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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVE ABSOLUTE VALUES ARE CONVEX AND CONCAVE

Abstract. In this paper, we derive general integral identity by establishing new Hermite-Hadamard type inequalities for functions whose absolute values of derivatives are convex and concave. Corresponding error estimates for midpoint formula are also included. Moreover, some applications to special means of real numbers are also provided.

Key words: Hermite-Hadamard inequality, convex functions, power-mean inequality, Holder's inequality, special means, midpoint formula.

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1. Introduction

Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is known as Hermite-Hadamard's inequality. Many inequalities have been established for convex functions out of which above inequality is most popular due to its rich geometrical significance and applications. Both inequalities hold in the reversed direction for the function f to be concave.

Recently, Kirmaci [5] obtained the following Hermite-Hadamard's type integral inequality.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) such that $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1}$$

In [3] S. Hussain et. al proved a variant of Hermite-Hadamard inequality as:

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. Assume that $q > 1$ such that $|f'|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \\ \leq \left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left(\frac{b-a}{4}\right)\left(\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right).\end{aligned} \tag{2}$$

In [9] C.E.M. Pearce and J.E. Pecaric proved the following result:

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. Assume that $q \geq 1$ such that $|f'|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left|f'(\frac{a+b}{2})\right|. \tag{3}$$

For recent results, generalizations and numerous applications concerning Hermite-Hadamard's inequality see ([1], [2], [3], [4], [7], [8]) and the references given therein. In [5], authors provided the right estimations of Hermite-Hadamard inequality for convex functions. In this paper, the left estimations of the Hermite-Hadamard inequality with applications will be investigated.

2. Results and discussions

In order to proceed towards our main results, we prove the following lemma:

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) such that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$, then for each $x \in [a, b]$, we have:*

$$\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)$$
$$= \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)f'(tb+(1-t)(a+b-x))dt$$
$$+ \frac{(b-x)^2}{(b-a)}\int_0^1 (t-1)f'(ta+(1-t)(a+b-x))dt.$$

Proof. Integrating by parts, we can state

$$I_1 = \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)f'(tb+(1-t)(a+b-x))dt$$
$$= -\frac{x-a}{b-a}f(a+b-x) + \frac{1}{b-a}\int_{a+b-x}^b f(u)\,du,$$

and

$$I_2 = \frac{(b-x)^2}{(b-a)}\int_0^1 (t-1)f'(ta+(1-t)(a+b-x))dt$$
$$= -\frac{b-x}{b-a}f(a+b-x) + \frac{1}{b-a}\int_a^{a+b-x} f(u)\,du.$$

Using the fact; if $a \le x \le b$ then $a \le a+b-x \le b$, we have

$$\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b+x) = I_1 + I_2.$$

We obtain the desired result. ■

Now we prove the following result by using above Lemma 1.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°. Assume that $p \in \mathbf{R}$, $p > 1$ such that $|f'|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$, then we have:*

$$(4)\quad \left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right|$$
$$\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2}{b-a}\left[\left|f'(b)\right|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \frac{(b-x)^2}{b-a}\left[\left|f'(a)\right|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right],$$

for each $x \in [a,b]$ and $q = \frac{p}{p-1}$.

Proof. Using Lemma 1, convexity of $|f'|^q$ and the Holder's inequality, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)f'(tb+(1-t)(a+b-x))dt \\
&\quad + \frac{(b-x)^2}{(b-a)}\int_0^1 (1-t)f'(ta+(1-t)(a+b-x))dt, \\
&\leq \frac{(x-a)^2}{(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^2}{(b-a)}\left(\int_0^1 (1-t)^p dt\right)^{1/p}\left(\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^{\frac{p}{p-1}}$ is convex, then Hermite-Hadamards inequality follows that

$$\int_0^1 \left|f'(tb+(1-t)(a+b-x))\right|^q dt \leq \frac{|f'(b)|^q + 2\left|f'(a+b-x)\right|^q}{2},$$

$$\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt \leq \frac{|f'(a)|^q + 2\left|f'(a+b-x)\right|^q}{2}.$$

Hence

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2}{b-a}\left[|f'(b)|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right. \\
&\qquad\qquad \left. + \frac{(b-x)^2}{b-a}\left[|f'(a)|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

The proof is completed. ■

Corollary 1. *In Theorem 4, put* $x = \frac{a+b}{2}$*, we get*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \times \left[\left(|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad \left. + \left(|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(|f'(a)| + |f'(b)|\right).
\end{aligned}$$

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°. Assume that $|f'|^q$ is convex function on I, for some $q \geq 1$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$, then for each $x \in [a, b]$, we have*

$$\begin{aligned}(5)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\ &\leq \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}} \times \left[\frac{(x-a)^2}{b-a}\left[\left|f'(b)\right|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right. \\ &\quad \left. + \frac{(b-x)^2}{b-a}\left[\left|f'(a)\right|^q + 2\left|f'(a+b-x)\right|^q\right]^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The Lemma 1 and well known power mean inequality follows that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\ &\leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)\left|f'(tb+(1-t)(a+b-x))\right|dt \\ &\quad + \frac{(b-x)^2}{(b-a)}\int_0^1 (1-t)\left|f'(ta+(1-t)(a+b-x))\right|dt, \\ &\leq \frac{(x-a)^2}{(b-a)}\left(\int_0^1 (1-t)dt\right)^{\frac{q-1}{q}}\left(\int_0^1 (1-t)\left|f'(tb+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^2}{(b-a)}\left(\int_0^1 (1-t)dt\right)^{\frac{q-1}{q}}\left(\int_0^1 (1-t)\left|f'(ta+(1-t)(a+b-x))\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

$|f'|^q$ is convex function implies that

$$\begin{aligned}&\int_0^1 (1-t)\left|f'(tb+(1-t)(a+b-x))\right|^q dt \\ &\leq \int_0^1 (1-t)\left[t\left|f'(b)\right|^q + (1-t)\left|f'(a+b-x)\right|^q\right]dt \\ &= \frac{|f'(b)|^q + 2|f'(a+b-x)|^q}{6}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 (1-t)\left|f'(ta+(1-t)(a+b-x))\right|^q dt \\ &\leq \frac{|f'(a)|^q + 2|f'(a+b-x)|^q}{6}\end{aligned}$$

If we combine above inequalities, we get the required result. The proof is completed. ■

Corollary 2. *Take $x = \frac{a+b}{2}$ in the above Theorem 5, we get*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(|f'(a)|^q + 2\left|f'(\frac{a+b}{2})\right|^q\right)^{\frac{1}{q}} + \left(|f'(b)|^q + 2\left|f'(\frac{a+b}{2})\right|^q\right)^{\frac{1}{q}}\right] \\
&\leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right)^{\frac{1}{q}}(b-a)\left(|f'(a)| + |f'(b)|\right).
\end{aligned}$$

Theorem 6. *Let f be defined as in Lemma 1 such that $|f'|$ is a convex function on I. Then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{(b-a)}\left[\frac{|f'(b)| + 2|f'(a+b-x)|}{6}\right] \\
&\quad + \frac{(b-x)^2}{(b-a)}\left[\frac{|f'(a)| + 2|f'(a+b-x)|}{6}\right],
\end{aligned} \tag{6}$$

for each $x \in [a,b]$.

Proof. Using Lemma 1, the convexity of $|f'|$ with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)f'(tb+(1-t)(a+b-x))dt \\
&\quad + \frac{(b-x)^2}{(b-a)}\int_0^1 (t-1)f'(ta+(1-t)(a+b-x))dt.
\end{aligned}$$

Since $|f'|$ is convex, then we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)\left[t|f'(b)| + (1-t)|f'(a+b-x)|\right]dt \\
&+ \frac{(b-x)^2}{(b-a)}\int_0^1 (t-1)\left[t|f'(a)| + (1-t)|f'(a+b-x)|\right]dt \\
&= \frac{(x-a)^2}{(b-a)}\left[\frac{|f'(b)| + 2|f'(a+b-x)|}{6}\right] + \frac{(b-x)^2}{(b-a)}\left[\frac{|f'(a)| + 2|f'(a+b-x)|}{6}\right].
\end{aligned}$$

Which completes the proof. ■

Corollary 3. *In Theorem 6, put* $x = \frac{a+b}{2}$, *we get*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{24}\left(|f'(a)| + 4\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right).$$

Remark 1. Using the convexity of $|f'|$ in Corollary 3, we get the inequality (1).

Theorem 7. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°. *Assume that* $q > 1$ *such that* $|f'|^q$ *is concave function on* I. *Suppose that* $a, b \in I^\circ$ *with* $a < b$ *and* $f' \in L[a,b]$, *then we have:*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \leq \left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}}\left[\frac{(x-a)^2\left|f'(\frac{a+2b-x}{2})\right| + (b-x)^2\left|f'(\frac{a+2b-x}{2})\right|}{b-a}\right], \tag{7}$$

for each $x \in [a,b]$ *and* $p = \frac{q}{q-1}$.

Proof. Using Lemma 1 and the Holder's integral inequality, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)\,|f'(tb+(1-t)(a+b-x))|\,dt \\
&\quad + \frac{(b-x)^2}{(b-a)}\int_0^1 (1-t)\,|f'(ta+(1-t)(a+b-x))|\,dt, \\
&\leq \frac{(x-a)^2}{(b-a)}\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_0^1 |f'(tb+(1-t)(a+b-x))|^q\,dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^2}{(b-a)}\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q}{q-1}}\left(\int_0^1 |f'(ta+(1-t)(a+b-x))|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, then Jensen's integral follows that

$$\begin{aligned}
&\int_0^1 |f'(tb+(1-t)(a+b-x))|^q\,dt \\
&= \int_0^1 t^0\,|f'(ta+(1-t)(a+b-x))|^q\,dt \\
&\leq \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{1}{\int_0^1 t^0 dt}\int_0^1 (tb+(1-t)(a+b-x))\,dt\right)\right|^q \\
&= \left|f'\left(\frac{a+2b-x}{2}\right)\right|^q.
\end{aligned}$$

Similarly

$$\int_0^1 \left|f'(ta+(1-t)(a+b-x))\right|^q dt \leq \left|f'\left(\frac{2a+b-x}{2}\right)\right|^q.$$

From the above inequalities, we get

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \leq \left[\frac{q-1}{2q-1}\right]^{\frac{q-1}{q}} \left[\frac{(x-a)^2\left|f'(\frac{a+2b-x}{2})\right| + (b-x)^2\left|f'(\frac{a+2b-x}{2})\right|}{b-a}\right].$$

The proof is completed. ■

Remark 2. Take $x = \frac{a+b}{2}$ in Theorem 7, we get the inequality (2).

Theorem 8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°. Assume that $q \geq 1$ with $q = \frac{p}{p-1}$, such that $|f'|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$, then for each $x \in [a,b]$ we have*

$$(8)\quad \left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \leq \frac{1}{2}\left[\frac{(x-a)^2}{b-a}\left|f'\left(\frac{2a+3b-2x}{3}\right)\right| + \frac{(b-x)^2}{b-a}\left|f'\left(\frac{3a+2b-2x}{2}\right)\right|\right].$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\left|f'(tb+(1-t)(a+b-x))\right|^q \geq t\left|f'(b)\right|^q + (1-t)\left|f'(a+b-x)\right|^q.$$

Thus

$$\left|f'(ta+(1-t)(a+b-x))\right|^q \geq t\left|f'(a)\right|^q + (1-t)\left|f'(a+b-x)\right|^q.$$

As $|f'|^q$ is concave, Jensen's integral inequality follows that

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f(a+b-x)\right| \leq \frac{(x-a)^2}{(b-a)}\int_0^1 (1-t)\left|f'(tb+(1-t)(a+b-x))\right| dt + \frac{(b-x)^2}{(b-a)}\int_0^1 (1-t)\left|f'(ta+(1-t)(a+b-x))\right| dt$$

$$\leq \frac{(x-a)^2}{(b-a)}\left(\int_0^1 (1-t)dt\right)\left|f'\left(\frac{\int_0^1 (1-t)(tb+(1-t)(a+b-x)\,dt}{\int_0^1 (1-t)dt}\right)\right|$$
$$+\frac{(b-x)^2}{(b-a)}\left(\int_0^1 (1-t)dt\right)\left|f'\left(\frac{\int_0^1 (1-t)(ta+(1-t)(a+b-x)\,dt}{\int_0^1 (1-t)dt}\right)\right|$$
$$=\frac{1}{2}\left[\frac{(x-a)^2\left|f'(\frac{2a+3b-2x}{3})\right|+(b-x)^2\left|f'(\frac{3a+2b-2x}{2})\right|}{b-a}\right].$$

The proof is completed. ■

Corollary 4. Take $x=\frac{a+b}{2}$ in Theorem 8, we get

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\left|f'\left(\frac{a+2b}{3}\right)\right|+\left|f'\left(\frac{2a+b}{3}\right)\right|\right].$$

Remark 3. Using the convexity of $|f'|$ in Corollary 4, we get inequality (3).

3. The midpoint formula

Let d be such that $a=x_0<x_1<x_3<\ldots<x_n=b$ is the division of the interval $[a,b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T(f,d)+E(f,d), \tag{9}$$

where

$$T(f,d)=\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

is the midpoint version and $E(f,d)$ denotes the error term. Here we derive some error estimates for midpoint formula.

Proposition 1. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable function on I°. Assume that $p\in\mathbf{R}$, $p>1$ such that $|f'|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$, then for every division d of $[a,b]$, we have*

$$E(f,d)\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\sum_{i=0}^{n-1} f\frac{(x_i+x_{i+1})^2}{2}\left(\left|f'(x_i)\right|+\left|f'(x_{i+1})\right|\right).$$

Proof. Using Corollary 1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, 2, 3, \ldots, n-1)$ of the division, we have

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right) \leq \frac{(x_i+x_{i+1})}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right).$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|^{\frac{p}{p-1}}$ is convex function, we obtain, by triangle inequality, that

$$\begin{aligned}\left|\int_a^b f(x)dx - T(f,d)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right| \\ &\leq \sum_{i=0}^{n-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\sum_{i=0}^{n-1} f\frac{(x_i+x_{i+1})^2}{2}\left(|f'(x_i)| + |f'(x_{i+1})|\right).\end{aligned}$$

■

The proof of the following proposition is similar to that of Proposition 1 and using Remark 3.

Proposition 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable function on I°. Assume that $q \geq 1$ with $q = \frac{p}{p-1}$, such that $|f'|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$, then by (9), for every division d of $[a,b]$, we have:*

$$E(f,d) \leq \sum_{i=0}^{n-1} f\frac{(x_i-x_{i+1})^2}{4}\left|f'(\frac{x_i+x_{i+1}}{2})\right|.$$

4. Applications to some special means

Consider the following special means. The arithmetic mean

$$A(a,b) = \frac{a+b}{2}, \quad a, b \in \mathbf{R}. \tag{10}$$

The harmonic mean

$$H(a,b) = \frac{2ab}{a+b}, \quad a, b \in \mathbf{R}\backslash\{0\}. \tag{11}$$

The logarithmic mean

$$L(a,b) = \begin{cases} a & \text{if } a = b, \\ \frac{b-a}{lnb-lna} & \text{if } a \neq b, \ a,b > 0. \end{cases} \tag{12}$$

The generalized logarithmic mean

$$L_n(a,b) = \begin{cases} a & \text{if } a = b, \\ \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & \text{if } a \neq b, \ n \in \mathbf{Z}\backslash\{-1,0\}, \ a,b > 0. \end{cases} \tag{13}$$

Some new inequalities are derived for the above means by using the results of Section 2.

Proposition 3. *Let $a, b \in \mathbf{R}, a < b, 0 \notin [a,b]$. Then for $q \geq 1$*

$$|L(a,b) - G(a,b)| \leq (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} A(a,b).$$

Proof. This follows by Corollary 1, taking $f(x) = e^x$. ■

Proposition 4. *Let $a, b \in \mathbf{R}$, $a < b$, $0 \notin [a,b]$. Then for $q \geq 1$*

$$\left|\frac{A(a,b)}{I(a,b)}\right| \leq \exp\left|(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} H^{-1}(a,b)\right|.$$

Proof. This follows by Corollary 1, taking $f(x) = -\ln x$. ■

Proposition 5. *Let $a, b \in \mathbf{R}$, $a < b$, $0 \notin [a,b]$. Then for $q \geq 1$*

$$\left|L^{-1}(a,b) - A^{-1}(a,b)\right| \leq (b-a)\left(\frac{3^{1-\frac{1}{q}}}{4}\right) H^{-1}\left(|a|^2, |b|^2\right).$$

Proof. This follows by Corollary 2, taking $f(x) = \frac{1}{x}$. ■

Proposition 6. *Let $a, b \in \mathbf{R}$, $a < b$, $0 \notin [a,b]$. Then for $q \geq 1$*

$$|L_n^n(a,b) - A^n(a,b)| \leq |n|(b-a)\frac{3^{1-\frac{1}{q}}}{4} A\left(|a|^{n-1}, |b|^{n-1}\right).$$

Proof. This follows by Corollary 2, taking $f(x) = e^x$. ■

Conclusion. We derived general integral identity by establishing new Hermite-Hadamard type inequalities for functions whose absolute values of derivatives are convex and concave. Corresponding error estimates for midpoint formula are also included. Moreover, some applications to special means of real numbers are also provided. These results give better estimates as presented earlier in the literature.

References

[1] AVCI M., KAVURMACI H., OZDEMIR M.E., New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, *Appl. Math. Comput.,* 217(2011), 5171-5176.

[2] DRAGOMIR S.S., PEARCE C.E.M., Selected topics on Hermite-Hadamard inequalities and applications, *RGMIA Monographs, Victoria University,* 2000, http://www.sta.vu.edu.au/RGMIA/monographs/hermite_ hadamard.html.

[3] HUSSAIN S., BHATTI M.I., IQBAL M., Hadamard-type inequalities for s-convex functions I, *Punjab Univ. Jour. Math.,* 41(2009), 51-60.

[4] HUSSAIN S., QAISAR S., More results on Hermite-Hadamard type inequality through (α, m)-preinvexity, *J. Appl. Anal. Comput.,* 6(2)(2016) 293-305.

[5] KIRMACI U.S., Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, *Appl. Math. Comp.,* 147 (2004), 137-146.

[6] KAVURMACI H., AVCI M., OZDEMIR M.E., New inequalities of Hermite-Hadamard type for convex functions with applications, *JIA,* 86(2011).

[7] QAISAR S., HUSSAIN S., HE C., On new inequalities of Hermite-Hadamard type for generalized convex functions, *Italian Journal of Pure and Applied Mathematics,* 33(2014), 139-148.

[8] QAISAR S., HUSSAIN S., HE C., On new inequalities of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex with applications, *Journal of the Egyptian Mathematical Society,* 22(1)(2014) 19-22.

[9] PEARCE C.E.M., PECARIC J., Inequalities for differentiable mappings with application to special means and quadrature formula, *Appl. Math. Lett.,* 13(2000), 51-55.

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