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Waghamore P. Harina and S. Rajeshwari

NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING SMALL FUNCTION WITH FINITE WEIGHT

Abstract. The purpose of the paper is to study the uniqueness of entire and meromorphic functions sharing a small function with finite weight. The results of the paper improve and extend some recent results due to Abhijit Banerjee and Pulak Sahoo [3].

Key words: entire and meromorphic function, weighted sharing, nonlinear differential polynomials.

AMS Mathematics Subject Classification: 30D35.

1. Introduction

In this paper by meromorphic functions we will always mean meromorphic function in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM.

We adopt the standard notations of value distribution theory (see [8]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Throughout this paper, we need the following definition.

$$\Theta(a;f)=1-\lim_{r\to\infty}\sup\frac{\bar{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman [7] proved the following result.

Theorem A. *Let f be a transcendental entire function, and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many zeros.*

In 2002, Fang and Fang [6] proved the following result.

Theorem B. *Let f and g be two non-constant entire functions, and let $n(\geq 8)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the same year Fang [5] investigated the value sharing of more general non-linear differential polynomial than that was considered in Theorem B and obtained the following result.

Theorem C. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM then $f \equiv g$.*

In 2004, Lin and Yi [14] considered the case of meromorphic function in Theorem B and obtained the following.

Theorem D. *Let f and g be two non-constant meromorphic functions with $\Theta(\infty, f) > \frac{2}{n+1}$, and let $n(\geq 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

Natural inquisition would be to investigate the situation for meromorphic function in Theorem C. In this direction in 2008, Zhang [20] proved the following result.

Theorem E. *Suppose that f is a transcendental meromorphic function with finite number of poles, g is a transcendental entire function, and let n, k be two positive integers with $n \geq 2k+6$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

To proceed further we require the following definition known as weighted sharing of values introduced by I. Lahiri [9] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1. *Let k be a non negative integer or infinity. For $a \in \mathbb{C} \bigcup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer

p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2009, using the notion of weighted sharing of values, Xu, Yi and Cao [15] proved the following result.

Theorem F. *Let f and g be two non-constant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 5k + 11$ or if $l = 1$ and $n > 7k + \frac{23}{2}$, then $f \equiv g$.*

Recently, Li [13] proved the following result which rectify and at the same time improve Theorem F.

Theorem G. *Let f and g be two non-constant meromorphic functions, and $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 3k + 11$ or if $l = 1$ and $n > 5k + 14$, then $f = g$ or $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} = 1$.*

In this direction recently Abhijith Banerjee [1] proved the following results first one of which improves Theorem G.

Theorem H. *Let f and g be two transcendental meromorphic function and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 3k + 9$ or if $l = 1$ and $n \geq 4k + 10$, or if $l = 0$ and $n \geq 9k + 18$, then $f = g$ or $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$. The possibility $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} = 1$ does not occur for $k = 1$.*

Theorem I. *Let f and g be two transcendental entire functions, and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq \frac{5k}{2} + 7$, or if $l = 0$ and $n \geq 5k + 12$, then $f = g$.*

In 2015, Abhijith Banerjee and Pulak Sahoo [3] obtained the following result.

Theorem J. *Let f and g be two non-entire transcendental meromorphic functions and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\not\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq 3k + 9$ or if $l = 1$ and $n \geq 4k + 10$ or if $l = 0$ and $n \geq 9k + 18$, then $f = g$.*

Theorem K. *Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\not\equiv 0)$*

is a polynomial. If $l \geq 2$ and $n \geq 2k+6$ or if $l=1$ and $n \geq \frac{5k}{2}+7$ or if $l=0$ and $n \geq 5k+12$, then $f=g$.

The following questions are inevitable.

Quation 1. *What can be said if the sharing value zero is replaced by a small function a in the above Theorems J and K?*

Quation 2. *Are the Theorems J and K also true for non-constant entire and meromorphic functions?*

In this paper, taking the possible answer of the above questions into background we obtain the following results.

Theorem 1. *Let f and g be two non-constant meromorphic functions, let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$, for a positive integer m or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1 \ldots a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ are complex constants. Also we suppose that $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share (a,l), and $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ are positive integers. Now*
(I) when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$, and one of the following conditions holds:
(a) $l \geq 2$ and $n > 3k+m+8$,
(b) $l=1$ and $n > 4k + \frac{3m+8}{2}$,
(c) $l=0$ and $n > 9k+4m+14$,
then one of the following three cases holds:
(I1) $f(z) \equiv tg(z)$ for a constant t such that $t^{d_1}=1$, where $d_1 = gcd(n+m, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i=0,1,2,\ldots,m$,
(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \ldots + a_1 w_2 + a_0)$, except for $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$,
(I3) $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv a^2$, except for $k=1$,
(II) when $P(w) \equiv c_0$, and one of the following conditions holds:
(a) $l \geq 2$ and $n > 3k+8$,
(b) $l=1$ and $n > 4k+9$,
(c) $l=0$ and $n > 9k+14$,
then one of the following two cases holds:
(II1) $f \equiv tg$ for some constant t such that $t^n=1$,
(II2) $c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$. In particular when $n > 2k$ and $a(z) = d_2 =$ constant, we get $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

Theorem 2. *Let f and g be two non-constant entire functions, let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$, for a positive integer m or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1 \ldots a_{m-1}, a_m(\neq 0)$, $c_0(\neq 0)$ are complex con-*

stants. Also we suppose that $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* (a,l)*, and* $n(\geq 1)$*,* $k(\geq 1)$*,* $l(\geq 0)$ *are positive integers. Now*
(I) *when* $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$*, and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > 2k + m + 5$,
(b) $l = 1$ *and* $n > \frac{5k}{2} + 2m + 5$,
(c) $l = 0$ *and* $n > 5k + 4m + 8$,
then the conclusion of Theorem 1 holds:
(II) *when* $P(w) \equiv c_0$*, and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > 2k + 5$,
(b) $l = 1$ *and* $n > \frac{5k}{2} + 5$,
(a) $l = 0$ *and* $n > 5k + 8$,
then the conclusion of Theorem 1 holds.

Theorem 3. *Let* f *and* g *be two non-constant meromorphic functions and* $a(z)(\not\equiv 0,\infty)$ *be a small function of* f *and* g*, let* $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$*, for a positive integer* m *or* $P(w) \equiv c_0$ *where* $a_0(\neq 0), a_1 \ldots a_{m-1}, a_m(\neq 0)$*,* $c_0(\neq 0)$ *are complex constants. Also we suppose that* $f^nP(f)f'$ *and* $g^nP(g)g'$ *share* (a,l)*, and* $n(\geq 1)$*,* $k(\geq 1)$*,* $l(\geq 0)$ *are positive integers and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > m + 10$,
(b) $l = 1$ *and* $n > \frac{3m}{2} + 12$,
(c) $l = 0$ *and* $n > 4m + 22$,
then one of the following two cases holds:
(I) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^{d_3} = 1$*, where* $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$*,* $a_{m-i} \neq 0$ *for some* $i = 0,1,2,\ldots,m$*,*
(II) f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$*, where* $R(w_1,w_2) = w_1^{n+1}(\frac{a_mw_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_mw_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1})$*,*

Theorem 4. *Let* f *and* g *be two non-constant entire functions and* $a(z)(\not\equiv 0,\infty)$ *be a small function of* f *and* g*, let* $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$*, for a positive integer* m *or* $P(w) \equiv c_0$ *where* $a_0(\neq 0), a_1 \ldots a_{m-1}$*,* $a_m(\neq 0)$*,* $c_0(\neq 0)$ *are complex constants. Also we suppose that* $f^nP(f)f'$ *and* $g^nP(g)g'$ *share* (a,l)*, and* $n(\geq 1)$*,* $k(\geq 1)$*,* $l(\geq 0)$ *are positive integers and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > m + 4$,
(b) $l = 1$ *and* $n > \frac{3m}{2} + 6$,
(c) $l = 0$ *and* $n > 4m + 11$,
then the conclusion of Theorem 3 holds.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1 ([16]). *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + ... + a_1 f(z) + a_0\,,$$

where $a_n(\neq 0), a_{n-1}...a_1, a_0$ are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2 ([21]). *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{1}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2}$$

Lemma 3 ([9]). *Let F and G be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following cases holds:*

(i) $T(r) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r)$,

(ii) $F = G$,

(iii) $FG = 1$. *where $T(r)$ denotes the maximum of $T(r, F)$ and $T(r, G)$ and $S(r) = o\{T(r)\}$ as $r \to \infty$, possibly outside a set of finite linear measure.*

Lemma 4 ([2]). *Let F and G be two non-constant meromorphic functions sharing $(1, 1)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 5 ([2]). *Let F and G be two non-constant meromorphic functions sharing $(1, 0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) \\ & + \overline{N}(r, \infty; G) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 6 ([4])**.** *Let f, g be two non-constant meromorphic functions, let n, k be two positive integers such that $n > 2k$. Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share d_2 CM. If $[f^n]^{(k)}[g^n]^{(k)} \equiv d_2^2$, then $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants such that $(-1)^k (c_1 c_2)^n (nc)^{2k} = d_2^2$.*

Lemma 7 ([18])**.** *If $H \equiv 0$, then F, G share 1CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 8. *Let f and g be two non-constant meromorphic(entire) functions. Let $P(w)$ be defined as in Theorem 1 and k, m, $n > 3k+m (> 2k+m)$ be three positive integers. If $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$, then $f^n P(f) \equiv g^n P(g)$.*

Proof. By the assumption $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^k$.

When $k \geq 2$, integrating we get

$$[f^n P(f)]^{(k-1)} \equiv [g^n P(g)]^{(k-1)} + C_{k-1}.$$

If possible we suppose $C_{k-1} \neq 0$.

Now in the view of the Lemma 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,f) \;&\leq\; T(r,[f^nP(f)]^{(k-1)}) - \overline{N}(r,0;[f^nP(f)]^{(k-1)}) \\
&\quad +N_k(r,0;f^nP(f)) + S(r,f) \\
&\leq\; \overline{N}(r,0;[f^nP(f)]^{(k-1)}) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,C_{k-1};[f^nP(f)]^{(k-1)}) \\
&\quad - \overline{N}(r,0;[f^nP(f)]^{(k-1)}) \\
&\quad + N_k(r,0;f^nP(f)) + S(r,f) \\
&\leq\; \overline{N}(r,\infty;f) + \overline{N}(r,0;[g^nP(g)]^{(k-1)}) + k\overline{N}(r,0;f) \\
&\quad + N(r,0;P(f)) + S(r,f) \\
&\leq\; (k+m+1)T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + N_k(r,0;g^nP(g)) + S(r,f) \\
&\leq\; (k+m+1)T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + k\overline{N}(r,0;g) + N(r,0;P(g)) + S(r,f) \\
&\leq\; (k+m+1)T(r,f) + (2k+m-1)T(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leq\; (3k+2m)T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)T(r,g) \leq (3k+2m)T(r) + S(r).$$

Where $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = \max\{S(r,f), S(r,g)\}$. Combining these we get

$$(n - m - 3k)T(r) \leq S(r).$$

Which is a contradiction since $n > 3k + m$.

Therefore $C_{k-1} = 0$ and so $[f^n P(f)]^{(k-1)} \equiv [g^n P(g)]^{(k-1)}$, Repeating $k-1$ times, we obtain

$$f^n P(f) = g^n P(g) + c_0.$$

If $k = 1$, clearly integrating one we obtain the above. If possible suppose $c_0 \neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,f) &\leq \overline{N}(r,0;f^n P(f)) + \overline{N}(r,\infty;f^n P(f)) \\
&\quad + \overline{N}(r,c_0;f^n P(f)) + S(r,f) \\
&\leq \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,0;g^n P(g)) + S(r,f) \\
&\leq (m+2)T(r,f) + \overline{N}(r,0;g) + mT(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leq (m+2)T(r,f) + (m+1)T(r,g) + S(r,f) + S(r,g) \\
&\leq (2m+3)T(r) + S(r).
\end{aligned}$$

similarly we get

$$(n+m)T(r,g) \leq (2m+3)T(r) + S(r)$$

combining these we get

$$(n - m - 3)T(r) \leq S(r).$$

which is a contradiction, since $n > m + 3$. Therefore $c_0 = 1$ and so

$$f^n P(f) \equiv g^n P(g).$$

This completes the lemma. ∎

Lemma 9. *Let f, g be two nonconstant meromorphic (entire functions) and $F = \frac{[f^n P(f)]^{(k)}}{a(z)}$, $G = \frac{[g^n P(g)]^{(k)}}{a(z)}$, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers such that $n > 3k + m + 3 (> 2k + m + 2)$ and $P(w)$ be defined as in Theorem 1. If $H \equiv 0$ then*

(I) when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$, one of the following three cases holds:

(I1) $f \equiv tg$ *for a constant* t *such that* $t^{d_1} = 1$*, where* $d_1 = gcd(n + m, \ldots, n+m-i, \ldots, n)$*,* $a_{m-i} \neq 0$ *for some* $i = 1, 2, \ldots, m$*,*

(I2) f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$*, where* $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_0) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \ldots + a_0)$*, except for* $P(w) = a_1 w + a_2$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$*,*

(I3) $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2$*,*

(II) when $P(w) \equiv c_0$*,, one of the following two case holds:*

(II1) $f \equiv tg$ *for some constant* t *such that* $t^n = 1$*,*

(II2) $c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$*. In particular, when* $n > 2k$ *and* $a(z) = d_2$ *we get* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$*, where* c_1*,* c_2 *and* c *are constants satisfying* $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$*.*

Proof. Since $H \equiv 0$, by Lemma 7, we get F and G share 1 CM. On integration we get,

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G - 1}, \tag{3}$$

where a, b are constants and $a \neq 0$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (3) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leq T(r,G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) \\
&\quad + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n P(g)) + \overline{N}(r,\infty;f) + S(r,g) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \\
&\quad + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) \\
&\quad + T(r,P(g)) + S(r,g) \\
&\leq T(r,f) + (k+m+2)T(r,g) + S(r,f) + S(r,g)
\end{aligned}$$

without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$.

So for $r \in I$ we have

$$(n-k-3)T(r,g) \leq S(r,g)$$

which is a contradiction since $n > k+3$.

If $b \neq -1$, from (3) we obtain that

$$F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So,

$$\overline{N}(r, \frac{(b-a)}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Using Lemma 2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv a^2(z),$$

where $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $a(z)$ CM.

Note that if $P(w) \equiv c_0$ then we have

$$c_0^2 [f^n]^{(k)} [g^n]^{(k)} \equiv a^2(z).$$

In particular when $n > 2k$ and $a(z) = d_2$ then we get by Lemma 6 that $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

If $b = -1$, from (3) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore,

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leq T(r,G) + N_{k+1}(r, 0; g^n P(g)) - \overline{N}(r, 0; G) + S(r,g) \\
&\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{a+b}; G) \\
&\quad + N_{k+1}(r, 0; g^n P(g)) - \overline{N}(r, 0; G) + S(r,g) \\
&\leq \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + T(r, P(g)) \\
&\quad + \overline{N}(r, 0; F) + S(r,g) \\
&\leq \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + T(r, P(g)) \\
&\quad + (k+1)\overline{N}(r, 0; f) + T(r, P(f)) + k\overline{N}(r, \infty; f) \\
&\quad + S(r,f) + S(r,g) \\
&\leq (k+m+2)T(r,g) + (2k+m+1)T(r,f) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(n - 3k - 3 - m)T(r, g) \leq S(r, g).$$

Which is a contradiction, since $n > 3k + m + 3$.

Case 3. Let $b = 0$. From (3) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{4}$$

If $a \neq 1$ then from (4) we obtain

$$\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in case 2. Therefore $a = 1$ and from (4) we obtain

$$F \equiv G,$$

i.e.,

$$[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}.$$

Note that $n > 3k + m + 3 > 3k + m$.

So by Lemma 8, we have

$$f^n P(f) \equiv g^n P(g). \tag{5}$$

Let $h = \frac{f}{g}$. If h is a constant, putting $f = gh$ in (5) we get

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + ... + a_0 g^n (h^n - 1) = 0,$$

which implies $h^{d_1} = 1$, where $d_1 = gcd(n + m, ..., n + m - i, ..., n + 1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. Thus $f = tg$ for a constant t such that $t^{d_1} = 1$. where $d_1 = gcd(n+m, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then from (5) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^n + a_{m-1} w_1^{m-1} + ... + a_0) - w_2^n(a_m w_2^n + a_{m-1} w_2^{m-1} + ... + a_0).$$

In particular when $P(w) = a_1 w + a_2$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ then $f \equiv g$. Note that when $P(w) \equiv c_0$ then we must have $f \equiv tg$ for some constant t such that $t^n = 1$. ■

Lemma 10. *Let f and g be two non constant meromorphic functions and $a(z)(\not\equiv 0, \infty)$ be a small function of f and g. Let n and m be two positive integers such that $n > \frac{4m}{t} - (m - 1)$, t denotes the number of distinct roots of the equation $P(w) \equiv 0$, where $P(w)$ is defined as in Theorem 3. Then*

$$f^n P(f) f' g^n P(g) g' \not\equiv a^2.$$

Proof. First suppose that

$$f^nP(f)f'g^nP(g)g' \equiv a^2(z). \tag{6}$$

Let d_1 be the distinct zeros of $P(w) = 0$ and multiplicity P_i, where $i = 1, 2, ..., t$, $1 \leq t \leq m$ and $\sum_{i=1}^{t} p_i = m$.

Now by the second fundamental theorem for f and g we get respectively

$$\begin{aligned} tT(r,f) \ &\leq \ \overline{N}(r,0;f) + \overline{N}(r,\infty;f) \\ &\quad + \sum_{i=1}^{t} \overline{N}(r,d_i;f) - \overline{N_0}(r,0;f') + S(r,f) \end{aligned} \tag{7}$$

and

$$\begin{aligned} tT(r,g) \ &\leq \ \overline{N}(r,0;g) + \overline{N}(r,\infty;g) \\ &\quad + \sum_{i=1}^{t} \overline{N}(r,d_i;g) - \overline{N_0}(r,0;g') + S(r,g), \end{aligned} \tag{8}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function of those zeros of f' which are not the zeros of f and $f - d_i$, $i = 1, 2, \ldots, t$ and $\overline{N}_0(r,0;g')$ can be similarly defined.

Let z_0 be a zero of f with multiplicity p but $a(z_0) \neq 0, \infty$. Clearly z_0 must be a pole of g with multiplicity q. Then from (6) we get $np + p - 1 = nq + mq + q + 1$. This gives

$$mq + 2 = (n+1)(p-q). \tag{9}$$

From (9) we get $p - q \geq 1$ and so $q \geq \frac{n-1}{m}$. Now $np + p - 1 = nq + mq + q + 1$ gives $p \geq \frac{n+m-1}{m}$. Thus we have

$$\overline{N}(r,0;f) \leq \frac{m}{n+m-1} N(r,0;f) \leq \frac{m}{n+m-1} T(r,f). \tag{10}$$

Let $z_1(a(z_1) \neq 0, \infty)$ be a zero of $f - d_i$ with multiplicity q_i, $i = 1, 2, ...t$. Obviously z_1 must be a pole of g with multiplicity $r(\geq 1)$. Then from (6) we get $p_iq_i + q_i - 1 = (n+m+1)r + 1 \leq n + m + 2$. This gives $q_i \geq \frac{n+m+2}{p_i+1}$ for $i = 1, 2, \ldots, t$ and so we get

$$\overline{N}(r,d_i;f) \leq \frac{p_i+1}{n+m+3} N(r,d_i;f) \leq \frac{p_i+1}{n+m+3} T(r,f).$$

Clearly

$$\sum_{i=1}^{d} \overline{N}(r,d_i;f) \leq \frac{m+t}{n+m+3} T(r,f). \tag{11}$$

Similarly we have

$$\overline{N}(r,0;g) \le \frac{m}{n+m-1}T(r,g) \tag{12}$$

and

$$\sum_{i=1}^{t}\overline{N}(r,d_i;g) \le \frac{m+t}{n+m+3}T(r,g). \tag{13}$$

Also it is clear that

$$\begin{aligned}\overline{N}(r,\infty;f) &\le \overline{N}(r,0;g) + \sum_{i=1}^{t}\overline{N}(r,d_i;g)\\ &\quad + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\ &\le \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)T(r,g) + \overline{N}_0(r,0;g')\\ &\quad + S(r,f) + S(r,g)\end{aligned} \tag{14}$$

then by (7), (10), (11) and (14) we get

$$\begin{aligned}tT(r,f) &\le \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad +\overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f) + S(r,g).\end{aligned} \tag{15}$$

Similarly we have

$$\begin{aligned}tT(r,g) &\le \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad + \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,f) + S(r,g).\end{aligned} \tag{16}$$

Then from (15) and (16) we get

$$\begin{aligned}t\{T(r,f)+T(r,g)\} &\le 2\left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad + S(r,f) + S(r,g)\end{aligned}$$

i.e.,

$$\left(t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g). \tag{17}$$

Since

$$\begin{aligned}&\left(t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3}\right)\\ &\qquad = \frac{(n+m-1)^2t + 2(n+m-1)(t-2m) - 8m}{(n+m-1)(n+m+3)}.\end{aligned}$$

We note that when $n+m-1 > \frac{4m}{t}$ i.e., when $n > \frac{4m}{t} - (m-1)$, then clearly $t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} > 0$ and so (17) leads to a contradiction. This completes the proof. ∎

Lemma 11. *Let f, g be two nonconstant meromorphic functions,let $F = \frac{f^n P(f) f'}{a}$, $G = \frac{g^n P(f) g'}{a}$, where $P(w)$ is defined as in Theorem 3, $a = a(z)(\not\equiv 0, \infty)$ is a small function with respect to f and g, and n is a positive integer such that $n > m+5$. If $H \equiv 0$ then one of the following three cases holds:*

(I) $f^n P(f) f' g^n P(g) g' \equiv a^2(z)$,

(II) $f(z) \equiv t g(z)$ for a constant t such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, ..., n+m+1-i, ..., n+1), a_{m-i} \neq 0$ for some $i = 1, 2, ..., m$,

(III) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^{n+1}(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + ... + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1})$.

Proof. Clearly

$$F = \frac{[f^{n+1}\{\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + ... + \frac{a_0}{n+1}\}]'}{a}$$

and

$$G = \frac{[g^{n+1}\{\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + ... + \frac{a_0}{n+1}\}]'}{a},$$

where

$$P_1(w) = \{\frac{a_m}{n+m+1} w^m + \frac{a_{m-1}}{n+m} w^{m-1} + ... + \frac{a_0}{n+1}\}$$

proceeding in the same way as the proof of Lemma 9, taking $k = 1$ and considering $n+1$ instead of n we get either

$$f^n P(f) f' g^n P(g) g' \equiv a^2(z)$$

or

$$f^n P(f) f' \equiv g^n P(g) g'.$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in the above equation we get

$$a_m g^m (h^{n+m+1} - 1) + a_{m-1} g^{m-1} (h^{n+m} - 1) + ... \\ + a_1 g (h^{n+2} - 1) + a_0 (h^{n+1} - 1) \equiv 0,$$

which implies that $h^{d_3} = 1$, where $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$. Thus $f \equiv tg$ for a constant t such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, ..., m\}$.

If h is not constant then f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = w_1^{n+1}(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - w_2^{n+1}(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1})$. ■

3. Proof of the theorems

Proof of Theorem 1. Let $F(z)$ and $G(z)$ be given as in Lemma 9. It follows that F and G share $(1,l)$ except for the zeros and poles of $P(z)$. So from (1) we obtain

$$(18) \quad \begin{aligned} N_2(r,0;F) &\leq N_2(r,0;[f^nP(f)]^{(k)}) + S(r,f) \\ &\leq T(r,[f^nP(f)]^{(k)}) - (n+m)T(r,f) \\ &\quad + N_{k+2}(r,0;f^nP(f)) + S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + O\{\log r\} + S(r,f). \end{aligned}$$

Again by (2) we have

$$(19) \quad N_2(r,0;G) \leq k\overline{N}(r,\infty;f) + N_{k+2}(r,0;g^nP(g)) + S(r,g).$$

From (18) we get

$$(20) \quad \begin{aligned} (n+m)T(r,f) &\leq T(r,F) + N_{k+2}(r,0;f^nP(f)) - N_2(r,0;F) \\ &\quad + O\{\log r\} + S(r,f). \end{aligned}$$

Case 1. Let $H \not\equiv 0$.

Subcase 1. Let $l \geq 2$. Let (i) of Lemma 3 holds. Then using (19) we obtain from (20),

$$(21) \quad \begin{aligned} (n+m)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ &\quad + N_{k+2}(r,0;g^nP(f)) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;f^nP(f)) + N_{k+2}(r,0;g^nP(g)) \\ &\quad +2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq (k+m+2)\{T(r,f) + T(r,g)\} + 2\overline{N}(r,\infty;f) \\ &\quad + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} \\ &\quad + S(r,f) + S(r,g) \\ &\leq [(k+m+4) - 2\Theta(\infty;f) + \epsilon]T(r,f) \\ &\quad + [(2k+m+4) - (k+2)\Theta(\infty,g) + \epsilon]T(r,g) \\ &\quad + S(r,f) + S(r,g) \end{aligned}$$

$$\leq [(3k+m+8)-2\Theta(\infty,f)-2\Theta(\infty,g) - k\text{min}\{\Theta(\infty,f),\Theta(\infty,g)\}+2\epsilon]T(r)+S(r).$$

In a similar way we can obtain

$$(22)\quad (n+m)T(r,g) \leq [(3k+m+8)-2\Theta(\infty,f)-2\Theta(\infty,g) - k\min\{\Theta(\infty,f),\Theta(\infty,g)\}+2\epsilon]T(r)+S(r).$$

From (21) and (22) we obtain

$$[n-3k-m-8+2\Theta(\infty,f)+2\Theta(\infty,g) + k\text{min}\{\Theta(\infty,f)\Theta(\infty,g)\}-2\epsilon]T(r)\leq S(r)$$

contradicting with the fact that $n \geq 3k+m+8$.

Subcase 2. Let $l = 1$, using Lemma 4 and (19) we obtain from (20),

$$
\begin{aligned}
(23)\quad (n+m)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\
&\quad + \frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)\\
&\quad + N_{k+2}(r,0;f^nP(f))+O\{\log r\}\\
&\quad + S(r,f)+S(r,g)\\
&\leq N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\
&\quad + \frac{1}{2}N_{k+1}(r,0;f^nP(f))+\frac{k+5}{2}\overline{N}(r,\infty;f)\\
&\quad + (k+2)\overline{N}(r,\infty;g)+O\{\log r\}\\
&\quad + S(r,f)+S(r,g)\\
&\leq (k+m+2)\{T(r,f)+T(r,g)\}\\
&\quad + \frac{k+m+1}{2}T(r,f)+\frac{k+5}{2}\overline{N}(r,\infty;f)\\
&\quad + (k+2)\overline{N}(r,\infty;g)+O\{\log r\}\\
&\quad + S(r,f)+S(r,g)\\
&\leq [(2k+\frac{3m+10}{2})-(\frac{k}{2}+3)\Theta(\infty,f)\\
&\quad - \frac{1}{2}\Theta(\infty,f)+\epsilon]T(r,f)+[(2k+m+4)\\
&\quad - (\frac{k}{2}+2g\Theta(\infty,g)-\frac{k}{2}\Theta(\infty,f)+\epsilon]T(r,g)\\
&\quad + O\{\log r\}+S(r,f)+S(r,g)\\
&\leq [4k+\frac{5m+18}{2}-(\frac{k+5}{2})\,(\Theta(\infty,f)\\
&\quad + \Theta(\infty,g))+2\epsilon]T(r)+S(r).
\end{aligned}
$$

Similarly

$$(24)\quad (n+m)T(r,g) \le [4k+\frac{5m+18}{2}-(\frac{k+5}{2})(\Theta(\infty,f) + \Theta(\infty,g))+2\epsilon]T(r)+S(r).$$

Combining (23) and (24) we obtain

$$[n-4k-\frac{5m+18}{2}+m+\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\epsilon]T(r)\le S(r),$$

contradiction. Since $n \ge 4k+\frac{3m+18}{2}$.

Subcase 3. Let $l=0$, using Lemma 5 and (19) we obtain from (20),

$$\begin{aligned}(25)\ (n+m)T(r,f) \le\ & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;f^nP(f))\\ &+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+O\{\log r\}+S(r,f)+S(r,g)\\ \le\ & N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &+2N_{k+2}(r,0;f^nP(f))+N_{k+1}(r,0;g^nP(g))\\ &+(2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)\\ &+O\{log\ r\}+S(r,f)+S(r,g)\\ \le\ & [(5k+3m+8)-(2k+4)\Theta(\infty;f)-\epsilon]T(r,f)\\ &+[(4k+2m+6)-(2k+3)\Theta(\infty;g)-\epsilon]T(r,g)\\ &+O\{log\ r\}+S(r,f)+S(r,g)\\ &+[(9k+5m+14)-(2k+3)[\Theta(\infty;f)\\ &+\Theta(\infty,g)]-\ \min\{\Theta(\infty,f)\Theta(\infty;g)\}\\ &+2\epsilon]T(r)+S(r).\end{aligned}$$

Similarly

$$\begin{aligned}(26)\ (n+m)T(r,g) \le\ & [(9k+5m+14)-(2k+3)[\Theta(\infty;f)+\Theta(\infty,g)]\\ &-\min\{\Theta(\infty,f)\Theta(\infty;g)\}+2\epsilon]T(r)+S(r).\end{aligned}$$

From (25) and (26) we get

$$\begin{aligned}&[n-9k-4m-14]+(2k+3)(\Theta(\infty,f)+\Theta(\infty;g))\\ &\qquad +\min\{\Theta(\infty;f)\Theta(\infty;g)\}-2\epsilon]T(r)\le S(r),\end{aligned}$$

contradicts with the facts that $n \ge 9k+4m+14$.

Case 3. Let $H \equiv 0$. Then the Theorem follows from Lemma 9. ■

Proof of Theorem 2. Noting that $\overline{N}(r,\infty;f)=0$, $\overline{N}(r,\infty;g)=0$ and proceeding in the like manner as the proof of Theorem 1 we obtain the result of the Theorem 2. ■

Proof of Theorem 3. Let $F=\frac{f^nP(f)f'}{a(z)}$ and $G=\frac{g^nP(g)g'}{a(z)}$. Then F, G share $(1,l)$, except the zeros and poles of $a(z)$. Clearly

$$F=\frac{[f^{n+1}\{\frac{a_m}{n+m+1}f^m+\frac{a_{m-1}}{n+m}f^{m-1}+...+\frac{a_0}{n+1}\}]'}{a}$$

and

$$G=\frac{[g^{n+1}\{\frac{a_m}{n+m+1}g^m+\frac{a_{m-1}}{n+m}g^{m-1}+...+\frac{a_0}{n+1}\}]'}{a},$$

where

$$P_1(w)=\{\frac{a_m}{n+m+1}w^m+\frac{a_{m-1}}{n+m}w^{m-1}+...+\frac{a_0}{n+1}\}.$$

Case 1. Let $H \not\equiv 0$. Now following the same procedure as adopted in the proof of Case 1 of Theorem 1 we can easily deduce a contradiction.

Case 2. Let $H \equiv 0$. Since $n>k_1$ and $n>m+5$ the theorem follows from Lemma 10 and 11. ■

Proof of Theorem 4. Noting that $\overline{N}(r,\infty;f)=0$, $\overline{N}(r,\infty;g)=0$ and proceeding in the like manner as the proof of Theorem 3 we obtain the result of the Theorem 4. ■

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WAGHAMORE P. HARINA
DEPARTMENT OF MATHEMATICS
CENTRAL COLLEGE CAMPUS
BANGALORE UNIVERSITY
BANGALORE-560 001, INDIA
e-mail: harinapw@gmail.com

S. RAJESHWARI
DEPARTMENT OF MATHEMATICS
CENTRAL COLLEGE CAMPUS
BANGALORE UNIVERSITY
BANGALORE-560 001, INDIA
e-mail: rajeshwaripreetham@gmail.com

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