sciendo Applied Mathematics and Nonlinear Sciences 8(1) (2023) 2357–2366

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

Good congruences on weakly U-abundant semigroups

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Submission Info

Communicated by Juan Luis García Guirao
Received November 17th 2021
Accepted April 10th 2022
Available online May 20th 2022

Abstract

In this paper, we study the congruences on w-U-a semigroups with s-transversals. The g-congruences on a w-U-a semigroup with an s-transversal T_V are described abstractly by the e-triple, which consists of equivalences on the structure component parts I, T and Λ. Also, it is shown that the set of all g-congruences on this kind of semigroups forms a complete lattice.

Keywords: weakly U-abundant semigroup, Ehresmann transversal, congruence, equivalence triple
2010 Mathematics Subject Classifications: 20M10.

1 Introduction

In 1982, Blyth and McFadden [2] first proposed the concept of inverse transversals. In 1993, Ei-Qallali [3] introduced ample transversals for abundant semigroups. Recently, Wang introduced E-transversals for E-semiabundant semigroups. According to Howie, the test of whether a structure theory for semigroups is truly satisfactory is to see if it could enable one to give an explicit description of the congruences, or not [6]. After the concept of inverse transversals was given, more and more authors initiated the study of congruencies for regular semigroups with inverse transversals and obtained some interesting results [1, 4, 10–12]. Recently, several researchers began to show an interest in congruences for abundant semigroups with ample transversals and obtained some meaningful results [8, 9, 15]. w-U-a semigroups (E-semiabundant semigroups) are a generalisation of an abundant semigroup, and include the class of abundant semigroups as a proper sub-class. The application of congruence theory on w-U-a semigroups is far more complex than that on regular (abundant) semigroups. For example, a homomorphic image of a w-U-a semigroup is not necessarily a w-U-a semigroup but a similar statement is true for a regular semigroup. This property happens to be very important in the study of semigroups. So we consider the g-congruences on a w-U-a semigroup with an s-transversal since this kind of congruence has the above-mentioned property [13].

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sciendo ISSN 2444-8656 doi:10.2478/amns.2021.2.00193

In this paper, we provide a construction method for g-congruences on w-U-a semigroups with (C) and E-transversals, and prove that the set of all g-congruences on these kinds of semigroups is a complete lattice.

2 Preliminaries

Let $E(S)$ be the set of all idempotents of a semigroup S, and U be any subset of $E(S)$.

For any $a,b \in S$, define $\tilde{\mathscr{L}}_U$ and $\tilde{\mathscr{R}}_U$ on S are as follows:

$$a\tilde{\mathscr{L}}_U b \text{ if and only if } (\forall e \in U)\,(ae = a \Leftrightarrow be = b)\,;$$
$$a\tilde{\mathscr{R}}_U b \text{ if and only if } (\forall e \in U)\,(ea = a \Leftrightarrow eb = b)\,.$$

For $U = E(S)$, we use $\tilde{\mathscr{L}}_U$ and $\tilde{\mathscr{R}}_U$ for $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{R}}$, respectively. It is easy to verify that $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \tilde{\mathscr{L}}_U$ and $\mathscr{R} \subseteq \mathscr{R}^* \subseteq \tilde{\mathscr{R}}_U$. For $U = E(S)$ and the regular elements a and b, we have $\mathscr{L} = \mathscr{L}^* = \tilde{\mathscr{L}}_U$ and $\mathscr{R} = \mathscr{R}^* = \tilde{\mathscr{R}}_U$. S is called weakly U-abundant (w-U-a for short) if each $\tilde{\mathscr{R}}_U$-class and each $\tilde{\mathscr{L}}_U$-class of S contain an idempotent of U. We always write S_U instead of S. We call a w-U-a semigroup S_U satisfying the congruence condition (c-c) if $\tilde{\mathscr{L}}_U$ is a right congruence and $\tilde{\mathscr{R}}_U$ is a left congruence ($\tilde{\mathscr{L}}_U$ and $\tilde{\mathscr{R}}_U$ are not always right and left congruences, respectively); consequently, we call S_U a w-U-a with (C). Further, if U forms a semilattice, then we call S_U an Ehresmann semigroup (E-semigroup for short). For an E-semigroup S_U and for any $a \in S$, there exists a unique idempotent $e \in U$ such that $e\tilde{\mathscr{L}}_U a$, and a unique idempotent $f \in U$ such that $f\tilde{\mathscr{R}}_U a$. We denote e and f by a^* and a^+, respectively.

For a semigroup denoted by another letter, say Q, we denote the set of its idempotents by $E(Q)$, the relations $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{L}}$ on Q_U by $\tilde{\mathscr{R}}_U(Q)$ and $\tilde{\mathscr{L}}_U(Q)$, and the $\tilde{\mathscr{R}}$-class and the $\tilde{\mathscr{L}}$-class of an element a by $\tilde{\mathscr{R}}^a_U(Q)$ and $\tilde{\mathscr{L}}^a_U(Q)$, respectively. The following results are immediately apparent from the definition of $\tilde{\mathscr{L}}_U$. Of course, a dual result holds for $\tilde{\mathscr{R}}_U$.

Lemma 1. *[5] Let $a \in S$ and $e, f \in U \subseteq E(S)$. Then*

1. $e\tilde{\mathscr{L}}_U a \Leftrightarrow (\forall f \in U) ae = a$ *and* $af = a$ *implies* $ef = e$;
2. $e\mathscr{L} f \Leftrightarrow e\tilde{\mathscr{L}}_U f$ *and* $e\mathscr{R} f \Leftrightarrow e\tilde{\mathscr{R}}_U f$.

Lemma 2. *[5] Let S_U be an E-semigroup. Then*

1. $(\forall a,b \in S)(ab)^* = (a^*b)^*$ *and* $(ab)^+ = (ab^+)^+$;
2. $(\forall a,b \in S)a\tilde{\mathscr{R}}_U b \Leftrightarrow a^+ = b^+$, $a\tilde{\mathscr{L}}_U b \Leftrightarrow a^* = b^*$.

Let T_V be a w-V-a semigroup and S_U be a w-U-a semigroup. We can call T_V a w-V-a subsemigroup of S_U if $T \subseteq S$ and $V \subseteq U \cap T$. For any $a \in T$, if there exist $e, f \in V$ such that $e\tilde{\mathscr{L}}_U a\tilde{\mathscr{R}}_U f$, then we call T_V $a \sim$ -w-V-a subsemigroup of S_U.

Lemma 3. *[14] If T_V is a $\sim$ -w-V-a subsemigroup of a w-U-a semigroup S_U, then*

$$\tilde{\mathscr{L}}_V \cap (T \times T) = \tilde{\mathscr{L}}_U \cup (T \times T)$$

and

$$\tilde{\mathscr{R}}_V \cap (T \times T) = \tilde{\mathscr{R}}_U \cup (T \times T),$$

and hence, if S_U satisfies c-c, then, so dose T_V.

Call a-w-V-a subsemigroup T_V of a w-U-a semigroup S_U with (C) a-E-subsemigroup if V forms a semilattice. Call a-E-subsemigroup T_V of S_U with (C) an Ehresmann transversal (E-transversal for short) of S_U if ($\forall x \in S$, $\exists e$, $f \in U$, $\exists! \overline{x} \in T$) $x = e\overline{x}f$, where $x^+ \mathscr{L} e$ and $\overline{x}^* \mathscr{R} f$. Based on the work of Yang [14], x can be used to determine the e and f uniquely. So, we denote e by e_x and f by f_x and put $I = \{e_x \in S\}$ and $\Lambda = \{f_x \in S\}$.

Lemma 4. *[14] Let S_U be a w-U-a semigroup with (C) and an E-transversal T_V. Then*

1. *if $x \in T$, then $\overline{x} = x$, $e_x = x^+$ and $f_x = x^*$;*
2. *if $x \in V$, then $\overline{x} = x = e_x = f_x$;*
3. *$I \cap \Lambda = V$;*
4. *$I = \{e \in U : (\exists \ell \in V) e \mathscr{L} \ell\}$ and $\Lambda = \{f \in U : (\exists h \in V) f \mathscr{L} h\}$;*
5. *$(\forall e \in I, \forall f \in \Lambda) e_e = e$ and $f_f = f$;*
6. *$(\forall e \in I, \exists! \ell \in V) e \mathscr{L} \ell$; $(\forall f \in \Lambda, \exists! h \in V) f \mathscr{R} h$.*

A subset T of S is called a *q-ideal* (*b-ideal*) of S if $ST \cap TS \subseteq T (TST \subseteq T)$. For a w-$U$-a semigroup S_U, we have the following result.

Lemma 5. *Let T_V be an E-transversal of a w-U-a semigroup S_U. Then T_V is a q-ideal of S_U if and only if T_V is a b-ideal of S_U.*

Proof. Suppose that T_V is a q-ideal, $x \in TST$, then there exist $a, b \in T$, $b \in S$ such that $x = abc = ab \cdot c = a \cdot bc \in T$. Thus T_V is a b-ideal of S_U. Conversely, if T_V is a b-ideal of S_U, then, for any $x \in ST \cap TS$, there exist $a, b \in S$, $c, d \in T$ such that

$$x = ac = db = e_a \overline{a} f_a c = d e_b \overline{b} f_b = d^+ \cdot e_a \overline{a} f_a \cdot c \in T.$$

Thus T_V is a q-ideal of S_U. $\square$

We call an E-transversal T_V of a w-U-a semigroup S_U a *q-ideal E-transversal* if T_V is a q-ideal of S_U. We call an q-ideal E-transversal T_V of a w-U-a semigroup S_U *strong* (s-transversal for short) if V is an order ideal of $E(T)$, and the idempotents of both IT and $T\Lambda$ are all sets of U.

Define φ and ψ as follows:

$$\varphi : \ I \to V, \ e \mapsto e\varphi = \ell, \text{ where } e \mathscr{L} \ell;$$

$$\psi : \ \Lambda \to V, \ f \mapsto f\psi = h, \text{ where } f \mathscr{R} h.$$

By Lemma 4, (6), both φ and ψ are well defined, and $\varphi|v = \psi|v = 1_V$.

Lemma 6. *[14] Let T_V be a s-transversal of a w-U-a semigroup S_U with (C). Then:*

1. *$(\forall f \in V, \forall e \in I) fe \in Reg_U(S) \Rightarrow fe = f \cdot e\varphi$;*
2. *$(\forall e \in V, \forall f \in \Lambda) fe \in Reg_U(S) \Rightarrow fe = f\psi \cdot e$.*

Let V be a semilattice and A and B be non-empty sets. We suppose that V, A and B are mutually disjoint. Putting $I = V \cup A$ and $\Lambda = V \cup B$, it is easy to see that $I \cap \Lambda = V$.

A (V, I, Λ)-*system* means that V, I, λ thus satisfying the conditions (B1)–(B3):

(B1) There exist two mappings $\varphi : I \to V$ and $\psi : \Lambda \to V$ such that

$$\varphi|v = \psi|v = 1|_V.$$

(B2) Mappings $\otimes : I \times V \to I$ and: $V \times \Lambda \to \Lambda$ satisfying

$$(\forall s,t \in V, \forall x \in I) x \otimes (st) = (x \otimes s) \otimes t,$$

$$(\forall s,t \in V, \forall y \in \Lambda)(st) * y = s * (t * y).$$

(B3) For any $x \in I$, $y \in \Lambda$, $s \in V$, we have

$$x \otimes x\varphi = x, y\psi * y = y,$$

$$(x \otimes s)\varphi = x\varphi \otimes s = x\varphi \cdot s,\ (s * y)\psi = s * y\psi = s \cdot y\psi,$$

that is, φ, ψ, $\otimes$,* are compatible.

In the rest of this paper, we use E to denote the set of all idempotents of W. For a semigroup S, we denote the set of all regular elements of S by Reg(S), and define

$$V_U(a) = \{a' \in V(a) | aa', a'a \in U\} \text{ and } \mathrm{Reg}_U(S) = \{a \in \mathrm{Reg}(S) | V_U(a) \neq \phi\}.$$

Lemma 7. *[14] Suppose that there is a (V, I, Λ)-system, T_V is an E-semigroup in which V is an order ideal of $E(T)$, and P is a $\Lambda \times I$ matrix over T satisfying (denote $P_{f,e}$ by [f, e])*

(B4) $(\forall a,b \in V) a[f,e]b = [a * f, e \otimes b]$,

(B5) $(\forall f \in \Lambda, \forall e \in I)[f\psi, e\varphi] = f\psi \cdot e\varphi;\ f, f\psi] = f\psi;\ [e\varphi, e] = e\varphi$ *and*

$$[f\psi, e] \in Reg_V(T) \Rightarrow [f\psi, e] = f\psi \cdot e\varphi,$$
$$[f, e\varphi] \in Reg_V(T) \Rightarrow [f, e\varphi] = f\psi \cdot e\varphi.$$

Put

$$W = \{(e,x,f) \in I \times T \times \Lambda : e\varphi = x^+, f\psi = x^*\}$$

with a multiplication

$$(e,x,f)(g,y,h) = (e \otimes a^+, a, a^* * h)$$

where $a = x[f,g]y \in T$. Then W_E is a w-E-a semigroup with (C) and an s-transversal isomorphic to T_V. Conversely, every w-U-a semigroup S_U with (C) and an s-transversal T_V can be constructed in the above manner.

For other notations and terminologies given in this paper, the reader is referred to other papers in the literature [6, 13, 14].

3 Good congruences

In this section, we describe the g-congruences on a w-U-a semigroup S_U with (C) and an s-transversal T_V. Let V, A and B be mutually disjoint sets, $I = V \cup A$ and $\Lambda = V \cup B$ (hence $I \cap \Lambda = V$). Assume that V, I, Λ form an (V, I, Λ)-system. We first give the following definitions.

Definition 1. A congruence ρ on S_U is called a *good congruence* (g-congruence for short) if, for any x, $y \in T$, $(x,y) \in \rho$ implies $(x^+, y^+) \in \rho$ and $(x^*, y^*) \in \rho$; an equivalence ρ on $I[\Lambda]$ is called a *normal equivalence* (n-equivalence for short), if for all m, $n \in V$, $i \in I[j \in \Lambda]$, $m\rho n$ implies $(i \otimes m)\rho(i \otimes n)[(m * j)\rho(n * j)]$. Let ρ^I and ρ^Λ be n-equivalences on I and Λ, respectively, and ρ^T is a g-congruence on T. We call (ρ^I, ρ^T, ρ^Λ) an *equivalence triple* (e-triple for short) on $I \times T \times \Lambda$ if the following conditions hold:

(C.1) $\rho^I|_V = \rho^\Lambda|_V = \rho^T|_V$;

(C.2) $(\forall j \in \Lambda)(\forall e, g \in I) e\rho^I g \Rightarrow [j,e]\rho^T[j,g]$;

(C.3) $(\forall i \in I)(\forall f,h \in \Lambda) f\rho^\Lambda h \Rightarrow [f,i]\rho^T[h,i]$.

Define a relation $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ on the semigroup W in Lemma 7 by

$$(e,x,f)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(g,y,h) \Leftrightarrow e\rho^I g, x\rho^T y, f\rho^\Lambda h.$$

Theorem 1. *Let W be a w-U-a semigroup S_U with (C) and a s-transversal T_V as in Lemma 7, and (ρ^I, ρ^T, ρ^Λ) be an e-triple on T. Then $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ is a g-congruence on W.*

Conversely, every g-congruence on W can be constructed in the above-mentioned manner.

Proof. For any e-triple $(\rho^I, \rho^T, \rho^\Lambda) \in W$. It is clear that $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ is an equivalence. Supposing that $(i,z,j),(e,x,f),(g,y,h) \in W$, and $(e,x,f)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(g,y,h)$, then, by $(*)$, $x\rho^T y$, $e\rho^I g$ and $f\rho^\Lambda h$, and so

$$(i,z,j)(e,x,f) = (i \otimes m^+, m, m^* * f),$$
$$(i,z,j)(g,y,h) = (i \otimes n^+, n, n^* * h),$$

where $m = z[j,e]x$ and $n = z[j,g]y$. Moreover, we have $[j,e]\rho^T[j,g]$, since $e\rho^I g$, (C.2) and $x\rho^T y$, and so $m\rho^T n$. On the other hand, since ρ^T is a g-congruence, we have $m^+\rho^T n^+$ and so, by (C.1), we get $m^+\rho^I n^+$. Therefore, $(i \otimes m^+)\rho^I(i \otimes n^+)$. Similarly, we can verify $(m^* * f)\rho^\Lambda(n^* * h)$. Thus, we have the following relation

$$(i \otimes m^+, m, m^* * f)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(i \otimes n^+, n, n^* * h),$$

that is to say,

$$(i,z,j)(e,x,f)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(i,z,j)(g,y,h).$$

Further, we can prove $(e,x,f)(i,z,j)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(g,y,h)(i,z,j)$ dually. So, to summarise, we proved that $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ is a congruence.

Supposing that $(x^+,x,x^*),(y^+,y,y^*) \in W'$ and $(x^+,x,x^*)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(y^+,y,y^*)$, then by $(*)$

$$x^+\rho^I y^+, x\rho^T y, x^*\rho^\Lambda x^*.$$

Moreover, we have $x^+\rho^T y^+$ since ρ^T is a g-congruence, and so by (C.1), $x^+\rho^\Lambda y^+$. Thus, we have the following relation

$$(x^+,x^+,x^+)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(y^+,y^+,y^+).$$

Similarly, we have the relation

$$(x^*,x^*,x^*)\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(y^*,y^*,y^*).$$

To sum up, it follows that $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ is a g-congruence.

Conversely, let ρ be a g-congruence on W. Define the equivalences on I, Λ and T, respectively:

(E1) $(\forall e,g\in I, e\varphi=x^+, g\varphi=y^+) e\rho_I g \Leftrightarrow (e,x^+,x^+)\rho(g,y^+,y^+)$;

(E2) $(\forall f,h\in\Lambda, f\psi=x^*, h\psi=y^*) f\rho_\Lambda h \Leftrightarrow (x^*,x^*,f)\rho(y^*,y^*,h)$;

(E3) $(\forall x,y\in T, x\rho_T y \Leftrightarrow (x^+,x,x^*)\rho(y^+,y,y^*)$.

We have demonstrated that ρ_I and ρ_Λ are equivalences on I and Λ, respectively, and that ρ_T is a g-congruence on T since ρ is a g-congruence on W.

For any $i,k\in V$, if $i\rho_I k$, then we have $(i,i,i),(k,k,k)\in T$ and $(i,i,i)\rho(k,k,k)$. Moreover, we have (e,x^+,x^+) $(i,i,i)\rho(e,x^+,x^+)(k,k,k)$ since it is a g-congruence, and so

$$\left(e\otimes\left(x^+\left[x^+,i\right]i\right)^+, x^+\left[x^+,i\right]i, \left(x^+\left[x^+,i\right]i\right)^* * i\right)\rho$$
$$\left(e\otimes\left(x^+\left[x^+,k\right]k\right)^+, x^+\left[x^+,k\right]k, \left(x^+\left[x^+,k\right]k\right)^* * k\right).$$

That is to say,

$$\left(e\otimes i, x^+ i, x^+ i\right)\rho\left(e\otimes k, x^+ k, x^+ k\right).$$

Thus we get $(e\otimes i)\rho_I(e\otimes k)$ and so ρ_I is an n-equivalence. By a similar argument, ρ_Λ it is also an n-equivalence.

Further we have the cases 1–3:

Case 1. Obviously, $\rho_I|_V=\rho_\Lambda|_V=\rho_T|_V$. So (C.1) holds.

Case 2. For any $e,g\in I$, if $e\varphi=x^+$, $g\varphi=y^+$ and $e\rho_I g$, then, by (E1), we have $(e,x^+,x^+)\rho(g,y^+,y^+)$. So, for any $(z^*,z^*,j)\in W$, we have

$$(z^*,z^*,j)(e,x^+,x^+)\rho(z^*,z^*,j)(g,y^+,y^+).$$

That is to say,

$$\left(z^*\otimes\left(z^*\left[j,e\right]x^+\right)^+, z^*\left[j,e\right]x^+, \left(z^*\left[j,e\right]x^+\right)^* * x^+\right)\rho$$
$$\left(z^*\otimes\left(z^*\left[j,g\right]y^+\right)^+, z^*\left[j,g\right]y^+, \left(z\left[j,g\right]y^+\right)^* * y^+\right).$$

Thus, we can get $z^*\left[j,e\right]x^+\rho_{S^\circ}z^*\left[j,g\right]y^+$. Moreover, by (B3) and (B4), we have $z^*\left[j,e\right]x^+=[j,e]$ and $z^*[j,g]y^+=[j,g]$. Thus, we have

$$[j,e]\rho_T[j,g].$$

So (C.2) holds.

Case 3. Similar to case 2, (C.3) also holds.

In conclusion, we have that $(\rho_I,\rho_T,\rho_\Lambda)$ is an e-triple on W.

By the direct part, $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$ is a g-congruence. On the other hand, for any $(e,x,f),(g,y,h)\in W$, (e,x,f) $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}(g,y,h)$, we have

$$e\rho_I g,\ x\rho_T y,\ f\rho_\Lambda h.$$

It follows that

$$\left(e,x^+,x^+\right)\rho\left(g,y^+,y^+\right), \left(x^+,x,x^*\right)\rho\left(y^+,y,y^*\right), \left(x^*,x^*,f\right)\rho\left(y^*,y^*,h\right).$$

Further

$$\left(e,x^+,x^+\right)\left(x^+,x,x^*\right), \left(x^*,x^*,f\right)\rho\left(g,y^+,y^+\right), \left(y^+,y,y^*\right)\left(y^*,y^*,h\right).$$

That is to say, $(e,x,f)\rho(g,y,h)$ and so $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}\subseteq\rho$. Obviously, $\rho\subseteq\rho^{(\rho^I,\rho^T,\rho^\Lambda)}$, and thus we have $\rho^{(\rho^I,\rho^T,\rho^\Lambda)}=\rho$. □

Next, we give an example of a g-congruence on a w-U-a semigroup Q_U with (C) and an s-transversal.

Example 1. *Let* $Q=\{e,g,h,w,f\}$ *with the following multiplication table*

	e	g	h	w	f
e	e	g	e	g	g
g	g	g	g	g	g
h	h	g	h	g	g
w	w	g	w	g	g
f	g	g	w	w	f

Associativity may be checked directly. And it is easy to verify that $E(Q)=\{e,g,h,f\}$ and that $T'=\{w,e,f,g\}$ is a subsemigroup of Q. Taking $U'=V'=\{e,g,f\}$, we then have

1. $e\tilde{\mathscr{L}}_U h\tilde{\mathscr{L}}_U w\tilde{\mathscr{R}}_U f$;
2. $\tilde{L}^g_{V'}(T')=\tilde{L}^g_{U'}(Q)=\tilde{R}^g_{V'}(T')=\tilde{R}^g_{U'}(Q)=\{g\}$;
3. $\tilde{L}^w_{V'}(T')=\{w,e\}$; $\tilde{L}^w_{U'}(Q)=\{w,e,h\}$;
4. $\tilde{R}^w_{V'}(T')=\tilde{R}^w_{U'}(Q)=\{w,f\}$.

It is easy to check that $Q_{U'}$ is a w-U$'$-a semigroup $Q_{U'}$ with (C) and an s-transversal $T'_{V'}$.

It is easy to verify that the identical relation is a g-congruence.

Let $gC(W)$ denote the set of all g-congruences on W and $eT(W)$ denote the set of all e-triples on W constructed as in Theorem 1. Then we have the following results.

Lemma 8. *If* $(\rho_1^I,\rho_1^T,\rho_1),(\rho_2^I,\rho_2^T,\rho_2^\Lambda)\in eT(W)$, *then*

$$\rho^{(\rho_1^I,\rho_1^T,\rho_1^\Lambda)}\subseteq\rho^{(\rho_2^I,\rho_2^T,\rho_2^\Lambda)}\Leftrightarrow\rho_1^I\subseteq\rho_2^I,\rho_1^T\subseteq\rho_2^T,\rho_1^\Lambda\subseteq\rho_2^\Lambda.$$

Proof. *Sufficiency*. Let us suppose that $e,g\in I$ such that $e\rho_1^I g$, $e\varphi=x^+$ and $g\varphi=y^+$. By the proof of Theorem 1, we have

$$\left((e,x^+,x^+),(g,y^+,y^+)\right)\in\rho^{(\rho_1^I,\rho_1^T,\rho_1^\Lambda)}\subseteq\rho^{(\rho_2^I,\rho_2^T,\rho_2^\Lambda)}.$$

It follows that $e\rho_2^I g$, and so $\rho_1^I\subseteq\rho_2^I$. Similarly, we can get $\rho_1^\Lambda\subseteq\rho_2^\Lambda$. On the other hand, if $x\rho_1^T y$, we have $x^+\rho_1^T y^+$ and $x^*\rho_1^T y^*$ since ρ_1^T is a g-congruence on T. Moreover, by (C.1), we have

$$\left((x^+,x,x^*),(y^+,y,y^*)\right)\in\rho^{(\rho_1^I,\rho_1^T,\rho_1^\Lambda)}\subseteq\rho^{(\rho_2^I,\rho_2^T,\rho_2^\Lambda)}.$$

Hence, $x\rho_2^T y$.

Necessity. It is obvious. □

Define relation $\le$ on $eT(W)$ by

$$(\rho_1^I,\rho_1^T,\rho_1^\Lambda)\le(\rho_2^I,\rho_2^T,\rho_2^\Lambda)\Leftrightarrow\rho_1^I\subseteq\rho_2^I,\rho_1^T\subseteq\rho_2^T,\rho_1^\Lambda\subseteq\rho_2^\Lambda.$$

Then $eT(W)$ is a partial ordered set with respect to $\le$. By Theorem 1 and Lemma 8, $gC(W)$ and $eT(W)$ are isomorphic as partial ordered sets. For any $\Omega\subseteq gC(W)$, $\overset{g}{\underset{\rho\in\Omega}{\vee}}\rho=\cap\{\sigma\in gC(W)|\rho\subseteq\sigma,\text{ for all }\rho\in\Omega\}$, that is, the smallest g-congruence on W containing every congruence of Ω.

Proposition 1. *Let $\Omega \subseteq gC(W)$ and $W_\rho = (\rho^I, \rho^T, \rho^\Lambda)$, where $\rho \in \Omega$. Then*

$$W_{\bigcap\limits_{\rho\in\Omega}\rho} = \left(\bigcap_{\rho\in\Omega}\rho^I, \bigcap_{\rho\in\Omega}\rho^T, \bigcap_{\rho\in\Omega}\rho^\Lambda\right)$$

and

$$W_{\overset{g}{\bigvee}\limits_{\rho\in\Omega}\rho} = \left(\overset{g}{\bigvee_{\rho\in\Omega}}\rho^I, \overset{g}{\bigvee_{\rho\in\Omega}}\rho^T, \overset{g}{\bigvee_{\rho\in\Omega}}\rho^\Lambda\right).$$

Proof. The first equality is pretty obvious, and so we just need to prove the second equality.

For any $e, g \in I$, $x, y \in T$, if $e\varphi = x^+$, $g\varphi = y^+$ and $e\left(\bigvee^g_{\rho\in\Omega}\rho\right)^I g$, then we have

$$i = (e, x^+, x^+) \overset{g}{\bigvee_{\rho\in\Omega}} \rho (g, y^+, y^+) = j.$$

Hence, there exist $\rho_i \in \Omega$ and $a_i = (e_i, x_i, f_i) \in W$ such that $i\rho_1 a_1 \rho_2 a_2 \cdots a_{n-1}\rho_n j$. This implies that $i^+\rho_1 a_1^+ \rho_2 a_2^+ \cdots a_{n-1}^+ \rho_n j^+$ and $e\rho_1^I e_1 \rho_2^I e_2 \cdots e_{n-1}\rho_n^I g$. Thus we have

$$\left(\overset{g}{\bigvee_{\rho\in\Omega}}\rho\right)^I \subseteq \overset{g}{\bigvee_{\rho\in\Omega}}\rho^I.$$

The converse is obvious. Similarly, we have $\left(\bigvee^g_{\rho\in\Omega}\rho\right)^\Lambda = \bigvee^g_{\rho\in\Omega}\rho^\Lambda$.

Next, assume that $x\left(\overset{g}{\bigvee_{\rho\in\Omega}}\rho\right)^T y$, then we have $s = (x^+, x, x^*) \overset{g}{\bigvee_{\rho\in\Omega}} \rho\, (y^+, y, y^*) = t$. Further, there exist $s_i = (x_i^+, x_i, x_i^*) \in W$ and $\rho_i \in \Omega$ such that $s\rho_1 s_1 \rho_2 s_2 \cdots \rho_{n-1} s_{n-1} \rho_n t$ and so $x\rho_1^T x_1 \rho_2^T x_2 \cdots \rho_{n-1}^T x_{n-1} \rho_n^T y$. Hence, we have

$$\left(\overset{g}{\bigvee_{\rho\in\Omega}}\rho\right)^T \subseteq \overset{g}{\bigvee_{\rho\in\Omega}}\rho^T.$$

On the other hand, $\bigvee^g_{\rho\in\Omega}\rho^T \subseteq \left(\bigvee^g_{\rho\in\Omega}\rho\right)^T$ is clear and so $\left(\bigvee^g_{\rho\in\Omega}\rho\right)^T \subseteq \bigvee^g_{\rho\in\Omega}\rho^T$. □

To sum up, we obtain the following theorem.

Theorem 2. *Let W be constructed as in Lemma 7. Then $eT(W)$ forms a complete lattice with respect to $\leq$ and $gC(W)$ is isomorphic to $eT(W)$ as a complete lattice.*

Acknowledgements.

This work was supported by the Natural Science Foundation of Chongqing (Nos. cstc2020jcyj-msxmX0272; cstc2021jcyj-msxmX0436); the Scientific and Technological Research Program of Chongqing Municipal Education Commission (No. KJQN202001107) and the Scientific Research Starting Foundation of Chongqing University of Technology (No. 2019ZD68).

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