) + S(r,f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function which only counts those points such that $f' = 0$ but $f \neq a_k, k = 1, \cdots, n$.

Lemma 4. *Let F, G and H_m be defined as in* (1.1) *and* (1.2), *$p(\geq 2)$ be a positive integer. If $E_p(S_m, F) = E_p(S_m, G)$, and $n > k+2$, $H_m \not\equiv 0$, then*

$$\begin{aligned} m[m(r,\frac{1}{F}) + m(r,\frac{1}{G})] \leq\ & N(r,\frac{1}{F^m}) + N(r,\frac{1}{G^m}) \\ & - 2(m(n-k)-2)[N(r,\frac{1}{f}) + N(r,\frac{1}{g})] \\ & + \frac{2m}{p+1}(T(r,F) + T(r,G)) + S(r), \end{aligned}$$

where $S(r) = \max\{S(r,f), S(r,g)\}$.

Proof. Since $E_p(S_m, F) = E_p(S_m, G)$,we have $E_p(1, F^m) = E_p(1, G^m)$. Suppose that z_0 is a common simple zero-point of $F^m - 1$ and $G^m - 1$. It follows from (1.2) that z_0 is a zero-point of H_m. Moreover, we know that the zero-points of $F^m - 1$ and $G^m - 1$ with multiplicity $q(\leq p)$ are not poles of H_m, the simple pole and simple zero-points of F^m or G^m also are not poles of H_m. Thus, we have

$$\begin{aligned} N_{1)}(r,\frac{1}{F^m - 1}) = N_{1)}(r,\frac{1}{G^m - 1}) & \leq N(r,\frac{1}{H_m}) \\ & \leq T(r,H_m) + O(1) \leq N(r,H_m) + S(r). \end{aligned}$$

Furthermore, by the definition of H_m, we obtain

$$\begin{aligned} N_{1)}(r,\frac{1}{F^m-1}) = N_{1)}(r,\frac{1}{G^m-1}) & \leq \overline{N}_{(2}(r,\frac{1}{F^m}) + \overline{N}_{(2}(r,\frac{1}{G^m}) \\ & + \overline{N}_0(r,\frac{1}{(F^m)'}) + \overline{N}_0(r,\frac{1}{(G^m)'}) \\ & + \overline{N}_{(p+1}(r,\frac{1}{F^m-1}) + \overline{N}_{(p+1}(r,\frac{1}{G^m-1}) + S(r) \\ & \leq \overline{N}_{(2}(r,\frac{1}{F^m}) + \overline{N}_{(2}(r,\frac{1}{G^m}) + \overline{N}_0(r,\frac{1}{(F^m)'}) + \overline{N}_0(r,\frac{1}{(G^m)'}) \\ & + \frac{m}{p+1}[T(r,F) + T(r,G)] + S(r). \end{aligned} \tag{2.1}$$

By the second fundamental theorem, we have

$$
\begin{aligned}
& m(T(r,F)+T(r,G)) = T(r,F^m)+T(r,G^m) \leq \overline{N}(r,F^m) \\
& + \overline{N}(r,\frac{1}{F^m}) + \overline{N}(r,\frac{1}{F^m-1}) + \overline{N}(r,G^m) + \overline{N}(r,\frac{1}{G^m}) \\
& + \overline{N}(r,\frac{1}{G^m-1}) - [N_0(r,\frac{1}{(F^m)'}) + N_0(r,\frac{1}{(G^m)'})] + S(r) \\
& = \overline{N}(r,\frac{1}{F^m}) + \overline{N}(r,\frac{1}{F^m-1}) + \overline{N}(r,\frac{1}{G^m}) + \overline{N}(r,\frac{1}{G^m-1}) \\
& - [N_0(r,\frac{1}{(F^m)'}) + N_0(r,\frac{1}{(G^m)'})] + S(r).
\end{aligned} \tag{2.2}
$$

By Lemma 2, we get $N(r,\frac{1}{(G^m)'}) \leq N(r,\frac{1}{G^m}) + S(r)$. Thus

$$
\begin{aligned}
& \overline{N}_0(r,\frac{1}{(G^m)'}) + \overline{N}_{(2}(r,\frac{1}{G^m-1}) + N_{(2}(r,\frac{1}{G^m}) - \overline{N}_{(2}(r,\frac{1}{G^m}) \\
& \leq N(r,\frac{1}{(G^m)'}) \leq N(r,\frac{1}{G^m}) + S(r).
\end{aligned}
$$

It follows that

$$
\overline{N}_0(r,\frac{1}{(G^m)'}) + \overline{N}_{(2}(r,\frac{1}{G^m-1}) \leq \overline{N}(r,\frac{1}{G^m}) + S(r). \tag{2.3}
$$

Similarly, we have

$$
\overline{N}_0(r,\frac{1}{(F^m)'}) + \overline{N}_{(2}(r,\frac{1}{F^m-1}) \leq \overline{N}(r,\frac{1}{F^m}) + S(r,f). \tag{2.4}
$$

From (2.1)-(2.4), we have

$$
\begin{aligned}
m(T(r,F)+T(r,G)) \leq {} & 2[\overline{N}_{(2}(r,\frac{1}{F^m}) + \overline{N}_{(2}(r,\frac{1}{G^m})] + 2[\overline{N}(r,\frac{1}{F^m}) \\
& + \overline{N}(r,\frac{1}{G^m})] + \frac{2m}{p+1}[T(r,F)+T(r,G)] + S(r).
\end{aligned} \tag{2.5}
$$

Since

$$
\overline{N}(r,\frac{1}{F^m}) + \overline{N}_{(2}(r,\frac{1}{F^m}) \leq N(r,\frac{1}{F^m}) - [N_{(3}(r,\frac{1}{F^m}) - 2\overline{N}_{(3}(r,\frac{1}{F^m})]
$$

and

$$
N_{(3}(r,\frac{1}{F^m}) - 2\overline{N}_{(3}(r,\frac{1}{F^m}) \geq [m(n-k)-2]N(r,\frac{1}{f}),
$$

we have

$$\overline{N}(r,\frac{1}{F^m})+\overline{N}_{(2}(r,\frac{1}{F^m})\le N(r,\frac{1}{F^m})-[m(n-k)-2]N(r,\frac{1}{f}). \tag{2.6}$$

Similarly,

$$\overline{N}(r,\frac{1}{G^m})+\overline{N}_{(2}(r,\frac{1}{G^m})\le N(r,\frac{1}{G^m})-[m(n-k)-2]N(r,\frac{1}{g}). \tag{2.7}$$

Combine (2.5)-(2.7), we have

$$\begin{aligned} m[T(r,F)+T(r,G)] &\le 2[N(r,\frac{1}{F^m})-(m(n-k)-2)N(r,\frac{1}{f})] \\ &\quad + 2[N(r,\frac{1}{G^m})-(m(n-k)-2)N(r,\frac{1}{g})] \\ &\quad + \frac{2m}{p+1}[T(r,F)+T(r,G)]+S(r), \end{aligned}$$

thus

$$\begin{aligned} m[m(r,\frac{1}{F})+m(r,\frac{1}{G})]\le N(r,\frac{1}{F^m})+N(r,\frac{1}{G^m})-2(m(n-k)-2)[N(r,\frac{1}{f}) \\ + N(r,\frac{1}{g})]+\frac{2m}{p+1}(T(r,F)+T(r,G))+S(r), \end{aligned}$$

which completes the proof of Lemma 4.

Lemma 5. *Let F, G and H_m be defined as in* (1.1) *and* (1.2)*, $k(\ge 2)$ be positive integer. If $E_p(S_m,F)=E_p(S_m,G)$, and $n>\frac{p+1}{p-1}(\frac{4}{m}+2k+tl+\frac{2tl}{p+1})$, then $H_m\equiv 0$.*

Proof. Let $F_1=f^n(f^l-1)^t$, $G_1=g^n(g^l-1)^t$. Since $E_p(S_m,F)=E_p(S_m,G)$, we get $E_p(1,F^m)=E_p(1,G^m)$.

If $H_m\not\equiv 0$, by Lemmas 1 and 4, we have

$$\begin{aligned} mT(r,F)+mT(r,G) &= T(r,F^m)+T(r,G^m), m[m(r,\frac{1}{F}) \\ &\quad + m(r,\frac{1}{G})]\le N(r,\frac{1}{F^m})+N(r,\frac{1}{G^m}) \\ &\quad - 2(m(n-k)-2)[N(r,\frac{1}{f})+N(r,\frac{1}{g})] \\ &\quad + \frac{2m}{p+1}[T(r,F)+T(r,G)]+S(r). \end{aligned} \tag{2.8}$$

Since $F_1^{(k)} = F$, $G_1^{(k)} = G$, thus

$$m(r, \frac{1}{F_1}) \leq m(r, \frac{1}{F}) + S(r, f), m(r, \frac{1}{G_1}) \leq m(r, \frac{1}{G}) + S(r, g). \tag{2.9}$$

By Lemma 2, we have

$$N(r, \frac{1}{F}) \leq N(r, \frac{1}{F_1}) + S(r, f), N(r, \frac{1}{G}) \leq N(r, \frac{1}{G_1}) + S(r, g). \tag{2.10}$$

Combining (2.8)-(2.10) we have

$$\begin{aligned} &m(1 - \frac{2}{p+1})[m(r, \frac{1}{F_1}) + m(r, \frac{1}{G_1})] \leq m(1 + \frac{2}{p+1})[N(r, \frac{1}{F_1}) + N(r, \frac{1}{G_1})] \\ &- 2(m(n-k) - 2)[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] + S(r). \end{aligned}$$

Thus

$$\begin{aligned} &m(1 - \frac{2}{p+1})[T(r, \frac{1}{F_1}) + T(r, \frac{1}{G_1})] \leq 2m[N(r, \frac{1}{F_1}) + N(r, \frac{1}{G_1})] \\ &- 2(m(n-k) - 2)[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] + S(r), \end{aligned}$$

we get

$$\begin{aligned} &m(1 - \frac{2}{p+1})(n + lt)[T(r, f) + T(r, g)] \leq (2mk + 4)[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] \\ &+ 2mt[N(r, \frac{1}{f^l - 1}) + N(r, \frac{1}{g^l - 1})] + S(r) \\ &\leq (2mk + 4 + 2mtl)[T(r, f) + T(r, g)] + S(r), \end{aligned}$$

which contradicts the assumption that $n > \frac{p+1}{p-1}(\frac{4}{m} + 2k + tl + \frac{2tl}{p+1})$. Therefore $H_m \equiv 0$, which completes the proof of Lemma 5.

Lemma 6 ([6])**.** *Let f be a transcendental entire function, k be a positive integer, and c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)} - c}) - N(r, \frac{1}{f^{(k+1)}}) + S(r, f) \\ &\leq N_{k+1}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)} - c}) - N_0(r, \frac{1}{f^{(k+1)}}) + S(r, f), \end{aligned}$$

where $N_0(r, 1/f^{(k+1)})$ *is the counting function which only counts those points such that* $f^{(k+1)} = 0$ *but* $f(f^k - c) \neq 0$.

Lemma 7 ([6])**.** *Let* f *be a transcendental meromorphic function,* a_1 *and* a_2 *be two meromorphic functions such that* $T(r, a_j) = S(r, f)(j = 1, 2)$ *and* $a_1 \not\equiv a_2$*, then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f - a_1}) + \overline{N}(r, \frac{1}{f - a_2}) + S(r, f).$$

Lemma 8 ([2])**.** *Let* f *and* g *be two entire functions. If there exists two nonconstant polynomials* p *and* q *such that* $p \circ f(z) = q \circ g(z)$*, then there exists entire function* h *and rational functions* $U(z)$ *and* $V(z)$ *such that* $f(z) = U \circ h(z), g(z) = V \circ h(z)$.

Lemma 9. *Let* U *and* V *be two rational functions,* n *and* t *be two positive integers such that* $n > \frac{3t}{2}$*, and set* $U^n(U - 1)^t \equiv aV^n(V - 1)^t$*. If there exists* z_0 *such that* $U(z_0) = 0$*, and* z_0 *is a zero of* $V - 1$ *with multiplicity* $q < 4$*, then* $U^j(U - 1) \equiv akV^j(V - 1)$*, where* $j = 2$ *or* $j = 3$*,* $k^t = 1$.

Proof. Suppose that z_0 is a zero of $U(z)$ with multiplicity p, by $U^n(U - 1)^t \equiv aV^n(V - 1)^t$, we have $np = qt$.

If $q = 1$, then $np = t$, which contradicts with $n > \frac{3t}{2}$.

If $q = 2$, then $np = 2t$. Since $n > \frac{3t}{2}$, we get $p = 1$, so $n = 2t$ and $U^2(U - 1) \equiv akV^2(V - 1)$, where $k^t = 1$.

If $q = 3$, then $np = 3t$. Since $n > \frac{3t}{2}$, we get $p = 1$, so $n = 3t$ and $U^3(U - 1) \equiv akV^3(V - 1)$, where $k^t = 1$. Which completes the proof of Lemma 9.

3. Proof of Theorem 1

Let F , G and H_m be defined as in (1.1) and (1.2). By Lemma 5, we have $H_m \equiv 0$, that is $\frac{(F^m)''}{(F^m)'} - 2\frac{(F^m)'}{F^m - 1} \equiv \frac{(G^m)''}{(G^m)'} - 2\frac{(G^m)'}{G^m - 1}$. Thus

$$\frac{1}{G^m - 1} \equiv \frac{A}{F^m - 1} + B, \tag{3.1}$$

where $A \neq 0$ and B be two constants. Hence $E(1, F^m) = E(1, G^m)$, $T(r, F) = T(r, G) + S(r, F)$.

We will prove the theorem by the following four steps.

Step I. We claim that

$$f^n(f^l-1)^t \equiv ag^n(g^l-1)^t. \tag{3.2}$$

To see this, we consider the following two cases.

Case 1. When $B=0$, by (3.1), we have

$$F^m = AG^m + (1-A). \tag{3.3}$$

Case 1.1. If $A=1$, by (3.3), we have $F^m = G^m$, and hence $f^n(f^l-1)^t \equiv ag^n(g^l-1)^t$.

Case 1.2. If $A \neq 1$, by (3.3) we have

$$F^{m-1}F' = AG^{m-1}G'. \tag{3.4}$$

From (3.3) and (3.4), we get:

when $F=0$, we have $G^m \neq 0,1$ and $G'=0$; when $G=0$, we have $F^m \neq 0,1$ and $F'=0$. Hence

$$\overline{N}(r,\frac{1}{F})-N_0(r,\frac{1}{(G^m)'}) = S(r,F), \overline{N}(r,\frac{1}{G})-N_0(r,\frac{1}{(F^m)'}) = S(r,F). \tag{3.5}$$

By the second fundamental theorem, we have

$$\begin{aligned} T(r,F^m) &\leq \overline{N}(r,F^m) + \overline{N}(r,\frac{1}{F^m}) \\ &\quad + \overline{N}(r,\frac{1}{F^m-(1-A)}) - N_0(r,\frac{1}{(F^m)'}) + S(r,F) \\ &\leq \overline{N}(r,\frac{1}{F^m}) + \overline{N}(r,\frac{1}{G^m}) - N_0(r,\frac{1}{(F^m)'}) + S(r,F) \\ &= \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) - N_0(r,\frac{1}{(F^m)'}) + S(r,F). \end{aligned}$$

Similarly, we have

$$T(r,G^m) \leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F}) - N_0(r,\frac{1}{(G^m)'}) + S(r,G).$$

Combining (3.5), we get

$$2mT(r,F) \leq [\overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F})] + S(r,F) \leq 2T(r,F) + S(r,F).$$

Hence $m = 1$. By (3.3) we get

$$f^n(f^l-1)^t \equiv ag^n(g^l-1)^t + P(z), \tag{3.6}$$

where $P(z)$ is a polynomial.

If $P(z) \equiv 0$, then by (3.6), we get $f^n(f^l-1)^t \equiv ag^n(g^l-1)^t$.

If $P(z) \not\equiv 0$, then by (3.6) and Lemma 6, we have

$$\begin{aligned} T(r, f^n(f^l-1)^t) &\le \overline{N}(r, f^n(f^l-1)^t) + \overline{N}(r, \frac{1}{f^n(f^l-1)^t}) \\ &+ \overline{N}(r, \frac{1}{f^n(f^l-1)^t - P}) + S(r,f) \\ &= \overline{N}(r, \frac{1}{f^n(f^l-1)^t}) + \overline{N}(r, \frac{1}{g^n(g^l-1)^t}) + S(r,f) \\ &\le \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^l-1}) + \overline{N}(r, \frac{1}{g}) \\ &+ \overline{N}(r, \frac{1}{g^l-1}) + S(r,f) \le 2(1+l)T(r,f) + S(r,f), \end{aligned}$$

thus $n + tl \le 2(1+l)$, which contradicts the assumption that $n > \frac{p+1}{p-1}(\frac{4}{m} + 2k + tl + \frac{2tl}{p+1})$.

Case 2. When $B \neq 0$, by (3.1), we have

$$\frac{1}{G^m - 1} = B\frac{F^m + (\frac{A}{B} - 1)}{F^m - 1}, \frac{A}{F^m - 1} = -B\frac{G^m - (\frac{1}{B} + 1)}{G^m - 1} \tag{3.7}$$

and $\frac{G^{m-1}G'}{(G^m-1)^2} = A\frac{F^{m-1}F'}{(F^m-1)^2}$. Thus

$$F^m + (\frac{A}{B} - 1) \neq 0, \quad G^m - (\frac{1}{B} + 1) \neq 0. \tag{3.8}$$

Case 2.1. If $A = B$, by (3.7), we have $F \neq 0$. Since $F = (f^n(f^l-1)^t)^{(k)}$ and $n > k$, thus $f \neq 0$. Let $f = e^\alpha$, where α is a nonconstant entire function. Thus $f^n(f^l-1)^t = e^{n\alpha}\sum_{j=0}^{t}(-1)^{t-j}C_t^j e^{lj\alpha} = \sum_{j=0}^{t}(-1)^{t-j}C_t^j e^{(n+lj)\alpha}$. Let

$$((-1)^{t-j}C_t^j e^{(n+lj)\alpha})^{(k)} = P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{(n+lj)\alpha},$$

where $P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})(j = 0, 1, 2, \cdots, t)$ are differential polynomials. Thus

$$F = \sum_{j=0}^{t} P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{(n+lj)\alpha} = e^{n\alpha}\sum_{j=0}^{t} P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{lj\alpha} = e^{n\alpha}F_0,$$

where $F_0 = \sum_{j=0}^{t} P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{lj\alpha}$.

Obviously, there exists $j(0 \le j \le t)$, such that $P_j(\alpha', \alpha'', \cdots, \alpha^{(k)}) \not\equiv 0$. Suppose $P_0(\alpha', \alpha'', \cdots, \alpha^{(k)}) \not\equiv 0$. Since $F \ne 0$, thus $F_0 \ne 0$. Since f is a nonconstant entire function, we use Lemma 7 and obtain

$$\begin{aligned} ltT(r, e^{\alpha}) = T(r, F_0) &\le \overline{N}(r, \frac{1}{F_0}) \\ &+ \overline{N}(r, \frac{1}{F_0 - P_0(\alpha', \alpha'', \cdots, \alpha^{(k)})}) + \overline{N}(r, F_0) + S(r, e^{\alpha}) \\ &= \overline{N}(r, \frac{1}{\sum_{j=1}^{t} P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{lj\alpha}}) + S(r, e^{\alpha}) \\ &= \overline{N}(r, \frac{1}{\sum_{j=1}^{t} P_j(\alpha', \alpha'', \cdots, \alpha^{(k)})e^{l(j-1)\alpha}}) \\ &+ S(r, e^{\alpha}) \le l(t-1)T(r, e^{\alpha}) + S(r, e^{\alpha}), \end{aligned}$$

which is a contradiction.

Case 2.2. If $A \ne B$ and $B = -1$, by (3.7), we have $G \ne 0$. Since $G = (g^n(g^l - 1)^t)^{(k)}$ and $n > k$, we have $g \ne 0$. Set $g = e^{\beta}$, where β is a nonconstant entire function. Similar to Case 2.1, we also have $ltT(r, e^{\beta}) \le l(t-1)T(r, e^{\beta}) + S(r, e^{\beta})$, which is a contradiction.

Case 2.3. If $A \ne B$ and $B \ne -1$, we consider the following two subcases.

Case 2.3.1. When $m > 1$, by (3.8) and the second fundamental theorem, we have

$$\begin{aligned} T(r, G^m) &\le \overline{N}(r, \frac{1}{G^m}) + \overline{N}(r, \frac{1}{G^m - (\frac{1}{B} + 1)}) + \overline{N}(r, G^m) - N_0(r, \frac{1}{(G^m)'}) \\ &+ S(r, G) \le \overline{N}(r, \frac{1}{G}) - N_0(r, \frac{1}{G'}) + S(r, G) \end{aligned}$$

and

$$\begin{aligned} T(r, F^m) &\le \overline{N}(r, \frac{1}{F^m}) + \overline{N}(r, \frac{1}{F^m - (1 - \frac{A}{B})}) + \overline{N}(r, F^m) - N_0(r, \frac{1}{(F^m)'}) \\ &+ S(r, F) \le \overline{N}(r, \frac{1}{F}) - N_0(r, \frac{1}{F'}) + S(r, F), \end{aligned}$$

thus $m[T(r, G) + T(r, F)] \le T(r, F) + T(r, G) + S(r, G)$, which is a contradiction.

Case 2.3.2. When $m=1$, by (3.8), we have $F+(\frac{A}{B}-1)\neq 0$, thus $(f^n(f^l-1)^t)^{(k)}+(\frac{A}{B}-1)\neq 0$. Since f is a nonconstant entire function, we use Lemma 6 to obtain

$$\begin{aligned}(n+lt)T(r,f)&=T(r,f^n(f^l-1)^t)\leq N_{k+1}(r,\frac{1}{f^n(f^l-1)^t})\\&+\overline{N}(r,\frac{1}{(f^n(f^l-1)^t)^{(k)}+(\frac{A}{B}-1)})\\&-N_0(r,\frac{1}{(f^n(f^l-1)^t)^{(k+1)}})+S(r,f)\\&\leq N_{k+1}(r,\frac{1}{f^n(f^l-1)^t})+S(r,f)\leq (k+1)\overline{N}(r,\frac{1}{f})\\&+N_{k+1}(r,\frac{1}{(f^l-1)^t})+S(r,f)\leq (k+1+lt)T(r,f)+S(r,f),\end{aligned}$$

thus $n\leq k+1$, which contradicts the assumption that $n>\frac{p+1}{p-1}(\frac{4}{m}+2k+tl+\frac{2tl}{p+1})$.

Combining case 1 and case 2, we get (3.2).

Step II. By the first step, we claim that if $f^l\not\equiv g^l$, then $l=1$.

By (3.2), we have

$$f^{n-1}(f^l-1)^{t-1}(f^l-\frac{n}{n+tl})f'=ag^{n-1}(g^l-1)^{t-1}(g^l-\frac{n}{n+tl})g'. \tag{3.9}$$

From (3.2) and (3.9), we obtain the following three cases:

(i) when $f=0$, we get $g=0$ or $g^l=1$,

(ii) when $f^l=1$, we get $g^l=1$ or $g=0$,

(iii) when $f^l=\frac{n}{n+tl}$, we get $g^l=\frac{n}{n+tl}$ or $g'=0$ (such that $g^l\neq\frac{n}{n+tl}$, $g\neq 0$, $g^l\neq 1$).

Combining (3.2), (i) and (ii) we have

$$\overline{N}(r,\frac{1}{f^l-1})-\overline{N}(r,\frac{1}{f^l-1},\frac{1}{g^l-1})\leq\frac{t}{n}N(r,\frac{1}{f^l-1}) \tag{3.10}$$

$$\overline{N}(r,\frac{1}{f})-\overline{N}(r,\frac{1}{f},\frac{1}{g})\leq\frac{t}{n}N(r,\frac{1}{g^l-1}). \tag{3.11}$$

Using the second fundamental theorem we have

$$\begin{aligned}2lT(r,f)&\leq\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^l-1})+\overline{N}(r,\frac{1}{f^l-\frac{n}{n+tl}})\\&+\overline{N}(r,f)-N_0(r,\frac{1}{f'})+S(r,f)\end{aligned} \tag{3.12}$$

and

$$2lT(r,g) \leq \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g^l-1}) + \overline{N}(r,\frac{1}{g^l-\frac{n}{n+tl}})$$
$$+ \overline{N}(r,g) - N_0(r,\frac{1}{g'}) + S(r,g). \tag{3.13}$$

If $f^l \not\equiv g^l$, then by (3.10)-(3.13),(i)-(iii), we have

$$\begin{aligned} 4lT(r,f) &= 2l[T(r,f)+T(r,g)] + S(r,f) \leq 2\overline{N}(r,\frac{1}{f},\frac{1}{g}) \\ &+ 2\overline{N}(r,\frac{1}{f^l-1},\frac{1}{g^l-1}) + 2\overline{N}(r,\frac{1}{f^l-\frac{n}{n+tl}},\frac{1}{g^l-\frac{n}{n+tl}}) \\ &+ \frac{2t}{n}N(r,\frac{1}{f^l-1}) + \frac{2t}{n}N(r,\frac{1}{g^l-1}) + 2\overline{N}(r,f) + S(r,f) \\ &\leq 2N(r,\frac{1}{f^l-g^l}) + \frac{2t}{n}N(r,\frac{1}{f^l-1}) + \frac{2t}{n}N(r,\frac{1}{g^l-1}) + S(r,g) \\ &\leq (2l+\frac{4tl}{n})T(r,f) + S(r,f). \end{aligned} \tag{3.14}$$

When $l \geq 2$, by (3.14), we obtain that $2l \leq \frac{4tl}{n}$, which contradicts the assumption that $n > \frac{p+1}{p-1}(\frac{4}{m} + 2k + tl + \frac{2tl}{p+1})$. Therefore, we get $l = 1$.

Step III. We claim that if $l = 1$, then $f \equiv g$.

In fact, we consider the following two cases.

Case 1. We shall prove that $f \equiv g$, or there exists positive integer j such that $f^j(f-1) \equiv ag^j(g-1)$, where $j = 2$ or $j = 3$. Since $l = 1$, by (3.2), we have

$$f^n(f-1)^t = ag^n(g-1)^t. \tag{3.15}$$

By Lemma 8 and (3.15), then there exists entire function h and rational functions $U(z)$ and $V(z)$ such that $f = U(h), g = V(h)$ and

$$U^n(U-1)^t \equiv aV^n(V-1)^t \tag{3.16}$$

and

$$U^{n-1}(U-1)^{t-1}(U-\frac{n}{n+t})U' \equiv aV^{n-1}(V-1)^{t-1}(V-\frac{n}{n+t})V'. \tag{3.17}$$

Hence $T(r,U) = T(r,V) + S(r,U)$.

Since f and g are entire functions, thus U and V are polynomials or be rational functions and only have one common pole.

By the second fundamental theorem, we have

$$\begin{aligned}2T(r,U) \leq{}& \overline{N}(r,\frac{1}{U})+\overline{N}(r,\frac{1}{U-1})+\overline{N}(r,U)\\ &+\overline{N}(r,\frac{1}{U-\frac{n}{n+t}})-N_0(r,\frac{1}{U'})+S(r,U)\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}2T(r,V) \leq{}& \overline{N}(r,\frac{1}{V})+\overline{N}(r,\frac{1}{V-1})+\overline{N}(r,V)\\ &+\overline{N}(r,\frac{1}{V-\frac{n}{n+t}})-N_0(r,\frac{1}{V'})+S(r,U).\end{aligned}\tag{3.19}$$

By Lemma 9 and $f=U(h), g=V(h)$, we get $f^j(f-1)\equiv akg^j(g-1)$, where $j=2$ or $j=3$, $k^t=1$, and

- when there exists z_0 such that $U(z_0)=0$, we have $V(z_0)=0$ or $V(z_0)=1$ with multiplicity $q\geq 3$;

- when there exists z_0 such that $U(z_0)=1$, we have $V(z_0)=1$ or $V(z_0)=0$ and $U(z_0)=1$ with multiplicity $q_1\geq 3$;

- when there exists z_0 such that $U(z_0)=\frac{n}{n+t}$, we have $V(z_0)=\frac{n}{n+t}$ or $U'(z_0)=0$ such that $V(z_0)\neq\frac{n}{n+t}$ and $U(z_0)\neq 0,1$.

If $U\not\equiv V$, by (3.18) and (3.19), we have

$$\begin{aligned}2T(r,U) \leq{}& \overline{N}(r,\frac{1}{U},\frac{1}{V})+\overline{N}(r,\frac{1}{U-1},\frac{1}{V-1})\\ &+\overline{N}(r,\frac{1}{U-\frac{n}{n+t}},\frac{1}{V-\frac{n}{n+t}})+\overline{N}(r,\frac{1}{U},\frac{1}{V-1})+\overline{N}(r,\frac{1}{V},\frac{1}{U-1})\\ &+\overline{N}(r,U)+S(r,U)\leq N(r,\frac{1}{U-V})+\frac{1}{3}(N(r,\frac{1}{V-1})\\ &+N(r,\frac{1}{U-1}))+\overline{N}(r,U)+S(r,U)\\ &\leq(1+\frac{2}{3})T(r,U)+\overline{N}(r,U)+S(r,U).\end{aligned}$$

Thus

$$T(r,U)\leq 3\overline{N}(r,U)+S(r,U).\tag{3.20}$$

Since U is a polynomial or rational function which has only one pole, then by (3.20), we have $d_U \leq 3$, where

$$U(z) = \sum_{k=0}^{m_1} a_k z^k / \sum_{j=0}^{m_2} b_j z^j, V(z) = \sum_{k=0}^{n_1} c_k z^k / \sum_{j=0}^{n_2} d_j z^j,$$

d_U =max$\{m_1, m_2\}$, d_V =max$\{n_1, n_2\}$. Combining (3.16), we get $d_V \leq 3$.

If there exists z_0, such that $U(z_0) = 0$ and $V(z_0) = 1$, since $d_V \leq 3$, by Lemma 8 and $f = U(h), g = V(h)$, we get $f^j(f-1) \equiv akg^j(g-1)$, where $j = 2$ or $j = 3$, $k^t = 1$.

If U and V IM 0, by (3.16), we obtain that U and V CM 0. Since U and V CM ∞, there exists constant A such that $U \equiv AV$, hence, we get $f \equiv Ag$. By (3.15), we have $A = 1$, thus $f \equiv g$.

Summarizing the above discussion we obtain $f \equiv g$ or there exists positive integer j, such that $f^j(f-1) \equiv ag^j(g-1)$, where $j = 2$ or $j = 3$, which completes the proof Case 1.

Case 2. We shall prove that if $U^j(U-1) \equiv akV^j(V-1)$, where $j = 2$ or $j = 3$, $k^t = 1$, then $f \equiv g$.

By $f = U(h), g = V(h)$ and $U^j(U-1) \equiv aV^j(V-1)$, we have

$$f^j(f-1) \equiv akg^j(g-1). \tag{3.21}$$

Let $h_1 = \frac{f}{g}$, then by (3.21) we have $h_1^j(h_1 - \frac{1}{g}) = c(1 - \frac{1}{g})$, thus $(h_1^3 - ak)g = h_1^j - ak$.

If h_1 is constant, we have $h_1 = 1$, thus $f \equiv g$.

If h_1 is nonconstant, we have $g = \frac{h_1^j - ak}{h_1^j - ak}$, which contradicts the assumption that g be transcendental entire function.

Step IV. If $f^l \equiv g^l$, then there exists constant b, such that $f \equiv bg$, where $b^l = 1$.

Summarizing the above discussion we obtain the proof of Theorem 1.

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