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APPROXIMATION BY MAX-PRODUCT OPERATORS

ABSTRACT. Here we study the approximation of functions by a great variety of Max-Product operators under differentiability. These are positive sublinear operators. Our study is based on our general results about positive sublinear operators. We produce Jackson type inequalities under initial conditions. So our approach is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of a high order derivative of the function under approximation. We improve known related results which do not use smoothness of functions..

KEY WORDS: positive sublinear operators, Max-product operators, modulus of continuity.

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1. Introduction

The main motivation here comes from the monograph by B. Bede, L. Coroianu and S. Gal [3], 2016.

We mention the interpolation Hermite-Fejer polynomials on Chebyshev knots of the first kind (see [3], p. 4): Let $f : [-1, 1] \to \mathbb{R}$ and based on the knots $x_{N,k} = \cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right) \in (-1, 1)$, $k \in \{0, ..., N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, which are the roots of the first kind Chebyshev polynomial $T_{N+1}(x) = \cos((N+1)\arccos x)$, we define (see Fejér [4])

$$(1) \qquad H_{2N+1}(f)(x) = \sum_{k=0}^{N} h_{N,k}(x) f(x_{N,k}),$$

where

$$(2) \qquad h_{N,k}(x) = (1 - x \cdot x_{N,k})\left(\frac{T_{N+1}(x)}{(N+1)(x - x_{N,k})}\right)^2,$$

the fundamental interpolation polynomials.

Denoting $A_{N+1}(f) = \|H_{2N+1} - f\|_\infty$, Fejer [4] proved that $\lim\limits_{N\to\infty} A_{N+1}(f)$ $= 0$, for all $f \in C([-1,1])$.

Popoviciu ([6]) also proved that $A_{N+1}(f) = O\left(\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right)\right)$, and Moldovan ([5]) improved it to

$$A_{N+1}(f) = O\left(\omega_1\left(f, \frac{\ln(N+1)}{N+1}\right)\right).$$

Here $\omega_1(f,\delta) = \sup\limits_{\substack{x,y\in[-1,1]:\\ |x-y|\leq\delta}} |f(x) - f(y)|$, $\delta > 0$, is the first modulus of continuity.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [3]) are defined by

$$(3) \qquad H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\, N \in \mathbb{N},$$

where $f : [-1,1] \to \mathbb{R}_+$ is continuous.

By [3], p. 286 we get that

$$(4) \qquad \left|H_{2N+1}^{(M)}(f)(x) - f(x)\right|$$

$$\leq 14\omega_1\left(f, \frac{1}{N+1}\right), \quad \forall\, N \in \mathbb{N}, \text{ any } x \in [-1,1].$$

Call

$$(5) \qquad E_N(x) := H_{2N+1}^{(M)}(|\cdot - x|)(x)$$

$$= \frac{\bigvee_{k=0}^{N} h_{N,k}(x)\,|x_{N,k} - x|}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad x \in [-1,1].$$

Then by [3], p. 287 we obtain that

$$(6) \qquad E_N(x) \leq \frac{2\pi}{N+1}, \quad \forall\, x \in [-1,1], \quad N \in \mathbb{N}.$$

For $m \in \mathbb{N}$, we get

$$(7) \quad H_{2N+1}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)\,|x_{N,k} - x|^m}{\bigvee_{k=0}^{N} h_{N,k}(x)}$$

$$= \frac{\bigvee_{k=0}^{N} h_{N,k}(x)\,|x_{N,k} - x|\,|x_{N,k} - x|^{m-1}}{\bigvee_{k=0}^{N} h_{N,k}(x)}$$

$$\leq 2^{m-1}\frac{\bigvee_{k=0}^{N} h_{N,k}(x)\,|x_{N,k} - x|}{\bigvee_{k=0}^{N} h_{N,k}(x)}$$

$$\leq \frac{2^m\pi}{N+1} \quad \forall\, x \in [-1,1], \quad N \in \mathbb{N}.$$

Hence it holds

(8) $\quad H_{2N+1}^{(M)}\left(|\cdot - x|^m\right)(x) \le \dfrac{2^m \pi}{N+1}, \quad \forall\, x \in [-1,1],\ m \in \mathbb{N},\ \forall\, N \in \mathbb{N}.$

Clearly it holds

(9) $\qquad\qquad\qquad H_{2N+1}^{(M)}\left(|\cdot - x|^m\right)(x) > 0,$

$\forall\, x \in [-1,1] : x \ne x_{N,k},\ \forall\, N \in \mathbb{N},$ any $k \in \{0,1,...,N\}$; any $m \in \mathbb{N}$.

Furthermore we have

$$H_{2N+1}^{(M)}(1)(x) = 1, \quad \forall\, x \in [-1,1],$$

and $H_{2N+1}^{(M)}$ maps continuous functions to continuous functions over $[-1,1]$ and for any $x \in \mathbb{R}$ we have $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$.

We also have $h_{N,k}(x_{N,k}) = 1$, and $h_{N,k}(x_{N,j}) = 0$, if $k \ne j$, furthermore it holds $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, for all $j \in \{0,...,N\}$, see [3], p. 282.

In this work we will improve (4) by assuming differentiability of f. Similar improvements, using the differentiability of f, will be presented for Max-product Lagrange interpolation operators, Max-product truncated sampling operators and Max-product Neural network operators.

2. Main results

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $n \in \mathbb{N}$, and

(10) $CB_+^n(I) = \Big\{ f : I \to \mathbb{R}_+ : f^{(i)} \text{ is continuous and bounded on } I,$
$$\text{for both } i = 0, n \Big\}.$$

We define for

(11) $\quad f \in CB_+(I) = \{f : I \to \mathbb{R}_+ : f \text{ is continuous and bounded on } I\},$

the first modulus of continuity

(12) $\qquad\qquad\qquad \omega_1(f,\delta) = \sup_{\substack{x,y \in I: \\ |x-y| \le \delta}} |f(x) - f(y)|,$

where $0 < \delta \le diameter\,(I)$.

Call $C_+(I) = \{f : I \to \mathbb{R}_+ : f \text{ is continuous on } I\}$.

Let $L_N : C_+(I) \to CB_+(I)$, $n, N \in \mathbb{N}$ be a sequence of operators satisfying the following properties (see also [3], p. 17):

(i) (positive homogeneous)

$$(13) \qquad L_N\left(\alpha f\right) = \alpha L_N\left(f\right), \ \ \forall\, \alpha \geq 0, \ \ f \in C_+\left(I\right),$$

(ii) (Monotonicity)

$$(14) \quad \text{if } f, g \in C_+\left(I\right) \text{ satisfy } f \leq g, \text{ then } L_N\left(f\right) \leq L_N\left(g\right), \ \ \forall\, N \in \mathbb{N},$$

and

(iii) (Subadditivity)

$$(15) \qquad L_N\left(f + g\right) \leq L_N\left(f\right) + L_N\left(g\right), \ \ \forall\, f, g \in C_+\left(I\right).$$

We call L_N positive sublinear operators.

In particular we will study the restrictions $L_N|_{CB^n_+(I)} : CB^n_+\left(I\right) \to CB_+\left(I\right)$.

The operators $H^{(M)}_{2N+1}$ are positive sublinear operators. From [1] we will be using the following result:

Theorem 1 ([1]). *Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+\left(I\right)$ into $CB_+\left(I\right)$, and $f \in CB^n_+\left(I\right)$, where $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume $L_N\left(1\right) = 1$, $\forall\, N \in \mathbb{N}$, and $f^{(i)}\left(x\right) = 0$, $i = 1, ..., n$, for a fixed $x \in I$, and $\delta > 0$. Then*

$$(16) \quad |L_N\left(f\right)\left(x\right) - f\left(x\right)|$$

$$\leq \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!}\left[L_N\left(|\cdot - x|^n\right)\left(x\right) + \frac{L_N\left(|\cdot - x|^{n+1}\right)\left(x\right)}{\left(n+1\right)\delta}\right],$$

$\forall\, N \in \mathbb{N}$.

We give

Theorem 2. *Let $f \in C^n\left(\left[-1, 1\right], \mathbb{R}_+\right)$, with $f^{(i)}\left(x\right) = 0$, $i = 1, ..., n \in \mathbb{N}$, for some fixed $x \in \left[-1, 1\right]$, $N \in \mathbb{N}$. Then*

$$(17) \quad \left|H^{(M)}_{2N+1}\left(f\right)\left(x\right) - f\left(x\right)\right|$$

$$\leq \frac{1}{n!}\omega_1\left(f^{(n)}, 2\sqrt[n+1]{\frac{\pi}{N+1}}\right)\left[\frac{2^n\pi}{N+1} + \frac{2^n}{\left(n+1\right)}\left(\sqrt[n+1]{\frac{\pi}{N+1}}\right)^n\right].$$

When $x = x_{N,k}$, the left hand side of (17) is zero.

Proof. Here we are using (8) and (16), namely we have

$$(18) \quad H^{(M)}_{2N+1}\left(|\cdot - x|^m\right)\left(x\right) \leq \frac{2^m\pi}{N+1}, \ \ \forall\, x \in \left[-1, 1\right], \ \ m \in \mathbb{N}, \ \ \forall\, N \in \mathbb{N}.$$

and

$$(19) \qquad \left| H_{2N+1}^{(M)} (f) (x) - f (x) \right|$$

$$\leq \frac{\omega_1 \left(f^{(n)}, \delta \right)}{n!} \left[\frac{2^n \pi}{N+1} + \frac{1}{(n+1) \delta} \left(\frac{2^{n+1} \pi}{N+1} \right) \right],$$

$\delta > 0$.

Then choose

$$(20) \qquad \delta := 2 \sqrt[n+1]{\frac{\pi}{N+1}},$$

we get

$$(21) \qquad \delta^{n+1} = 2^{n+1} \frac{\pi}{N+1},$$

and

$$(22) \left| H_{2N+1}^{(M)} (f) (x) - f (x) \right| \leq \frac{\omega_1 \left(f^{(n)}, 2 \sqrt[n+1]{\frac{\pi}{N+1}} \right)}{n!} \left[\frac{2^n \pi}{N+1} + \frac{1}{(n+1)} \delta^n \right]$$

$$= \frac{1}{n!} \omega_1 \left(f^{(n)}, 2 \sqrt[n+1]{\frac{\pi}{N+1}} \right) \left[\frac{2^n \pi}{N+1} + \frac{1}{(n+1)} 2^n \left(\sqrt[n+1]{\frac{\pi}{N+1}} \right)^n \right],$$

proving the claim. ∎

It follows the $n = 1$ case.

Corollary 1. *Let $f \in C^1 ([-1, 1], \mathbb{R}_+)$, with $f' (x) = 0$, for some fixed $x \in [-1, 1]$, $N \in \mathbb{N}$. Then*

$$(23) \qquad \left| H_{2N+1}^{(M)} (f) (x) - f (x) \right|$$

$$\leq \omega_1 \left(f', 2 \sqrt{\frac{\pi}{N+1}} \right) \left[\frac{2\pi}{N+1} + \sqrt{\frac{\pi}{N+1}} \right].$$

From (17) and/or (23), as $N \to \infty$, we get that $H_{2N+1}^{(M)} (f) (x) \to f (x)$.

Proof. By (17). ∎

We make

Remark 1. We compare (23) to (4). We prove that (23) gives a sharper estimate and speed than (4). We observe that

$$(24) \qquad \frac{2\sqrt{\pi}}{\sqrt{N+1}} \left(\frac{2\pi}{N+1} + \frac{\sqrt{\pi}}{\sqrt{N+1}} \right) \leq \frac{14}{N+1}$$

$$\Leftrightarrow$$

$$(25) \qquad \frac{1}{\sqrt{N+1}} \le \frac{7-\pi}{2\pi\sqrt{\pi}},$$

true for large enough $N \in \mathbb{N}$.

We also make

Remark 2. Here we compare (17) to (4). We prove that (17) gives a better estimate and speed than (4). We see that

$$(26) \qquad \frac{2\sqrt[n+1]{\pi}}{n!\sqrt[n+1]{N+1}} \left[\frac{2^n \pi}{N+1} + \frac{2^n}{(n+1)} \left(\frac{\sqrt[n+1]{\pi}}{\sqrt[n+1]{N+1}} \right)^n \right] \le \frac{14}{N+1},$$

$$\Leftrightarrow$$

$$(27) \qquad \frac{1}{\sqrt[n+1]{N+1}} \le \left(7 - \frac{2^n \pi}{(n+1)!} \right) \frac{n!}{2^n \pi \sqrt[n+1]{\pi}},$$

true for large enough $N \in \mathbb{N}$.

About notice that $7 - \frac{2^n \pi}{(n+1)!} > 0$.

We continue with

Remark 3. Here we deal with Lagrange interpolation polynomials on Chebyshev knots of second kind plus the endpoints ± 1 (see [3], p. 5). These polynomials are linear operators attached to $f : [-1,1] \to \mathbb{R}$ and to the knots $x_{N,k} = \cos \left(\left(\frac{N-k}{N-1} \right) \pi \right) \in [-1,1]$, $k = 1, ..., N$, $N \in \mathbb{N}$, which are the roots of $\omega_N(x) = \sin(N-1) t \sin t$, $x = \cos t$. Notice that $x_{N,1} = -1$ and $x_{N,N} = 1$. Their formula is given by ([3], p. 377)

$$(28) \qquad L_N(f)(x) = \sum_{k=1}^{N} l_{N,k}(x) f(x_{N,k}),$$

where

$$(29) \qquad l_{N,k}(x) = \frac{(-1)^{k-1} \omega_N(x)}{(1 + \delta_{k,1} + \delta_{k,N})(N-1)(x - x_{N,k})},$$

$N \ge 2$, $k = 1, ..., N$, and $\omega_N(x) = \prod_{k=1}^{N} (x - x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j} = 1$, if $i = j$, and $\delta_{i,j} = 0$, if $i \ne j$. Then (see [3], p. 5)

$$(30) \qquad \|L_N(f) - f\|_{\infty,[-1,1]} \le c\omega_1 \left(f, \frac{1}{N} \right) \ln N.$$

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, are defined by ([3], p. 12)

$$(31) \qquad L_N^{(M)}(f)(x) = \frac{\bigvee_{k=1}^N l_{N,k}(x) f(x_{N,k})}{\bigvee_{k=1}^N l_{N,k}(x)}, \quad x \in [-1, 1],$$

where $f : [-1, 1] \to \mathbb{R}_+$ is continuous.

First we see that $L_N^{(M)}(f)(x)$ is well defined and continuous for any $x \in [-1, 1]$. Following [3], p. 289, because $\sum_{k=1}^N l_{N,k}(x) = 1$, $\forall\ x \in \mathbb{R}$, for any x there exists $k \in \{1, ..., N\} : l_{N,k}(x) > 0$, hence $\bigvee_{k=1}^N l_{N,k}(x) > 0$. We have that $l_{N,k}(x_{N,k}) = 1$, and $l_{N,k}(x_{N,j}) = 0$, if $k \neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, all $j \in \{1, ..., N\}$, and $L_N^{(M)}(1) = 1$.

Call $I_N^+(x) = \{k \in \{1, ..., N\} ; l_{N,k}(x) > 0\}$, then $I_N^+(x) \neq \emptyset$.

So for $f \in CB_+([-1, 1])$ we get

$$(32) \qquad L_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \geq 0.$$

By [3], p. 295, we have:

Let $f \in C([-1, 1], \mathbb{R}_+)$, $N \in \mathbb{N}$, $N \geq 3$, N is odd, then

$$(33) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right| \leq 4\omega_1\left(f, \frac{1}{N-1}\right), \quad \forall\ x \in [-1, 1].$$

Notice here that $|x_{N,k} - x| \leq 2, \forall\ x \in [-1, 1]$.

By [3], p. 297, we get that

$$(34) \qquad L_N^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=1}^N l_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k=1}^N l_{N,k}(x)}$$
$$= \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{\pi^2}{6(N-1)},$$

$N \geq 3$, $\forall\ x \in (-1, 1)$, N is odd.

We get that $(m \in \mathbb{N})$

$$(35) \qquad L_N^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) |x_{N,k} - x|^m}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{2^{m-1}\pi^2}{6(N-1)},$$

$N \geq 3$ odd, $\forall\ x \in (-1, 1)$.

We present

Theorem 3. *Let* $f \in C^n([-1, 1], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in [-1, 1]$, $f^{(i)}(x) = 0$,
$i = 1, ..., n$. *Here* $N \in \mathbb{N}$, $N \geq 3$ *is odd. Then*

$$(36) \quad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \frac{\omega_1 \left(f^{(n)}, \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{1}{n+1}} \right)}{n!} \left[\frac{2^{n-1} \pi^2}{6(N-1)} + \frac{1}{(n+1)} \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{n}{n+1}} \right].$$

Proof. When $x = \pm 1$, the left hand side of (36) is zero, hence (36) is
trivially true. Let now $x \in (-1, 1)$, by Theorem 1 and (16), (35), we obtain

$$(37) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \frac{\omega_1 \left(f^{(n)}, \delta \right)}{n!} \left[\frac{2^{n-1} \pi^2}{6(N-1)} + \frac{1}{(n+1)\delta} \frac{2^n \pi^2}{6(N-1)} \right]$$

(setting $\delta := \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{1}{n+1}}$, i.e. $\delta^{n+1} = \frac{2^n \pi^2}{6(N-1)}$)

$$= \frac{\omega_1 \left(f^{(n)}, \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{1}{n+1}} \right)}{n!} \left[\frac{2^{n-1} \pi^2}{6(N-1)} + \frac{1}{(n+1)} \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{n}{n+1}} \right],$$

proving the claim.   ∎

The case $n = 1$ follows:

Corollary 2. *Let* $f \in C^1([-1, 1], \mathbb{R}_+)$, $x \in [-1, 1]$, $f'(x) = 0$. *Here*
$N \in \mathbb{N}$, $N \geq 3$ *is odd. Then*

$$(38) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \omega_1 \left(f', \frac{\pi}{\sqrt{3(N-1)}} \right) \left[\frac{\pi^2}{6(N-1)} + \frac{\pi}{2\sqrt{3(N-1)}} \right].$$

By (36) and/or (38), we get that $L_N^{(M)}(f)(x) \to f(x)$, *as* $N \to \infty$.

We make

Remark 4. Here we compare (38) to (33), and we prove that (38) gives
better estimates and speeds than (33). We observe that

$$(39) \qquad \frac{\pi}{\sqrt{3(N-1)}} \left[\frac{\pi^2}{6(N-1)} + \frac{\pi}{2\sqrt{3(N-1)}} \right] \leq \frac{4}{N-1}$$

$$\Leftrightarrow$$

$$(40) \qquad \frac{1}{\sqrt{3(N-1)}} \leq \frac{24 - \pi^2}{\pi^3},$$

true for large enough $N \geq 3$ odd.

Remark 5. Here we compare (36) to (33), and we prove that (36) gives better estimates and speeds that (33). We see that

$$(41) \qquad \frac{1}{n!} \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{1}{n+1}} \left[\frac{2^{n-1} \pi^2}{6(N-1)} + \frac{1}{(n+1)} \left(\frac{2^n \pi^2}{6(N-1)} \right)^{\frac{n}{n+1}} \right]$$

$$\leq \frac{4}{N-1}$$

$$\Leftrightarrow$$

$$(42) \qquad \frac{1}{N-1} \leq \frac{3}{2^{n-1} \pi^2} \left(\frac{24(n+1)! - 2^n \pi^2}{(n+1) 2^{n-1} \pi^2} \right)^{n+1},$$

true for large enough $N \, (\text{odd}) \geq 3$.

We continue with

Remark 6. From [3], p. 297, we have: Let $f \in C([-1,1], \mathbb{R}_+)$, $N \geq 4$, $N \in \mathbb{N}$, N even. Then

$$(43) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right| \leq 28 \omega_1 \left(f, \frac{1}{N-1} \right), \quad \forall \, x \in [-1,1].$$

From [3], p. 298, we get

$$(44) \qquad L_N^{(M)}(|\cdot - x|)(x) \leq \frac{4\pi^2}{3(N-1)} = \frac{2^2 \pi^2}{3(N-1)}, \quad \forall \, x \in (-1,1).$$

Hence $(m \in \mathbb{N})$

$$(45) \qquad L_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^{m+1} \pi^2}{3(N-1)}, \quad \forall \, x \in (-1,1).$$

We present

Theorem 4. Let $f \in C^n([-1,1], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in [-1,1]$, $f^{(i)}(x) = 0$, $i = 1, ..., n$. Here $N \in \mathbb{N}$, $N \geq 4$, N is even. Then

$$(46) \quad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \frac{\omega_1 \left(f^{(n)}, \left(\frac{2^{n+2} \pi^2}{3(N-1)} \right)^{\frac{1}{n+1}} \right)}{n!} \left[\frac{2^{n+1} \pi^2}{3(N-1)} + \frac{1}{(n+1)} \left(\frac{2^{n+2} \pi^2}{3(N-1)} \right)^{\frac{n}{n+1}} \right].$$

Proof. When $x = \pm 1$, the left hand side of (46) is zero, thus (46) is trivially true.

Let now $x \in (-1, 1)$, by Theorem 1 and (16), (45), we obtain

$$(47) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!} \left[\frac{2^{n+1}\pi^2}{3(N-1)} + \frac{1}{(n+1)\,\delta} \frac{2^{n+2}\pi^2}{3(N-1)} \right]$$

(setting $\delta := \left(\frac{2^{n+2}\pi^2}{3(N-1)}\right)^{\frac{1}{n+1}}$, i.e. $\delta^{n+1} = \frac{2^{n+2}\pi^2}{3(N-1)}$)

$$= \frac{\omega_1\left(f^{(n)}, \left(\frac{2^{n+2}\pi^2}{3(N-1)}\right)^{\frac{1}{n+1}}\right)}{n!} \left[\frac{2^{n+1}\pi^2}{3(N-1)} + \frac{1}{(n+1)}\left(\frac{2^{n+2}\pi^2}{3(N-1)}\right)^{\frac{n}{n+1}} \right],$$

proving the claim. ∎

The case $n = 1$ follows:

Corollary 3. Let $f \in C^1\left([-1, 1], \mathbb{R}_+\right)$, $x \in [-1, 1]$, $f'(x) = 0$. Here $N \in \mathbb{N}$, $N \geq 4$, N is even. Then

$$(48) \qquad \left| L_N^{(M)}(f)(x) - f(x) \right|$$

$$\leq \omega_1\left(f', \frac{2\pi\sqrt{2}}{\sqrt{3(N-1)}}\right) \left[\frac{4\pi^2}{3(N-1)} + \frac{\pi\sqrt{2}}{\sqrt{3(N-1)}} \right].$$

By (46) and/or (48), we get that $L_N^{(M)}(f)(x) \to f(x)$, as $N \to \infty$.

We make

Remark 7. Here we compare (48) to (43). We prove that (48) gives better estimates and speeds that (43). Indeed we have

$$(49) \qquad \frac{2\pi\sqrt{2}}{\sqrt{3(N-1)}} \left[\frac{4\pi^2}{3(N-1)} + \frac{\pi\sqrt{2}}{\sqrt{3(N-1)}} \right] \leq \frac{28}{N-1}$$

$$\Leftrightarrow$$

$$(50) \qquad \frac{1}{N-1} \leq \frac{3}{8\pi^2} \left(\frac{42 - 2\pi^2}{2\pi^2}\right)^2,$$

true for large enough $N \geq 4$, even.

We make

Remark 8. Here we compare (46) to (43). We prove that (46) gives better estimates and speeds that (43). We observe that

$$(51) \qquad \frac{1}{n!} \left(\frac{2^{n+2}\pi^2}{3(N-1)} \right)^{\frac{1}{n+1}} \left[\frac{2^{n+1}\pi^2}{3(N-1)} + \frac{1}{(n+1)} \left(\frac{2^{n+2}\pi^2}{3(N-1)} \right)^{\frac{n}{n+1}} \right]$$
$$\leq \frac{28}{N-1}$$
$$\Leftrightarrow$$

$$(52) \qquad \frac{1}{N-1} \leq \left(\frac{3}{2^{n+2}\pi^2} \right) \left[\frac{42(n+1)! - 2^{n+1}\pi^2}{(n+1)2^n\pi^2} \right]^{n+1},$$

true for large enough $N \geq 4$. N even.

We continue with

Remark 9. The sampling truncated linear operators (see [3], p. 7) are defined by

$$(53) \qquad W_N(f)(x) = \sum_{k=0}^{N} \frac{\sin(Nx - k\pi)}{Nx - k\pi} f\left(\frac{k\pi}{N} \right), \forall\, x \in [0, \pi],$$

and

$$(54) \qquad T_N(f)(x) = \sum_{k=0}^{N} \frac{\sin^2(Nx - k\pi)}{(Nx - k\pi)^2} f\left(\frac{k\pi}{N} \right),$$

$\forall\, x \in [0, \pi]$; $f \in C([0, \pi], \mathbb{R})$ and they are used as approximators.

Here we deal with the Max-product truncated sampling operators (see [3], p. 13) defined by

$$(55) \qquad W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx - k\pi)}{Nx - k\pi} f\left(\frac{k\pi}{N} \right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx - k\pi)}{Nx - k\pi}}, \quad x \in [0, \pi],$$

$f : [0, \pi] \to \mathbb{R}_+$, continuous, and

$$(56) \qquad T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx - k\pi)}{(Nx - k\pi)^2} f\left(\frac{k\pi}{N} \right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx - k\pi)}{(Nx - k\pi)^2}}, \quad x \in [0, \pi],$$

$f : [0, \pi] \to \mathbb{R}_+$, continuous.

Following [3], p. 343, and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx - k\pi)}{Nx - k\pi}$, we get that $s_{N,k}\left(\frac{k\pi}{N} \right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N} \right) = 0$, if $k \neq j$, furthermore $W_N^{(M)}(f)\left(\frac{j\pi}{N} \right) = f\left(\frac{j\pi}{N} \right)$, for all $j \in \{0, ..., N\}$.

Clearly $W_N^{(M)}(f)$ is a well-defined function for all $x \in [0, \pi]$, and it is continuous on $[0, \pi]$, also $W_N^{(M)}(1) = 1$.

By [3], p. 344, $W_N^{(M)}$ are positive sublinear operators.

Call $I_N^+(x) = \{k \in \{0, 1, ..., N\}; s_{N,k}(x) > 0\}$, and set $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$.

We see that

$$(57) \qquad W_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)}.$$

We call

$$(58) \qquad F_N(x) := W_N^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k=0}^{N} s_{N,k}(x)}$$

$$= \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)}.$$

By Theorem 8.2.8 ([3], p. 345) we get: Let $f \in C([0, \pi], \mathbb{R}_+)$. Then

$$(59) \qquad \left| W_N^{(M)}(f)(x) - f(x) \right| \leq 4\omega_1 \left(f, \frac{1}{N} \right)_{[0,\pi]}, \quad \forall\, N \in \mathbb{N},\, x \in [0, \pi].$$

We have that ([3], p. 346)

$$(60) \qquad F_N(x) \leq \frac{\pi}{2N}.$$

Notice also $|x_{N,k} - x| \leq \pi$, $\forall\, x \in [0, \pi]$.

Therefore ($m \in \mathbb{N}$) it holds

$$(61) \qquad W_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{\pi^{m-1}\pi}{2N} = \frac{\pi^m}{2N}.$$

We present

Theorem 5. *Let $f \in C^n([0, \pi], \mathbb{R}_+)$, $x \in [0, \pi]$ fixed, $f^{(i)}(x) = 0$, $i = 1, ..., n$. Then*

$$(62) \quad \left| W_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1 \left(f^{(n)}, \left(\frac{\pi^{n+1}}{2N} \right)^{\frac{1}{n+1}} \right)}{n!} \left[\frac{\pi^n}{2N} + \frac{\left(\frac{\pi^{n+1}}{2N} \right)^{\frac{n}{n+1}}}{n+1} \right]$$

$$(63) \qquad = \frac{1}{n!} \omega_1 \left(f^{(n)}, \frac{\pi}{\sqrt[n+1]{2N}} \right) \left[\frac{\pi^n}{2N} + \frac{1}{n+1} \left(\frac{\pi}{\sqrt[n+1]{2N}} \right)^n \right].$$

Proof. Using Theorem 1, (16) and (61), we get

$$(64) \quad \left| W_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(f^{(n)}, \delta\right)}{n!} \left[\frac{\pi^n}{2N} + \frac{1}{(n+1)\delta}\left(\frac{\pi^{n+1}}{2N}\right) \right]$$

(choosing $\delta := \left(\frac{\pi^{n+1}}{2N}\right)^{\frac{1}{n+1}}$, i.e. $\delta^{n+1} = \frac{\pi^{n+1}}{2N}$)

$$= \frac{1}{n!}\omega_1\left(f^{(n)}, \left(\frac{\pi^{n+1}}{2N}\right)^{\frac{1}{n+1}}\right)\left[\frac{\pi^n}{2N} + \frac{1}{(n+1)}\left(\frac{\pi^{n+1}}{2N}\right)^{\frac{n}{n+1}} \right],$$

proving the claim. ∎

The case $n = 1$ follows:

Corollary 4. *Let $f \in C^1\left([0,\pi], \mathbb{R}_+\right)$, $x \in [0,\pi]$ fixed, $f'(x) = 0$. Then*

$$(65) \quad \left| W_N^{(M)}(f)(x) - f(x) \right| \leq \omega_1\left(f', \frac{\pi}{\sqrt{2N}}\right)\left[\frac{\pi}{2N} + \frac{\pi}{2\sqrt{2N}} \right].$$

By (62)-(63) and/or (65), we get that $W_N^{(M)}(f)(x) \to f(x)$, as $N \to +\infty$.

We make

Remark 10. Here we compare (65) to (59) and we prove that (65) gives better estimates and speeds that (59). Indeed we have

$$(66) \quad \frac{\pi}{\sqrt{2N}}\left(\frac{\pi}{2N} + \frac{\pi}{2\sqrt{2N}} \right) \leq \frac{4}{N}$$

$$\Leftrightarrow$$

$$(67) \quad \frac{1}{\sqrt{2N}} \leq \frac{16 - \pi^2}{2\pi^2},$$

true for large enough $N \in \mathbb{N}$.

We also make

Remark 11. Here we compare (62)-(63) to (59), and we prove that (62)-(63) gives better estimates and speeds that (59). We observe that

$$(68) \quad \frac{1}{n!}\left(\frac{\pi^{n+1}}{2N}\right)^{\frac{1}{n+1}}\left[\frac{\pi^n}{2N} + \frac{\left(\frac{\pi^{n+1}}{2N}\right)^{\frac{n}{n+1}}}{(n+1)} \right] \leq \frac{4}{N}$$

$$\Leftrightarrow$$

$$(69) \quad \frac{1}{\sqrt[n+1]{2N}} \leq \frac{8(n+1)! - \pi^{n+1}}{(n+1)\pi^{n+1}},$$

true for large enough $N \in \mathbb{N}$.

Notice here that $8(n+1)! - \pi^{n+1} > 0$, $\forall \, n \in \mathbb{N}$.

We continue with

Remark 12. Here we study $T_N(f)(x)$, see (54).

By Theorem 8.2.13, [3], p. 352, we get: Let $f \in C([0,\pi], \mathbb{R}_+)$, then

$$(70) \qquad \left| T_N^{(M)}(f)(x) - f(x) \right| \le 4\omega_1 \left(f, \frac{1}{N} \right)_{[0,\pi]}, \quad \forall\, N \in \mathbb{N},\ x \in [0,\pi].$$

By [3], p. 352, we get

$$(71) \qquad T_N^{(M)}(|\cdot - x|)(x) \le \frac{\pi}{2N},$$

hence $(m \in \mathbb{N})$ we find

$$(72) \qquad T_N^{(M)}(|\cdot - x|^m)(x) \le \frac{\pi^m}{2N}.$$

Here again $x_{N,k} = \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$.

The operators $T_N^{(M)}$ are positive sublinear operators, mapping $C([0,\pi], \mathbb{R}_+)$ into itself, and $T_N^{(M)}(1) = 1$. So we can apply again Theorem 1. We obtain the same results as before with $W_N^{(M)}$, we state them:

Theorem 6. Let $f \in C^1([0,\pi], \mathbb{R}_+)$, $x \in [0,\pi]$ fixed, $f'(x) = 0$. Then

$$(73) \quad \left| T_N^{(M)}(f)(x) - f(x) \right| \le \omega_1 \left(f', \frac{\pi}{\sqrt{2N}} \right) \left[\frac{\pi}{2N} + \frac{\pi}{2\sqrt{2N}} \right], \quad \forall\, N \in \mathbb{N}.$$

Theorem 7. Let $f \in C^n([0,\pi], \mathbb{R}_+)$, $x \in [0,\pi]$ fixed, $f^{(i)}(x) = 0$, $i = 1, ..., n$. Then

$$(74) \qquad \left| T_N^{(M)}(f)(x) - f(x) \right|$$
$$\le \frac{\omega_1 \left(f^{(n)}, \frac{\pi}{n+\sqrt[1]{2N}} \right)}{n!} \left[\frac{\pi^n}{2N} + \frac{\left(\frac{\pi}{n+\sqrt[1]{2N}} \right)^n}{n+1} \right], \quad \forall\, N \in \mathbb{N}.$$

Clearly (73), (74) can perform better than (70), the same study as for $W_N^{(M)}$.

Furthermore we derive $T_N^{(M)}(f)(x) \to f(x)$, as $N \to +\infty$.

We continue with

Remark 13. Let $b : \mathbb{R} \to \mathbb{R}_+$ be a centered (it takes a global maximum at 0) bell-shaped function, with compact support $[-T, T]$, $T > 0$ (that is $b(x) > 0$ for all $x \in (-T, T)$) and $I = \int_{-T}^{T} b(x)\, dx > 0$.

The Cardaliaguet-Euvrard neural network operators are defined by (see [2])

$$(75) \qquad C_{N,\alpha}(f)(x) = \sum_{k=-N^2}^{N^2} \frac{f\left(\frac{k}{N}\right)}{I N^{1-\alpha}} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right),$$

$0 < \alpha < 1$, $N \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$.

$CB(\mathbb{R})$ denotes the continuous and bounded function on $\mathbb{R}$, and

$$CB_+(\mathbb{R}) = \{f : \mathbb{R} \to [0, \infty); \ f \in CB(\mathbb{R})\}.$$

The corresponding max-product Cardaliaguet-Euvrard neural network operators will be given by

$$(76) \qquad C_{N,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)},$$

$f \in \mathbb{R}$, $f \in CB_+(\mathbb{R})$, see also [2].

Next we follow [2].

For any $x \in \mathbb{R}$, denoting

$$J_{T,N}(x) = \left\{k \in \mathbb{Z}; \ -N^2 \le k \le N^2, N^{1-\alpha}\left(x - \frac{k}{N}\right) \in (-T, T)\right\},$$

we can write

$$(77) \qquad C_{N,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)},$$

$x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, where $J_{T,N}(x) \ne \emptyset$. Indeed, we have $\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) > 0$, $\forall \ x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We have that $C_{N,\alpha}^{(M)}(1)(x) = 1$, $\forall \ x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

See in [2] there: Lemma 2.1, Corollary 2.2 and Remarks.

We need

Theorem 8 ([2]). *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (76).*

(i) If $|f(x)| \le c$ for all $x \in \mathbb{R}$ then $\left|C_{N,\alpha}^{(M)}(f)(x)\right| \le c$, for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $C_{N,\alpha}^{(M)}(f)(x)$ is continuous at any point $x \in \mathbb{R}$, for all $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(ii) If $f, g \in CB_+(\mathbb{R})$ satisfy $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $C_{N,\alpha}^{(M)}(f)(x) \leq C_{N,\alpha}^{(M)}(g)(x)$ for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iii) $C_{N,\alpha}^{(M)}(f + g)(x) \leq C_{N,\alpha}^{(M)}(f)(x) + C_{N,\alpha}^{(M)}(g)(x)$ for all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iv) For all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, we have
$$\left| C_{N,\alpha}^{(M)}(f)(x) - C_{N,\alpha}^{(M)}(g)(x) \right| \leq C_{N,\alpha}^{(M)}(|f - g|)(x);$$

(v) $C_{N,\alpha}^{(M)}$ is positive homogeneous, that is $C_{N,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{N,\alpha}^{(M)}(f)(x)$ for all $\lambda \geq 0$, $x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $f \in CB_+(\mathbb{R})$.

We make

Remark 14. We have

$$(78) \qquad E_{N,\alpha}(x) := C_{N,\alpha}^{(M)}(|\cdot - x|)(x)$$
$$= \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) \left|x - \frac{k}{N}\right|}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)},$$

$\forall\, x \in \mathbb{R}$, and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

By (77), $C_{N,\alpha}^{(M)}$ satisfies

$$C_{N,\alpha}^{(M)}(f \vee g)(x) = C_{N,\alpha}^{(M)}(f)(x) \vee C_{N,\alpha}^{(M)}(g)(x),$$

$\forall\, f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

Notice that

$$(79) \qquad \bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) = \bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right).$$

By [2], Lemma 3.1 there, we have: Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$. Then for any $j \in \mathbb{Z}$ with $-N^2 \leq j \leq N^2$, all $x \in \left[\frac{j}{N}, \frac{j+1}{N}\right]$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, we have

$$(80) \qquad \bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)$$
$$= \max\left\{b\left(N^{1-\alpha}\left(x - \frac{j}{N}\right)\right), b\left(N^{1-\alpha}\left(x - \frac{j+1}{N}\right)\right)\right\} > 0.$$

Lemma 3.1 ([2]), is valid only for all $x \in [-N, N]$.

We mention from [2] the following:

Theorem 9 ([2])**.** *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$. In addition, suppose that the following requirements are fulfilled:*

(i) There exist $0 < m_1 \leq M_1 < \infty$ such that $m_1(T - x) \leq b(x) \leq M_1(T - x)$, $\forall \, x \in [0, T]$;

(ii) There exist $0 < m_2 \leq M_2 < \infty$ such that $m_2(x + T) \leq b(x) \leq M_2(x + T)$, $\forall \, x \in [-T, 0]$.

Then for all $f \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and for all $N \in \mathbb{N}$ satisying $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, we have the estimate

$$(81) \qquad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq c\omega_1\left(f, N^{\alpha-1}\right)_{\mathbb{R}},$$

where

$$c := 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right),$$

and

$$\omega_1(f, \delta)_{\mathbb{R}} := \sup_{\substack{x,y\in\mathbb{R}: \\ |x-y|\leq\delta}} |f(x) - f(y)|.$$

We make

Remark 15. In [2], was proved that

$$(82) \qquad E_{N,\alpha}(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1},$$

$$\forall \, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

That is

$$(83) \qquad C_{N,\alpha}^{(M)}(|\cdot - x|)(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1},$$

$$\forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

From (78) we have that $\left|x - \frac{k}{N}\right| \leq \frac{T}{N^{1-\alpha}}$.

Hence $(m \in \mathbb{N})$ $(\forall\, x \in \mathbb{R}$ and $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\})$

$$(84) \quad C_{N,\alpha}^{(M)}\left(|\cdot - x|^m\right)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|^m}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}$$

$$\leq \left(\frac{T}{N^{1-\alpha}}\right)^{m-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1},$$

$$\forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

Then $(m \in \mathbb{N})$ it holds

$$(85) \qquad C_{N,\alpha}^{(M)}\left(|\cdot - x|^m\right)(x) \leq T^{m-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} \frac{1}{N^{m(1-\alpha)}},$$

$$\forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

Call

$$(86) \qquad\qquad\qquad \lambda := \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} > 0.$$

Consequently $(m \in \mathbb{N})$ we derive

$$(87) \quad C_{N,\alpha}^{(M)}\left(|\cdot - x|^m\right)(x) \leq \frac{\lambda T^{m-1}}{N^{m(1-\alpha)}}, \quad \forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

We need

Theorem 10. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (76).*

Let $f \in CB_+^n(\mathbb{R})$, $n \in \mathbb{N}$. Let $x \in \mathbb{R} : f^{(i)}(x) = 0$, $i = 1, ..., n$, and $\delta > 0$. Then

$$(88) \quad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right|$$

$$\leq \frac{\omega_1\left(f^{(n)}, \delta\right)_{\mathbb{R}}}{n!} \left[C_{N,\alpha}^{(M)}\left(|\cdot - x|^n\right)(x) + \frac{C_{N,\alpha}^{(M)}\left(|\cdot - x|^{n+1}\right)(x)}{(n+1)\,\delta} \right],$$

$$\forall\, N \in \mathbb{N} : N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}.$$

Proof. By [1], we get that

$$(89) \qquad |f(x) - f(y)| \leq \frac{\omega_1\left(f^{(n)}, \delta\right)_{\mathbb{R}}}{n!} \left[|x - y|^n + \frac{|x - y|^{n+1}}{(n+1)\delta} \right],$$

$\forall\, y \in \mathbb{R},\, \delta > 0$. Using Theorem 8 and $C_{N,\alpha}^{(M)}(1) = 1$, we get

$$(90) \quad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq C_{N,\alpha}^{(M)}\left(|f(\cdot) - f(x)| \right)(x)$$

$$\leq \frac{\omega_1\left(f^{(n)}, \delta\right)_{\mathbb{R}}}{n!} \left[C_{N,\alpha}^{(M)}\left(|\cdot - x|^n \right)(x) + \frac{C_{N,\alpha}^{(M)}\left(|\cdot - x|^{n+1} \right)(x)}{(n+1)\delta} \right],$$

$\forall\, N \in \mathbb{N} : N > \max\left\{ T + |x|, T^{-\frac{1}{\alpha}} \right\}.$ ∎

We give

Theorem 11. *Same assumptions as in Theorem 9. Let* $f \in CB_+^n(\mathbb{R})$, $n \in \mathbb{N}$, $x \in \mathbb{R} : f^{(i)}(x) = 0$, $i = 1, ..., n$. *Then*

$$(91) \qquad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(f^{(n)}, \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{1}{n+1}} \right)_{\mathbb{R}}}{n!}$$

$$\times \left[\frac{\lambda T^{n-1}}{N^{n(1-\alpha)}} + \frac{1}{(n+1)} \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{n}{n+1}} \right],$$

$\forall\, N > \max\left\{ T + |x|, \left(\frac{2}{T} \right)^{\frac{1}{\alpha}} \right\}.$

Proof. We use (88) and we choose

$$(92) \qquad \qquad \delta := \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{1}{n+1}},$$

i.e. $\delta^{n+1} = \frac{\lambda T^n}{N^{(n+1)(1-\alpha)}}$. Hence

$$(93) \quad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \overset{((87),\,(88))}{\leq} \frac{1}{n!} \omega_1\left(f^{(n)}, \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{1}{n+1}} \right)_{\mathbb{R}}$$

$$\times \left[\frac{\lambda T^{n-1}}{N^{n(1-\alpha)}} + \frac{1}{(n+1)\delta} \frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right]$$

$$= \frac{1}{n!} \omega_1\left(f^{(n)}, \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{1}{n+1}} \right)_{\mathbb{R}}$$

$$\times \left[\frac{\lambda T^{n-1}}{N^{n(1-\alpha)}} + \frac{1}{(n+1)} \left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}} \right)^{\frac{n}{n+1}} \right],$$

$\forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, proving the claim.   ∎

It follows the case $n = 1$.

Corollary 5. *Same assumptions as in Theorem 9. Let* $f \in CB_+^1(\mathbb{R})$, $x \in \mathbb{R} : f'(x) = 0$. *Then*

$$(94) \qquad \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right|$$

$$\leq \omega_1 \left(f', \sqrt{\frac{\lambda T}{N^{2(1-\alpha)}}} \right)_{\mathbb{R}} \left[\frac{\lambda}{N^{1-\alpha}} + \frac{1}{2}\sqrt{\frac{\lambda T}{N^{2(1-\alpha)}}} \right],$$

$\forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$.

By (91) and/or (94) we get that $C_{N,\alpha}^{(M)}(f)(x) \to f(x)$, as $N \to +\infty$.

We make

Remark 16. We prove that (94) performs better than (81).

Indeed we have that

$$(95) \qquad \sqrt{\frac{\lambda T}{N^{2(1-\alpha)}}} \left[\frac{\lambda}{N^{1-\alpha}} + \frac{1}{2}\sqrt{\frac{\lambda T}{N^{2(1-\alpha)}}} \right] \leq \frac{2(\lambda+1)}{N^{1-\alpha}},$$

$$\Leftrightarrow$$

$$(96) \qquad \frac{1}{N^{(1-\alpha)}} \leq \frac{2(\lambda+1)}{\lambda\left(\sqrt{\lambda T} + \frac{T}{2}\right)},$$

true $\forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, large enough.

We also make

Remark 17. We prove that (91) performs better than (81). We observe that

$$(97) \quad \frac{1}{n!}\left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}}\right)^{\frac{1}{n+1}} \left[\frac{\lambda T^{n-1}}{N^{n(1-\alpha)}} + \frac{1}{(n+1)}\left(\frac{\lambda T^n}{N^{(n+1)(1-\alpha)}}\right)^{\frac{n}{n+1}} \right]$$

$$\leq \frac{2(\lambda+1)}{N^{1-\alpha}}$$

$$\Leftrightarrow$$

$$(98) \qquad \frac{1}{N^{n(1-\alpha)}} \leq \frac{2(\lambda+1)}{\left[\frac{\lambda\,^{n+1}\!\sqrt{\lambda T}\,T^{\frac{n^2+n-1}{n+1}}}{n!} + \frac{\lambda T^n}{(n+1)!} \right]},$$

true $\forall\, N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, large enough.

Here using Theorem 1 we extend the domain of the application results of [1].

Remark 18. We start with the Max-product Bernstein operators ([3], p. 10)

$$(99) \qquad B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad \forall\, N \in \mathbb{N},$$

$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-1}$, $x \in [0,1]$, $\bigvee$ stands for maximum, and $f \in C_+([0,1]) = \{f : [0,1] \to \mathbb{R}_+ \text{ is continuous}\}$.

From [1] we get

$$(100) \qquad B_N^{(M)}(|\cdot - x|^m)(x) \le \frac{6}{\sqrt{N+1}}, \quad \forall\, x \in [0,1],\ m, N \in \mathbb{N}.$$

Denote by

$$C_+^n([0,1]) = \{f : [0,1] \to \mathbb{R}_+,\ n\text{-times continuously differentiable}\},\quad n \in \mathbb{N}.$$

We give

Theorem 12. *Let $f \in C_+^n([0,1])$, a fixed $x \in [0,1]$ such that $f^{(i)}(x) = 0$, $i = 1, ..., n$. Then*

$$(101)\quad \left| B_N^{(M)}(f)(x) - f(x) \right| \le \frac{\omega_1\left(f^{(n)}, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\right)}{n!}$$

$$\times \left[\frac{6}{\sqrt{N+1}} + \frac{1}{(n+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{n}{n+1}} \right], \quad \forall\, N \in \mathbb{N}.$$

We get $B_N^{(M)}(f)(x) \to f(x)$, as $N \to \infty$.

Proof. Use of (16) for $\delta = \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}$. ∎

We continue with

Remark 19. Here we focus on the truncated Favard-Szász-Mirakjan operators

$$(102) \qquad K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1],$$

$N \in \mathbb{N}$, $f \in C_+([0,1])$, $s_{N,k}(x) = \frac{(Nx)^k}{k!}$, see [3], p. 11.

From [1] we get

$$(103) \qquad K_N^{(M)} \left(|\cdot - x|^m \right) (x) \le \frac{3}{\sqrt{N}}, \quad \forall \, x \in [0,1] \,, \ N \in \mathbb{N}, \, m \in \mathbb{N}.$$

We give

Theorem 13. *Let $f \in C_+^n ([0,1])$, x fixed in $[0,1]$ such that $f^{(i)}(x) = 0$, $i = 1, ..., n$. Then*

$$(104) \qquad \left| K_N^{(M)} (f) (x) - f(x) \right| \le \frac{\omega_1 \left(f^{(n)}, \left(\frac{3}{\sqrt{N}} \right)^{\frac{1}{n+1}} \right)}{n!}$$

$$\times \left[\frac{3}{\sqrt{N}} + \frac{1}{(n+1)} \left(\frac{3}{\sqrt{N}} \right)^{\frac{n}{n+1}} \right], \quad \forall \, N \in \mathbb{N}.$$

Proof. Use of (16) for $\delta = \left(\frac{3}{\sqrt{N}} \right)^{\frac{1}{n+1}}$. ∎

We make

Remark 20. Next we study the truncated Max-product Baskakov operators (see [3], p. 11)

$$(105) \qquad U_N^{(M)} (f) (x) = \frac{\bigvee_{k=0}^{N} b_{N,k} (x) f \left(\frac{k}{N} \right)}{\bigvee_{k=0}^{N} b_{N,k} (x)},$$

$x \in [0,1]$, $f \in C_+ ([0,1])$, $N \in \mathbb{N}$, where

$$b_{N,k} (x) = \left(\begin{array}{c} N+k-1 \\ k \end{array} \right) \frac{x^k}{(1+x)^{N+k}}.$$

We give

Theorem 14. *Let $f \in C_+^n ([0,1])$, $x \in [0,1]$ fixed, such that $f^{(i)}(x) = 0$, $i = 1, ..., n$, $n \in \mathbb{N}$. Then*

$$(106) \quad \left| U_N^{(M)} (f) (x) - f(x) \right| \le \frac{\omega_1 \left(f^{(n)}, \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}} \right)^{\frac{1}{n+1}} \right)}{n!}$$

$$\times \left[\frac{2\sqrt{3} \left(\sqrt{2} + 2 \right)}{\sqrt{N+1}} + \frac{1}{(n+1)} \left(\frac{2\sqrt{3} \left(\sqrt{2} + 2 \right)}{\sqrt{N+1}} \right)^{\frac{n}{n+1}} \right],$$

$\forall \, N \in \mathbb{N} - \{1\}$.

Proof. Use of (16) for $\delta = \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}} \right)^{\frac{1}{n+1}}$, we use that (see [1])

$$(107) \qquad \left(U_N^{(M)} \left(|\cdot - x|^m \right) \right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2} + 2 \right)}{\sqrt{N+1}}, \;\; N \geq 2, \; N \in \mathbb{N}. \qquad \blacksquare$$

We make

Remark 21. Here we study Max-product Meyer-Köning and Zeller operators (see [3], p. 11) defined by

$$(108) \quad Z_N^{(M)} \left(f \right)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}\left(x \right) f\left(\frac{k}{N+k} \right)}{\bigvee_{k=0}^{\infty} s_{N,k}\left(x \right)}, \;\; \forall \; N \in \mathbb{N}, \; f \in C_+ \left([0,1] \right),$$

$s_{N,k}\left(x \right) = \begin{pmatrix} N + k \\ k \end{pmatrix} x^k$, $x \in [0,1]$.

From [1] we get

$$(109) \qquad Z_N^{(M)} \left(|\cdot - x|^m \right)(x) \leq \frac{8\left(1 + \sqrt{5} \right)}{3} \frac{\sqrt{x}\left(1 - x \right)}{\sqrt{N}} =: \rho\left(x \right),$$

$\forall \; x \in [0,1]$, $N \geq 4$, $\forall \; m \in \mathbb{N}$.

We finish with

Theorem 15. *Let* $f \in C_+^n \left([0,1] \right)$, $n \in \mathbb{N}$, $x \in [0,1]$, $f^{(i)}\left(x \right) = 0$, $i = 1, ..., n$. *Then*

$$(110) \qquad \left| Z_N^{(M)} \left(f \right)(x) - f\left(x \right) \right| \leq \frac{\omega_1 \left(f^{(n)}, \left(\rho\left(x \right) \right)^{\frac{1}{n+1}} \right)}{n!}$$

$$\times \left[\rho\left(x \right) + \frac{\left(\rho\left(x \right) \right)^{\frac{n}{n+1}}}{\left(n + 1 \right)} \right], \;\; \forall \; N \geq 4, \; N \in \mathbb{N}.$$

Proof. Use of (16) with $\delta = \left(\rho\left(x \right) \right)^{\frac{1}{n+1}}$. $\blacksquare$

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