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Orthogonal Polynomials Approximation and Balanced Truncation for a Lowpass Filter

K. Perev

Key Words: *Legendre; Laguerre and Chebyshev orthogonal polynomials; balanced truncation; reachability and observability gramians; lowpass analog filter.*

Abstract. *This paper considers the problem of orthogonal polynomial approximation based balanced truncation for a lowpass filter. The proposed method combines the system properties of balanced truncation, the computational effectiveness of proper orthogonal decomposition and the approximation capability of the orthogonal polynomials approximation. Orthogonal polynomials series expansion of the reachability and observability gramians is used in order to avoid solving large-scale Lyapunov equations and thus, significantly reducing the computational effort for obtaining the balancing transformation. The proposed method is applied for model reduction of a lowpass analog filter. Different sets of orthonormal functions are obtained from Legendre, Laguerre and Chebyshev orthogonal polynomials and the corresponding reduced order models are compared. The approximation precision is measured by the relative mean square error between the outputs of the full order model and the obtained reduced order models.*

1. Introduction

Dynamical system modeling is an irrevocable component of system analysis and design. There always exists a trade-off between the complexity of the model and the accuracy of the described relations. As higher is the requirement for model accuracy, as more are the differential and algebraic equations describing system dynamics. A main part of the analysis problem is to use the created model for simulating the behavior of the explored physical processes. When the complexity of the model increases significantly, the simulation of the full size model can often become an infeasible task. Such large-scale system models are for example VLSI electrical circuits, weather change prediction, air quality transformation and wave propagation evolution. Large dimensional system models are difficult to handle and require large computational resources. There emerges a need for reducing the size of the system description and therefore, for approximation of the existing physical relations. The basic motivation for system approximation comes from the demand for simplified models of dynamical systems that capture the main features of the original complex model. Approximation of dynamical systems and model order reduction in particular includes techniques such that the original model is replaced by a lower order one, while preserving the main features of the input/output behavior of the described phenomenon.

One of the main approaches for dynamical system approximation is by using orthogonal polynomial descriptions and is based on the fact that every continuous function defined on a bounded interval can be represented by orthogonal polynomial series with arbitrary degree of accuracy. Some of the most popular complete orthogonal sets are built from Laguerre, Legendre and Chebyshev polynomials. The wide application of Laguerre orthogonal polynomials for dynamical system approximation is due to the ability to work with Laguerre series both in time and frequency domain. In time domain the Laguerre polynomials form a complete set of orthogonal polynomials in the Hilbert space $L_2[0, \infty)$ with a certain exponential weighting function. In the frequency domain Laguerre polynomials are represented by lowpass filters consisting of a first-order lowpass filter and a higher order allpass factor. In [7] is proved the equivalence between the Laguerre based approximation method and the classical moment matching approach. The problem of selecting an optimal expansion point in the rational Krylov subspace reduction method is also discussed. The approximation of infinite dimensional transfer functions in terms of Laguerre series expansion is studied in [2]. The computational procedure is performed in Laplace domain and is based on minimization of a quadratic performance criterion. In [19] is presented an adaptive multiple model method for control of stochastic systems. The Laguerre polynomials are used in a procedure of system identification for approximation of the transfer function model. A common feature of the presented Laguerre polynomial series applications is the statement of the approximation problem in frequency domain, yet the choice of the expansion frequency point is not clearly justified. A particular drawback of the frequency domain methods is the requirement to work with complex valued functions, where the complexity of the computations is not suited for large-scale systems. A time domain model order reduction method based on Laguerre function expansion of the impulse response is presented in [6], where the state trajectory is projected onto a subspace spanned by part of the columns of the Fourier coefficients matrix. It is shown that the impulse response Fourier coefficients of the reduced order system match the first coefficients in the series representation of the original system. A simplified algorithm for balanced realization of Laguerre network models is presented in [20]. The proposed algorithm does not require the computation of the reachability and observability gramians and proceeds by forming a generalized Hankel matrix for constructing the balanced transformations. A specific feature of the Laguerre series approximation in time domain is the infinite time interval, where the basis functions are defined. Orthogonal series approximation on a bounded time interval is carried out by utilizing Legendre and Chebyshev polynomials. The Legendre polynomials constitute a complete orthogonal set on the Hilbert space $L_2[-1,1]$. There is no weighting function to support the inner product integral and therefore, Legendre polynomials satisfy the orthogonality polynomial condition on the definition interval. The application of Legendre polynomials for system analysis and identification is studied in [13]. The proposed method reduces the dynamical system problem to that of solving a system of algebraic equations, thus greatly simplifying the computational procedure. It is shown that the Legendre polynomials have a good rate of convergence and can be used to approximate the exact solution in various control problems. A numerical method for solving optimal control problems by approximating the system dynamics and performance index with Legendre orthogonal polynomials is presented in [8]. The Legendre polynomials are used to eliminate the integration in the performance index by expanding the state and input variables in orthogonal polynomial series. The proposed method avoids solving matrix Riccati equations or inverting ill-conditioned matrices. The major drawback of the proposed method is the dependence of the solution convergence rate on the shape of the input and state variables. [17] presents an efficient algorithm for Galerkin-Legendre approximation of the solution to second and fourth order elliptic differential equations. The basic idea of the proposed algorithm is to construct a polynomial basis for the elliptic differential equations with a relatively small number of unknown parameters. The main drawback of the Legendre approximation is the lack of a fast transform between the physical space and the spectral space [17]. Another popular complete orthogonal set in Hilbert spaces is generated from Chebyshev polynomials. The advantage of using Chebyshev polynomials is in the opportunity to solve the approximation problem in two different geometric measures. From one side is their orthogonality with a certain weighting function in the Hilbert space $L_2[-1,1]$ and therefore, the best approximation is achieved by minimizing a quadratic performance criterion. From the other side is their best approximation property with respect to the uniform norm in the real $C[-1,1]$ space. Chebyshev polynomials are widely used for obtaining numerical solutions of nonlinear Volterra and Volterra-Fredholm integral equations [5,10]. The application of Chebyshev polynomials for solving optimal control problems is presented in [12,14]. A continuous-time identification problem is solved in [9], where Chebyshev polynomials series approximation is used in combination with the instrumental variable method for compensation of the parameter estimation bias.

The techniques of model order reduction serve as the ground for another major attempt in dynamical system approximation. There exist two main approaches for model order reduction: the singular value decomposition (SVD) based approach and the Krylov subspace projection based approach. Two of the most popular methods from the SVD based approach for model order reduction are balanced truncation (BT) and proper orthogonal decomposition (POD) [3]. The main idea of balanced truncation is to construct a change of basis transformation in state space, where the reachability and observability gramians are equal diagonal matrices with diagonal elements, the Hankel singular values of the system [11]. The transformed states, which correspond to small Hankel singular values, are truncated from the state space description since they have small effect on the input/output behavior of the system. The method of balanced truncation preserves stability of the reduced order system and gives an upper bound on the error of approximation. However, it requires solving a large scale Lyapunov equations for computing the gramians, which often is computationally unfeasible task. The proper orthogonal decomposition method is based on discretization of system state trajectories called snapshots and their projection onto a lower dimensional subspace of the state space [18]. The main advantage of proper orthogonal decomposition is its computational effectiveness. However, proper orthogonal decomposition does not guarantee stability and is less accurate than the balanced truncation method.

The method proposed in this paper combines the procedures of balanced truncation with orthogonal polynomial approximation of system gramians for the purpose of model order reduction. The computational algorithm is based on collecting system trajectories data, where a data snapshots matrix is utilized to compute the Fourier coefficients vectors for a polynomial series approximation of the system state impulse response. The computational procedure is practical in sense that the trajectories data collection can be performed either from experiment or from simulations. The proposed method is applied for model order reduction of a lowpass analog filter. Different sets of orthogonal polynomials for computing the gramians are compared and the approximation error of the corresponding orthogonal representations is explored.

2. Orthogonal Polynomials Approximation

The complete set of orthogonal polynomials is widely used for approximation of continuous functions. Its application is determined from the fact that the set of polynomials with real coefficients is dense in the space of real valued continuous functions. Therefore, every continuous function f in a Hilbert vector space H can be uniquely represented on the interval $[a, b]$ as $f(t)=\sum_{n=0}^{\infty}c_n\varphi_n(t)$, where c_n, $n = 0,1,...$ are the Fourier coefficients of the function $f(t)$ with respect to the complete orthogonal set $\{\varphi_n\}_{n=0}^{\infty}$. Usually the orthogonal functions are normalized, such that the following orthonormal condition is satisfied:

$$\int_a^b \varphi_m(t)\varphi_n(t)dt = \begin{cases} 0 & for \quad m \neq n \\ 1 & for \quad m = n \end{cases}, \tag{1}$$

where a and b can be finite or infinite numbers. The Hilbert space considered in the derivations to follow is $L_2[a, b]$ and therefore, every function that is subjected to approximation $f \in L_2[a,b]$ satisfies the condition $\int_a^b f^2(t)dt < \infty$. The function f can be approximated in orthogonal series expansion as $f(t) \approx \sum_{n=0}^{N} c_n \varphi_n(t)$, where N is the order of truncation of the orthogonal series and the Fourier coefficients c_n, n = 1,2,3 ... are computed by the expression $c_n = \int_a^b f(t)\varphi_n(t)dt$. The error of approximationis ε_N determined from the expression [16]

(2) $$\varepsilon_N = \left[\int_a^b f^2(t)dt - \sum_{n=1}^{N} c_n^2\right]^{1/2}.$$

Remark. *The Weierstrass approximation theorem states that for every continuous functions f defined on a closed, bounded interval holds that* $\lim_{N\to\infty} \varepsilon_N(f) = 0$ [4]. *However, there are many functions which are not at all suited for approximation by a single polynomial in the entire interval of interest. Functions whose graphs have sharp rises surrounded by weakly curved stretches are one such example. Therefore, the size of the function derivative influences the error of approximation. For* $(n + 1)^{st}$ *order continuously differentiable functions, the following result holds for the approximation error* [4].

(3) *If* $\left|f^{(n+1)}(t)\right| \le M$ *for all* $t \in [a, b]$,

then $$\varepsilon_n(f) \le \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1} \text{ on } [a, b].$$

Laguerre polynomials form a complete orthogonal set in the Hilbert space $L_2[0, \infty)$ with weighting function $w(t) = e^{-t}$, $0 \le t < \infty$. The orthogonal set of Laguerre functions is obtained by applying the Gram-Schmidt orthogonalization process to the sequence of functions $\{e^{-t/2}, te^{-t/2}, t^2e^{-t/2}, ...\}$. The orthonormal Laguerre functions are defined by the following expression:

(4) $\varphi_n(t) = e^{-t/2} L_n(t)$,

where the Laguerre polynomial of order n is defined as [1]

(5) $$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n! t^k}{(k!)^2 (n-k)!}, \; n = 0,1,2,...$$

The Laguerre polynomials can also be obtained as a solution to the Laguerre differential equation

(6) $tL_n''(t) + (1-t)L_n'(t) + nL_n(t) = 0$,

or by applying the Rodrigues' formula

(7) $$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right), \; n = 0,1,2,...$$

The first several members of the Laguerre polynomials orthogonal set defined on the interval $[0, \infty)$ are obtained as follows:

$L_0(t) = 1$, $L_1(t) = 1 - t$, $L_2(t) = 1 - 2t + \frac{1}{2}t^2$,

$L_3(t) = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$, $L_4(t) = 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4$,

$L_5(t) = 1 - 5t + 5t^2 - \frac{5}{3}t^3 + \frac{5}{24}t^4 - \frac{1}{120}t^5$.

Figure 1 represents the Laguerre functions $\varphi_n(t)$ corresponding to the Laguerre polynomials presented above on a bounded interval $t \in [0,10]$.

Approximation of a function $f(t) \in L_2[0,\infty)$ in terms of Laguerre orthogonal series can be computed as

(8) $f(t) \approx \sum_{n=0}^{N} c_n \varphi_n(t)$, $t \in [0,\infty)$,

where $c_n = \int_0^\infty f(t)\varphi_n(t)dt$.

Laguerre functions have important application in frequency domain. Each Laguerre function is realizable as the unit impulse response of a linear time-invariant system [16]. The Laplace transform of the Laguerre function is given by the expression $\hat{\varphi}_n(s,p) = \int_0^\infty \varphi_n(t)e^{-st}dt$, n = 0,1,2,... and is known as the Laguerre filter $\hat{\varphi}_n(s,p) = \frac{\sqrt{2p}}{s+p}\frac{(s-p)^n}{(s+p)^n}$, $p > 0$, where the first term is a first order lowpass filter, the second term is a n^{th} order allpass factor and p is a parameter. The transfer function of every stable linear time-invariant system can be represented as a series of Laguerre filters as $G(s) = \sum_{n=0}^{\infty} c_n \hat{\varphi}(s,p)$.

Legendre polynomials form a complete orthogonal set in the Hilbert space $L_2[-1,1]$ with weighting function $w(t) = 1$, $t \in [-1,1]$. Therefore, the Legendre polynomials satisfy the polynomial orthogonality condition. The polynomials are obtained by applying the Gram-Schmidt orthogonalization procedure of the functions 1, t, t^2 ... on the definition interval $[-1,1]$. The n-th order Legendre polynomial is obtained by using the Rodrigues' formula as follows [1]:

(9) $$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2 - 1\right)^n, \; n = 1,2,...$$

The Legendre polynomials satisfy the recurrence relations

(10) $$P_{n+1}(t) = \frac{(2n+1)tP_n(t) - nP_{n-1}(t)}{n+1}, \; P_0(t) = 1,$$

$P_1(t) = t$, n = 1,2,...

and also the differential equations

(11) $$\frac{d}{dt}P_{n+1} - \frac{d}{dt}P_{n-1} = (2n+1)P_n, \; n = 2,3,...$$

The Legendre polynomials can be normalized in order to satisfy the orthonormal condition (1) with the orthonormal function $\varphi_n = \sqrt{\frac{2n+1}{2}}P_n$. The first several members of the Legendre polynomials orthogonal set are as follows:

$P_0(t) = 1,\ P_1(t) = t,\ P_2(t)=\frac{3}{2}\left(t^2-\frac{1}{3}\right),$

$P_3(t)=\frac{5}{2}\left(t^3-\frac{3}{5}t\right),\ P_4(t)=\frac{35}{8}\left(t^4-\frac{6}{7}t^2+\frac{3}{35}\right),$

$P_5(t)=\frac{63}{8}\left(t^5-\frac{10}{9}t^3+\frac{5}{21}t\right).$

Figure 2 shows the first five Legendre polynomials on the definition interval. Every function $f(t) \in L_2[-1,1]$ can be approximated in terms of Legendre polynomials as

(12) $f(t)\approx\sum_{n=1}^{N}c_n\sqrt{\frac{2n+1}{2}}P_n(t),$

where

$c_n=\sqrt{\frac{2n+1}{2}}\int_{-1}^{1}f(t)P_n(t)dt.$

The Legendre polynomials are defined on a bounded interval [–1,1]. However, in many problems the definition interval is bounded but with different boundary points. In such cases a polynomial rescaling and change of variables is necessary. For example, a complete set of Legendre orthogonal polynomials in the vector space $L_2[0,T]$ is obtained by introducing a change of variables as follows:

(13) $\varphi_n(\tau)=\sqrt{\frac{2n+1}{2}}P_n(\tau),$

where

$\tau=\frac{2}{T}t-1, \quad t \in [0, T].$

In such cases, expressions (12) take on the following representation: $f(t)\approx\sum_{n=1}^{N}c_n\sqrt{\frac{2n+1}{2}}P_n\left(\frac{2}{T}t-1\right),\ t \in [0, T],$

where

(14) $c_n=\sqrt{\frac{2n+1}{2}}\int_{0}^{T}f(t)P_n\left(\frac{2}{T}t-1\right)dt.$

Another set of orthogonal polynomials defined on the same bounded interval [–1,1] are Chebyshev polynomials. Chebyshev polynomials solve the approximation problem in two different geometric measures. From one side Chebyshev polynomials form a complete orthogonal set in the Hilbert space $L_2[-1,1]$ with a certain weighting function on the definition interval. From the other side Chebyshev polynomials yield the best approximation to elements of the real $C[-1,1]$ space with respect to the uniform norm. This problem is a minimax problem, since the obtained solution achieves the minimum value of the maximum deviation between the given function and the approximating polynomial [4]. Chebyshev polynomials of first kind have the following representation:

(15) $T_n(t) = \cos n\theta,\ \theta = \arccos(t),\ t \in [-1,1],$
$n = 0,1,2,...$

Chebyshev polynomials of second kind are given by the expressions

(16) $U_n(t)=\frac{[\sin(n+1)\theta]}{\sqrt{1-t^2}},\theta = \arccos(t),\ t \in [-1,1],$
$n = 0,1,2,...$

Chebyshev polynomial of first kind of order n can be obtained by the formula

(17) $T_n(t)=\frac{n}{2}\sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k-1)!}{k!(n-2k)!}(2t)^{n-2k},\ t \in [-1,1],$

where

$[n/2] = n/2$ for even n and $[n/2] = (n-1)/2$ for odd n. Expression (17) determines the following recursion formula:

(18) $T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t),\ T_0(t) = 1,\ T_1(t) = t,$
$n = 1,2,3,...$

The following relationships hold between the two kinds of Chebyshev polynomials [1]:

(19) $T_n(t) = U_n(t) - t\,U_{n-1}(t),$
$(1-t^2)U_{n-1}(t) = tT_n(t) - T_{n+1}(t),\ n = 1,2,3,...$

(20) $T_n(t) = t\,U_{n-1}(t) - U_{n-2}(t),$
$U_n(t) = 2tT_n(t) + U_{n-2}(t),\ n = 2,3,4,...$

These formulas give the following first several Chebyshev polynomials:

$T_0(t) = 1,\ T_1(t) = t,\ T_2(t) = 2t^2 - 1,$
$T_3(t) = 4t^3 - 3t,\ \ T_4(t) = 8t^4 - 8t^2 + 1,$
$T_5(t) = 16t^5 - 20t^3 + 5t,$
$T_6(t) = 32t^6 - 48t^4 + 18t^2 - 1,$
$T_7(t) = 64t^7 - 112t^5 + 56t^3 - 7t,$
$U_0(t) = 1,\ U_1(t) = 2t,\ U_2(t) = 4t^2 - 1,$
$U_3(t) = 8t^3 - 4t,\ \ U_4(t) = 16t^4 - 12t^2 + 1,$
$U_5(t) = 32t^5 - 32t^3 + 6t,$
$U_6(t) = 64t^6 - 80t^4 + 24t^2 - 1,$
$U_7(t) = 128t^7 - 192t^5 + 80t^3 - 8t.$

Figures 3 and *4* represent Chebyshev polynomials of first and second kind on [– 1,1].

The leading coefficient for Chebyshev polynomials of first kind is 2^{n-1} for $n \geq 1$ and 1 for $n = 0$. The leading coefficient for Chebyshev polynomials of second kind is 2^n. The Chebyshev polynomials possess the symmetry property: $T_n(-t) = (-1)^n T_n(t)$ and $U_n(-t) = (-1)^n U_n(t)$. The Rodrigues' formula for Chebyshev polynomials of first kind is presented as

(21) $T_n(t)=\frac{(-2)^n}{(2n)!}\sqrt{1-t^2}\frac{d^n}{dt^n}(1-t^2)^{n-0.5},\ t \in [-1,1].$

The differential equations governing Chebyshev polynomials are

(22) $(1-t^2)T_n''(t)-tT_n'(t)+n^2T_n(t)=0,\ t \in [-1,1],$

(23) $(1-t^2)U_n''(t)-3tU_n'(t)+n(n+2)U_n(t)=0,\ t \in [-1,1].$

Chebyshev polynomials of first kind are orthogonal with weighting function $w(t)=\frac{1}{\sqrt{1-t^2}}$ and Chebyshev polynomials of second kind are orthogonal with weighting function $w(t)=\sqrt{1-t^2}$. The continuous orthogonality condition for Chebyshev polynomials of first kind is given by the expression

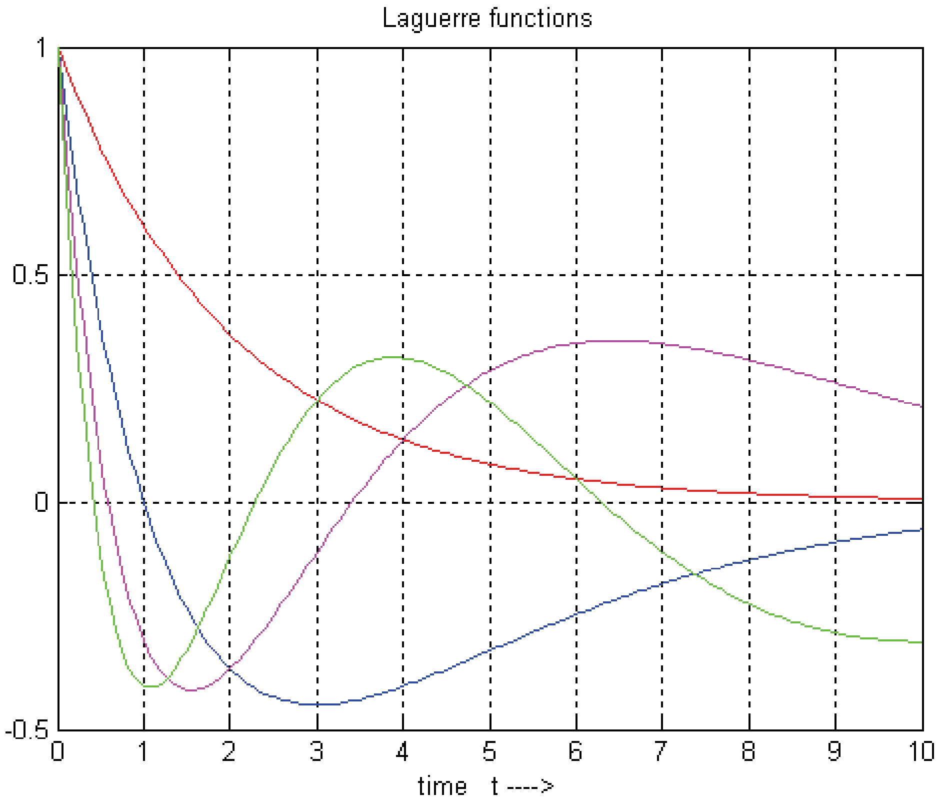

Figure 1. Laguerre functions $\varphi_0(t) \div \varphi_3(t)$

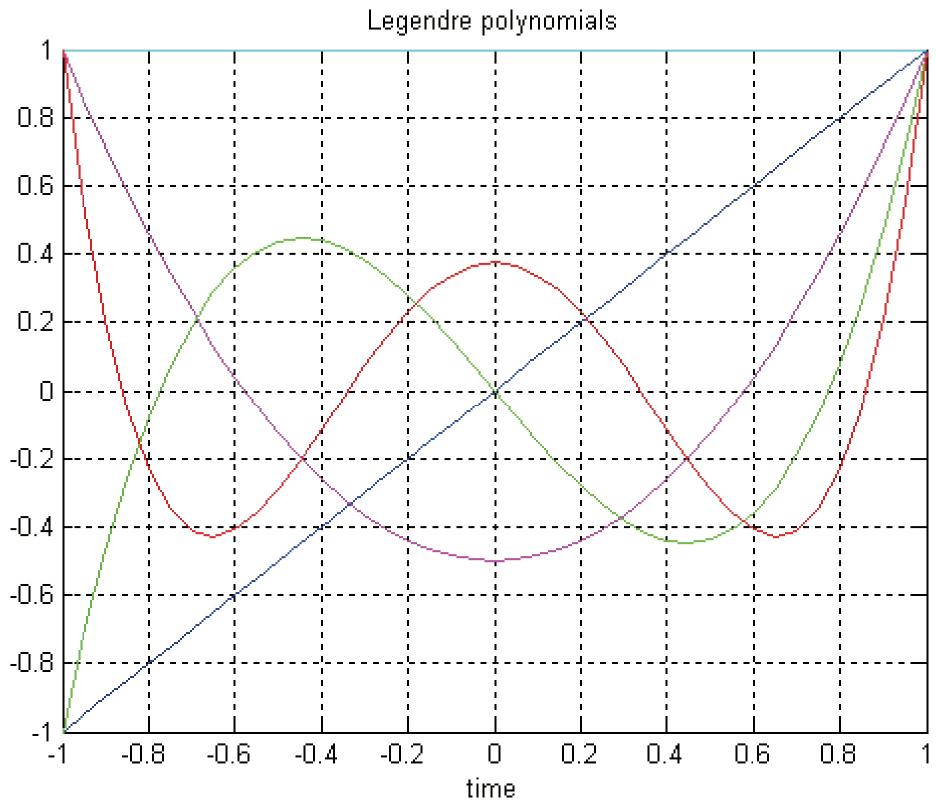

Figure 2. Legendre polynomials $P_0(t) \div P_4(t)$

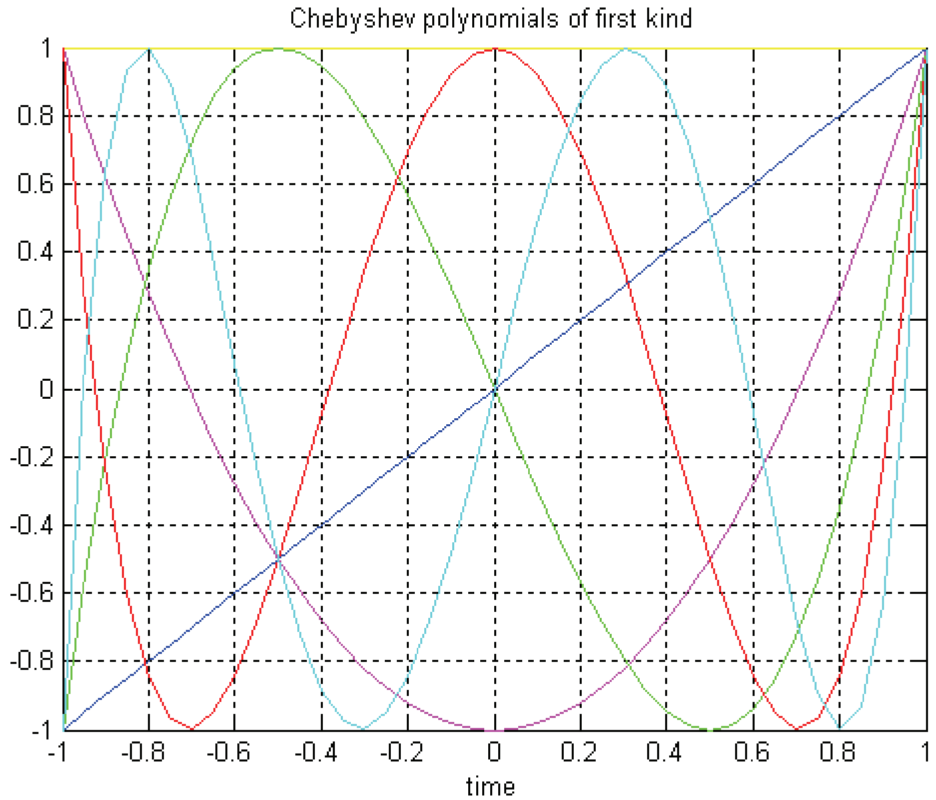

Figure 3. Chebyshev polynomials I $T_0(t) \div T_5(t)$

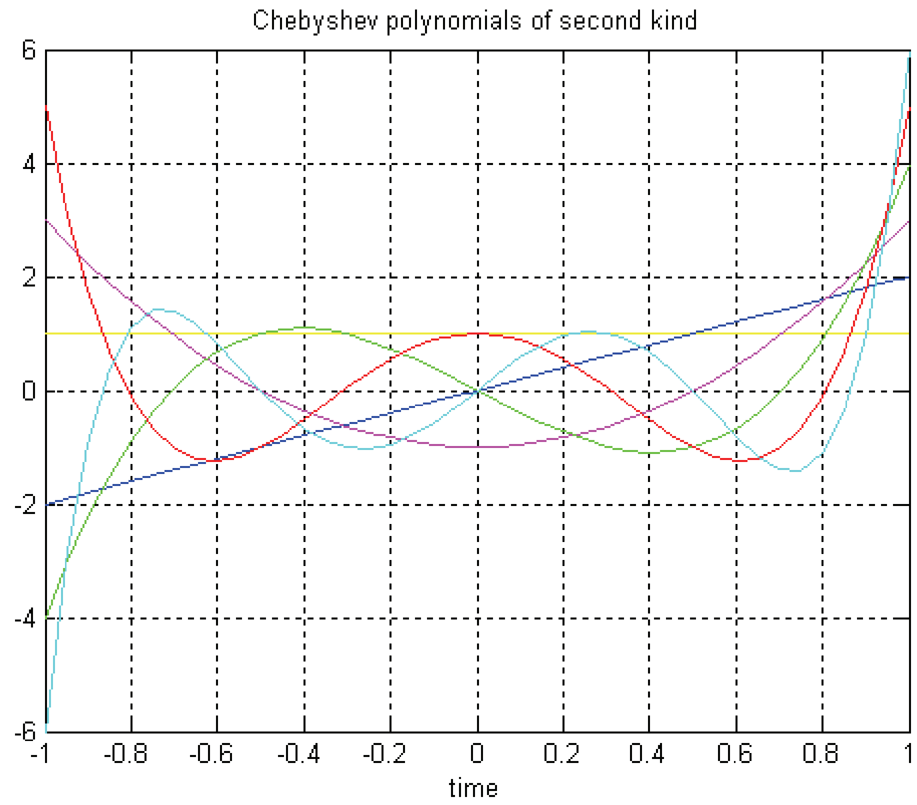

Figure 4. Chebyshev polynomials II $U_0(t) \div U_5(t)$

$$\left(T_p,T_q\right)=\int_{-1}^{1}\frac{T_p(t)T_q(t)}{\sqrt{1-t^2}}dt=\begin{cases}0 & for \quad p\neq q\\ \dfrac{\pi}{2} & for \quad p=q\neq 0\\ \pi & for \quad p=q=0\end{cases}.$$

A complete set of orthonormal functions in the Hilbert space $L_2[-1,1]$ can be constructed from Chebyshev polynomials of first kind and is defined as

(24) $\varphi_n(t)=\sqrt{\frac{2}{\pi}}\frac{T_n(t)}{\left(1-t^2\right)^{1/4}}$, n = 1,2,3,...

and $\varphi_0(t)=\sqrt{\frac{1}{\pi}}\frac{1}{\left(1-t^2\right)^{1/4}}$, $t \in [-1,1]$.

Similarly, the continuous orthogonality condition for Chebyshev polynomials of second kind is given as

$$\left(U_p,U_q\right)=\int_{-1}^{1}U_p(t)U_q(t)\sqrt{1-t^2}\,dt=\begin{cases}0 & for \quad p\neq q\\ \dfrac{\pi}{2} & for \quad p=q\end{cases}$$

A complete set of orthonormal functions in the Hilbert space L_2 [−1,1] can be obtained from Chebyshev polynomials of second kind as follows:

(25) $\varphi_n(t)=\sqrt{\frac{2}{\pi}}\left(1-t^2\right)^{1/4}U_n(t)$, $t \in [-1,1]$, n = 0,1,2,...

Every function $f(t) \in L_2$ [−1,1] can be approximated in terms of Chebyshev polynomials of first or second kind as

(26) $f(t)\approx\sum_{n=0}^{N}c_n\varphi_n(t),\quad c_n=\int_{-1}^{1}f(t)\varphi_n(t)dt$, n = 0,1,2,...

where N is the order of truncation of the series. When the interval of approximation is different from [−1,1], a change of variables becomes necessary and the so called shifted Chebyshev polynomials are used [9]. For example on the interval [0, T], the complete orthonormal set of Chebyshev functions in the Hilbert space L_2 [0, T] are defined as follows:

(27) $\varphi_n(\tau)=\sqrt{\frac{2}{\pi}}\frac{T_n(\tau)}{\left(1-\tau^2\right)^{1/4}}$ or $\varphi_n(\tau)=\sqrt{\frac{2}{\pi}}\left(1-\tau^2\right)^{1/4}U_n(\tau)$,

$\tau=\frac{2}{T}t-1,\quad t\in[0,T]$.

The Fourier coefficients for the Chebyshev series expansion on the interval [0, T] are computed from the expression

(28) $c_n=\frac{2}{T}\int_0^T f(t)\varphi_n\left(\frac{2}{T}t-1\right)dt$, n = 0,1,2,...

3. Orthogonal Polynomials Based Balanced Truncation

Consider the stable, linear time-invariant dynamical system

(29.1) $\sum: \dot{x}(t)=Ax(t)+Bu(t)$, $t \geq 0$,

(29.2) $y(t) = Cx(t)$, $x(0) = x_0$,

where $x(t) \in R^n$, $u(t) \in R$ and $y(t) \in R$. The reachability gramian on the interval [0, T] is given by the expression

(30) $W_r(0,T)=\int_0^T e^{At}BB^Te^{A^Tt}dt$.

The observability gramian on the interval is defined as follows:

(31) $W_o(0,T)=\int_0^T e^{A^Tt}C^TCe^{At}dt$.

Assume that the initial condition x_0 = 0 and that a delta impulse is applied at the input, i.e. $u(t) = \delta(t)$. The obtained state impulse response is given by the expression $x(t) = e^{At}B$. The reachability gramian (30) on the interval [0, T] can be written in the form $W_r(0,T)=\int_0^T x(t)x^T(t)dt$. Assume that the state impulse response is represented by the corresponding orthogonal series approximation $x(t)=\sum_{k=0}^{N}f_k\varphi_k\left(\frac{2}{T}t-1\right)$, where $\left\{\varphi_k\left(\frac{2}{T}t-1\right)\right\}_{k=0}^{N}$ is Legendre or Chebyshev set of orthonormal functions defined on the finite interval [0, T], f_k, k = 1.2,...,N, are the Fourier coefficients and N is the order of truncation of the orthogonal series. Satisfying the orthonormal condition (1), the reachability gramian can be approximated by the expression

(32) $W_r(0,T)\approx\frac{T}{2}\int_0^T\sum_{k=0}^{N}f_k\varphi_k\left(\frac{2}{T}t-1\right)\sum_{k=0}^{N}f_k^T\varphi_k\left(\frac{2}{T}t-1\right)dt=\frac{T}{2}\sum_{k=0}^{N}f_kf_k^T$,

where the coefficient $\frac{T}{2}$ is due to the change of variables in integration. The Fourier coefficients f_k, k = 1.2,...,N, can be computed from the snapshots data matrix of system trajectories.

(33) $X=\left[x_1\quad x_2\quad\cdots\quad x_L\right]=\begin{bmatrix}x_{11} & x_{12} & \cdots & x_{1L}\\ x_{21} & x_{22} & \cdots & x_{2L}\\ \vdots & \vdots & \ddots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nL}\end{bmatrix}$, $x_j=x(t_j)$,

j = 1,2,...,L

where L is the snapshots number ($\Delta t \cdot L = T$) and the state response signal can be obtained either by experiment or by simulation. Corresponding to the state trajectory snapshots, we obtain the orthogonal functions snapshots in a matrix form

(34) $\Phi=\begin{bmatrix}\varphi_{01} & \varphi_{02} & \cdots & \varphi_{0L}\\ \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1L}\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_{N,1} & \varphi_{N,2} & \cdots & \varphi_{N,L}\end{bmatrix}$, $\varphi_{kj} = \varphi_k(t_j)$, j = 1,2, ...,L,

k = 0,1, ..., N

The Fourier coefficients vectors can be calculated as $f_k=\frac{2}{T}\sum_{j=1}^{L}\varphi_{kj}x_j\Delta t$, k = 0,1,...,N. In order to compute the observability gramian, we consider the adjoint (dual) system [3]

Σ^*: $\dot{p}(t)=-A^*p(t)-C^*\tilde{y}(t)$,

$\tilde{u}(t) = B^*p(t)$,

where the star superscript denotes the adjoint operator, p denotes the state, and $\tilde{y}$ and $\tilde{u}$ are the input and the output of the adjoint system, respectively. The time of the adjoint system is running backwards and a change of variables

$\tau = -t$ is appropriate. Since the system matrices are with real elements we can write the following equations:

(35.1) Σ^*: $\dot{p}(\tau) = A^T p(\tau) + C^T \tilde{y}(\tau)$,

(35.2) $\tilde{u}(\tau) = B^T p(\tau)$.

If the system Σ is reachable, then the adjoint system Σ* is observable and vice versa. Furthermore, the reachability gramian of the adjoint system is the same as the observability gramian of the original system. Therefore, in order to obtain the observability gramian of Σ we need to compute the reachability gramian of Σ*. The state impulse response of the adjoint system is given by the expression $p(t) = e^{A^T t} C^T$. The observability gramian (31) on the interval [0, *T*] can be written in the form $W_o(0,T) = \int_0^T e^{A^T t} C^T C e^{At} dt = \int_0^T p(t) p^T(t) dt$. Similarly to the original system (29), we apply orthogonal series approximation on the state trajectory of the adjoint system (35) $p(t) = \sum_{k=0}^{N} g_k \varphi_k \left(\frac{2}{T} t - 1\right)$, where $\left\{\varphi_k\left(\frac{2}{T}t-1\right)\right\}_{k=0}^{N}$ is Legendre or Chebyshev set of orthogonal functions defined on the finite interval [0, *T*], g_k, $k = 1,2,...,N$ are the corresponding Fourier coefficients and *N* is the order of truncation of the orthogonal series. The observability gramian of system (29) is approximated as:

(36) $W_o(0,T) \approx \frac{T}{2} \int_0^T \sum_{k=0}^{N} g_k \varphi_k \left(\frac{2}{T}t-1\right) \sum_{k=0}^{N} g_k^T \varphi_k \left(\frac{2}{T}t-1\right) dt = \frac{T}{2} \sum_{k=0}^{N} g_k g_k^T$.

We can represent the snapshots data matrix for the state trajectory of the adjoint system as

(37) $P = [p_1 \ p_2 \ \cdots \ p_L] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1L} \\ p_{21} & p_{22} & \cdots & p_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nL} \end{bmatrix}$, $p_j = p(t_j)$,

$j = 1,2,...,L$

where *L* is the number of data snapshots. Using the orthogonal functions snapshots we calculate the Fourier coefficients for the observability gramian as $g_k = \frac{2}{T} \sum_{j=1}^{L} \varphi_{kj} p_j \Delta t$, $k = 0,1,...,N$. If the system (29) is stable, the reachability and observability gramians at infinity $W_r = \lim_{T\to\infty} W_r(0,T)$ and $W_o = \lim_{T\to\infty} W_o(0,T)$ exist and are the unique solutions of the Lyapunov equations

(38) $AW_r + W_r A^T + BB^T = 0$,

(39) $A^T W_0 + W_0 A + C^T C = 0$.

The major difficulty for performing balancing transformations and balanced model order reduction of linear dynamical systems is computation of the gramians at infinity W_r and W_0 that can be done by solving the equations of Lyapunov (38) and (39). Since approximation of the gramians at infinity by orthogonal series avoids solving the equations of Lyapunov, it reduces the computational burden for model reduction of large-scale systems. To approximate the gramians at infinity is convenient to choose Laguerre orthogonal series, since the Laguerre functions are orthogonal on the infinite interval [0, ∞). The advantage of using Laguerre orthogonal series is that, there is no need to change the definition interval for the orthogonal functions and therefore to change the variable of integration. The smallest amount of energy needed to move the system from zero to the state *x* is given by the quantity $E_r = x^T W_r^{-1} x$, while the energy obtained by observing the output of the system with initial condition *x* and no input function is given as $E_o = x^T W_o x$. Therefore, the reachabiliy and observability gramians play an essential role in forming the input/output behavior of the system. One way to reduce the number of states is to eliminate those which require a large amount of input energy E_r to be reached and yield small amount of observation energy E_o at the output. The transformation, which changes the basis of system description such that the reachability and observability gramians are equal diagonal matrices with the Hankel singuar values on the diagonal is called a balancing transformation. The method related to reducing these system states, which correspond to small Hankel singular values and therefore, have small contribution to the input/output behavior is called balanced truncation. Approximation by balanced truncation preserves stability and the H_∞ norm of the error between the original and the truncated system is given by the expression [3]

(40) $\left\|\Sigma - \tilde{\Sigma}\right\|_\infty \le 2(\sigma_{k+1} + \ldots + \sigma_n)$.

A main algorithm for balanced truncation is the square root algorithm. The square root algorithm consists of the following steps: *i)* perform a Cholesky decomposition on the reachability gramian $W_r = UU^T$, *ii)* perform a Cholesky decomposition on the observability gramian $W_o = LL^T$, *iii)* perform singular value decomposition on the product of Cholesky factors $U^T L = W \Sigma V^T$, *iv)* compute the similarity transformation matrices for change of basis $P = \Sigma^{-1/2} V^T L^T$ and $P^{-1} = U W \Sigma^{-1/2}$, *v)* apply the similarity transformation to system matrices and obtain the system model into a balanced form $A_b = PAP^{-1}$, $B_b = PB$ and $C_b = CP^{-1}$.

4. Model Order Reduction for a Lowpass Filter

Consider an analog lowpass filter obtained by a cascade connection of *RLC* circuits (*figure 5*). Assume that all elements of the circuit have the same values of the inductances, capacitances and resistances $L = 100H$, $R = 5{*}10^3\ \Omega$ and $C = 5{*}10^{-5} F$.

If we ignore the influence of the input and output impedances of the *RLC* elements, the transfer function of a cascade connection of *N* such elements is the product of the corresponding transfer functions, i.e. $G(s) = G_1(s)G_2(s) \ldots G_N(s)$, where

(41) $G_i(s) = \frac{U_{i,изх}(s)}{U_{i,вх}(s)} = \frac{1}{LCs^2 + RCs + 1}$, $i = 1,2,...,N$

is the transfer function of the ith element and represents the ratio between the corresponding output and input voltages.

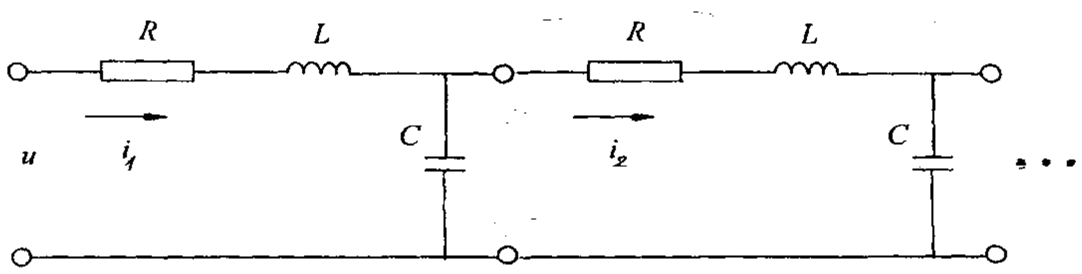

Figure 5. Analog low pass filter built by a cascade connection of *RLC* elements

The state space model of this element is given by the following equations:

(42.1) $\dot{x}_1(t) = \frac{1}{C} x_2(t)$,

(42.2) $\dot{x}_2(t) = -\frac{1}{L} x_1(t) - \frac{R}{L} x_2(t) + \frac{1}{L} u(t)$,

(42.3) $y(t) = x_1(t)$,

where $x_1(t) = u_C(t)$ is the voltage across the capacitor and $x_2(t) = i_L(t)$ is the current through the inductor for each element. The analog filter model for two cascade connected elements in state space $G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ is represented as [15]

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right].$$

Consider a lowpass filter built from three cascade connected *RLC* circuits, which determines the order of the filter to be $n = 6$.

Figure 6 and *figure 7* show the state impulse responses $x_1(t)$ and $x_2(t)$ for the analog filter model of full order $n = 6$. The other state impulse responses of the filter are similar to the presented responses and are typical lowpass filter characteristics. The state impulse responses constitute the input data for performing the orthogonal

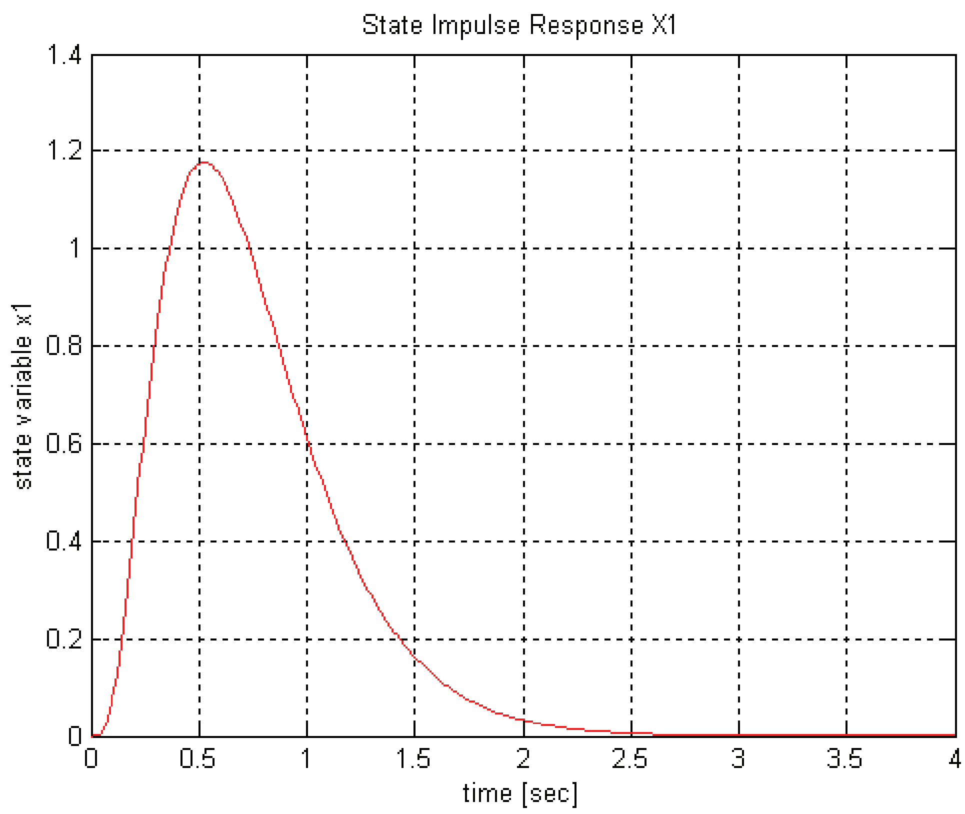

Figure 6. State impulse response x1 of the filter

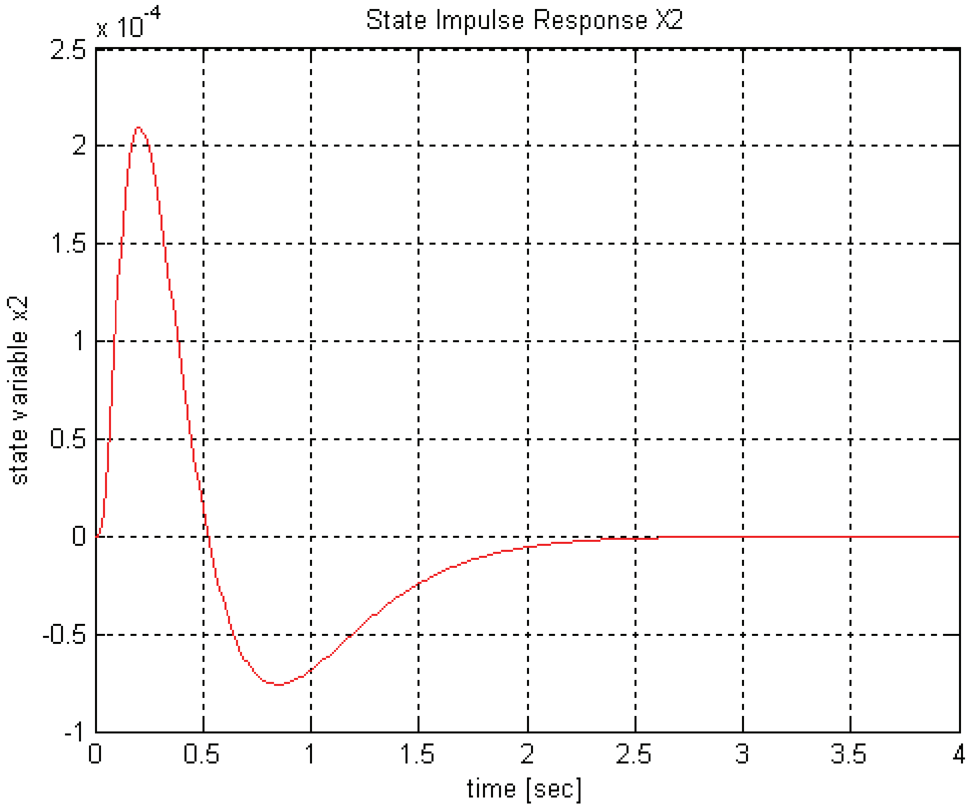

Figure 7. State impulse response x2 of the filter

polynomials approximation based balanced truncation procedure for model order reduction of the lowpass filter. The Hankel singular values of the full order lowpass filter are computed as follows:

$$\Sigma = \begin{bmatrix} 0.687 & 0.216 & 0.299\cdot 10^{-1} & 0.134\cdot 10^{-2} & 0.578\cdot 10^{-4} & 0.149\cdot 10^{-5} \end{bmatrix}$$

The system impulse responses are simulated on the time interval [0, T], T = 4 sec with discretization step Δ = 0.01 sec. The order of series expansion for all cases is N = 12.

Figure 8 shows the unit step responses of the full order model and the reduced third order models obtained by Legendre, Laguerre and Chebyshev of first and second kind approximations. It is clearly seen that all step responses almost coincide. The relative mean quadratic error of approximation between the step responses of the full order model and the reduced third order models is computed as follows:

$$\varepsilon_{Ldr} = \frac{\|y - y_{Ldr}\|_2}{\|y\|_2} = 0.00246, \quad \varepsilon_{Lgr} = \frac{\|y - y_{Lgr}\|_2}{\|y\|_2} = 0.000482,$$

$$\varepsilon_{ChI} = \frac{\|y - y_{ChI}\|_2}{\|y\|_2} = 0.00181, \quad \varepsilon_{ChII} = \frac{\|y - y_{ChII}\|_2}{\|y\|_2} = 0.00179$$

From the computed relative error norms between the step responses of the full order model and the reduced third order models is clear that all third order approximations are very close to each other and the error of approximation is quite small. This fact can be explained with the small values of the Hankel singular values corresponding to the truncated states of the balanced system. This conclusion can be confirmed from the logarithmic magnitude responses shown on *figure 9*. It is apparent that the magnitude responses of the approximated systems are closely related for all four polynomial types. From the figure can also be seen that the reduced order systems have magnitude responses, which deviate from the response of the full order model only in the high frequency range. Therefore, the presented method of approximation can be successfully applied in the case of slowly changing signal characteristics.

Next, we explore the approximating properties of the model reduction method for different orthogonal polynomials. Consider the Legendre polynomial approximation based balanced truncation of the analog filter model consisting of a cascade connection of four *RLC* circuits and thus, presenting a system of order n = 8. The Hankel singular values of the full order filter model are computed as follows:

$$\Sigma = \begin{bmatrix} 0.724 & 0.275 & 0.563\cdot 10^{-1} & 0.605\cdot 10^{-2} & 0.256\cdot 10^{-3} & 0.113\cdot 10^{-4} & 0.411\cdot 10^{-6} & 0.81\cdot 10^{-8} \end{bmatrix}$$

We apply Legendre series approximation to system state impulse responses with order of series approximation N = 12.

Figure 10 presents the unit step responses of the full order model and the reduced fourth and second order models. As can be seen from the figure, the unit step responses of the full order and the reduced fourth order models are closely related. The difference appears in the step response of the reduced second order model and it is due to the relatively large Hankel singular value for the truncated third state variable. The figure relations can be quantified by computing the relative mean quadratic error between the outputs of the full order and reduced order models.

$$\varepsilon_{Ldr,4} = \frac{\|y - y_4\|_2}{\|y\|_2} = 2.483 \cdot 10^{-4} \quad \text{and} \quad \varepsilon_{Ldr,2} = \frac{\|y - y_2\|_2}{\|y\|_2} = 0.051 .$$

Similar information can be obtained from the logarithmic magnitude responses shown on *figure 11*. It is observed that the logarithmic magnitude response of the second order approximation deviates from the response of the full order model in the low frequency range and this is the reason for the larger approximation error in the step responses on *figure 10*. The magnitude response of the fourth order approximation follows closely the response of the full order model and demonstrates good approximation capability of the reduced order model in the low frequency range.

Figure 12 shows the step responses of the full order model and the reduced fourth and second order models when the approximation is performed by using Chebyshev polynomials of first kind.

The step response of the reduced fourth order model obtained by Chebyshev series approximation is closely related to the step response of the full order model. The step response of the reduced second order approximation clearly deviates from the step response of the original filter model. The relative mean square error quantifies the observed differences between the responses

$$\varepsilon_{ChbI,4} = \frac{\|y - y_4\|_2}{\|y\|_2} = 2.004 \cdot 10^{-4} \quad \text{and} \quad \varepsilon_{ChbI,2} = \frac{\|y - y_2\|_2}{\|y\|_2} = 0.055 .$$

The corresponding logarithmic magnitude responses are shown on *figure 13*. The observation made is that the reduced second order characteristic is much different from the characteristics of the full order and reduced fourth order models. While the magnitude response of the fourth order model deviates from the full order model only in the high frequency range, the second order model differs from the full order model in the area of low frequencies as well. The performed numerical experiments show that the observed orthogonal polynomials approximations are very alike as concerned reducing the order of the system model. The step responses and logarithmic magnitude responses are closely related for the same order of polynomial series approximation for the explored four orthogonal polynomial sets. The order of model reduction is clearly determined from the magnitude of the truncated Hankel singular values. As larger are these values, as greater is the deviation of the system characteristics from the corresponding ones of the full order model.

5. Conclusion

This paper considers the problem of model order reduction by applying a polynomial series approximation based balanced truncation method. The proposed method combines the system properties of balanced truncation and the computational effectiveness of proper orthogonal decomposition with the approximation power of orthogonal polynomials. Four types of orthogonal polynomial sets are used to build a complete set of orthonormal functions for approximating certain system characteristics. The main result of this paper is based on the orthogonal series representation of the state impulse responses of the linear system, as the reachability and observability gramian approximations are computed by using the Fourier coefficients vectors from the orthogonal series expansion. In this way the procedure for solving large-scale Lyapunov equations is avoided and thus, the computational effort for obtaining the gramians is largely reduced. Further, the balanced truncation algorithm for model order reduction is applied. Different experiments are performed for exploring the approximation properties of the presented method. It is confirmed that the approximation capability of the proposed method depends on the size of the truncated Hankel singular values. It is also shown that all four orthogonal polynomials possess almost identical precision in approximating the dynamical system. The relative mean quadratic error is computed for all four cases as a quantitative measure for the approximation power of the proposed method.

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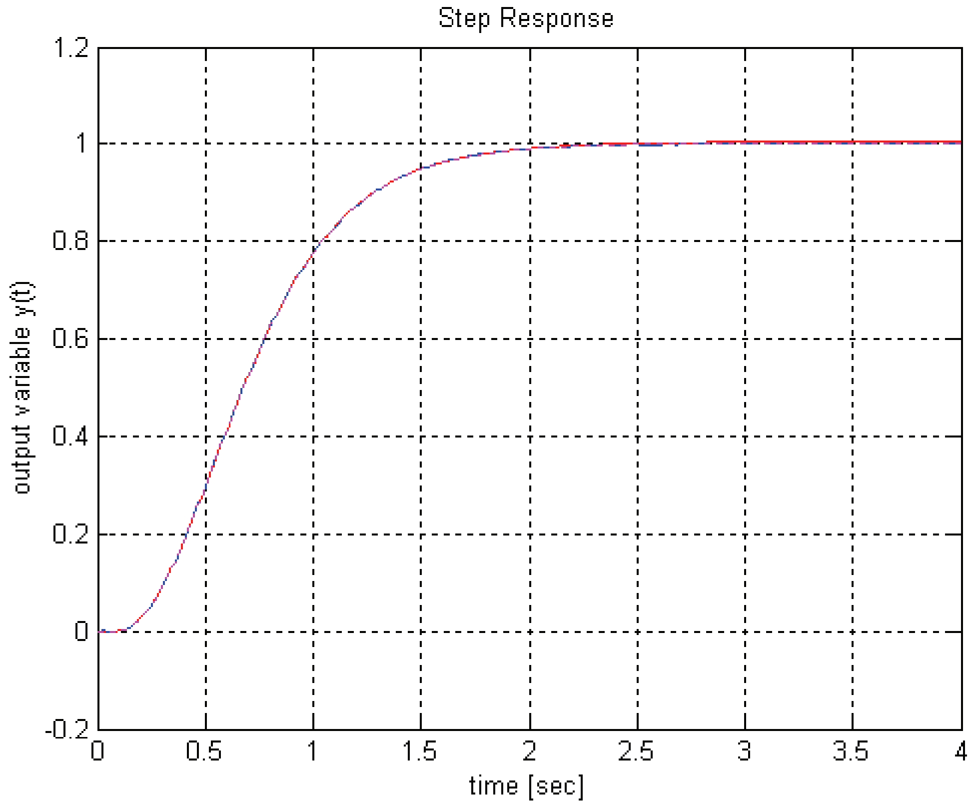

Figure 8. Step response of the full order model ---, Legendre -.-, Laguerre ..., Chebyshev I, II ....

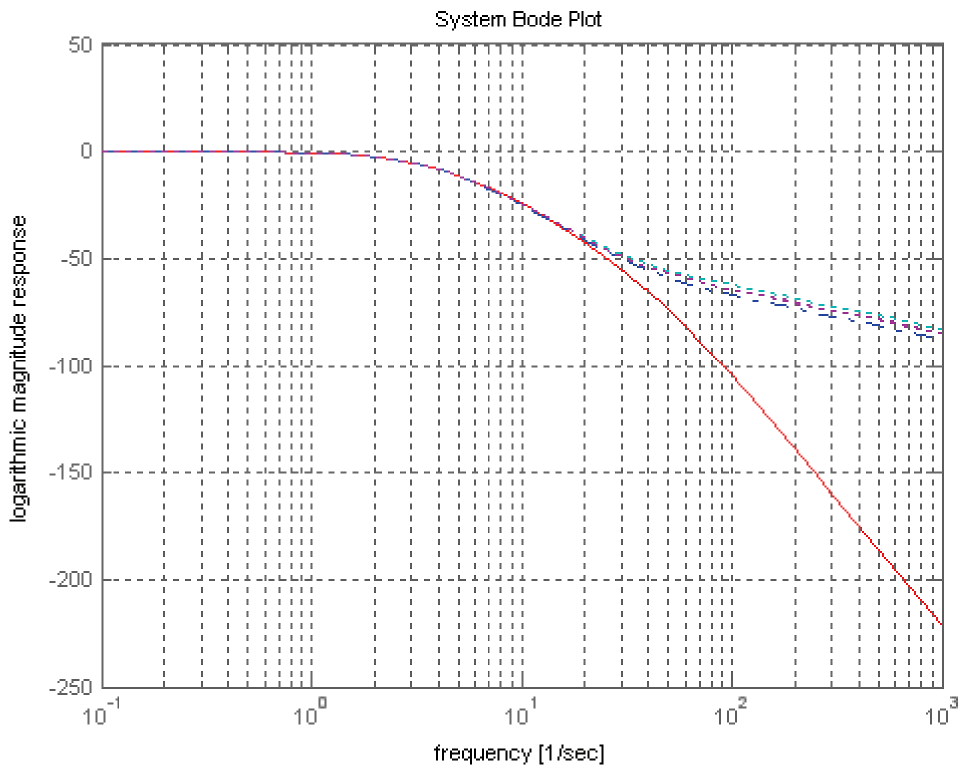

Figure 9. Logarithmic magnitude response full order ---, Legendre -.-, Laguerre ..., Chebyshev I, II ....

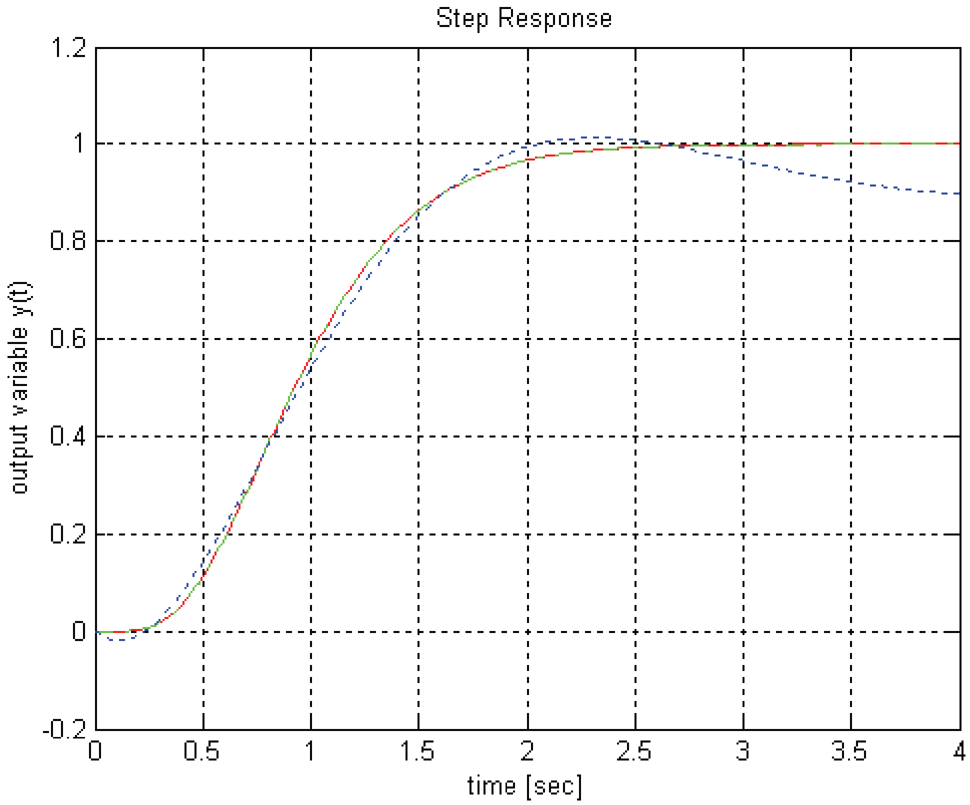

Figure 10. Step responses of full order model ---, Legendre 4th order -.-, and 2nd order.....

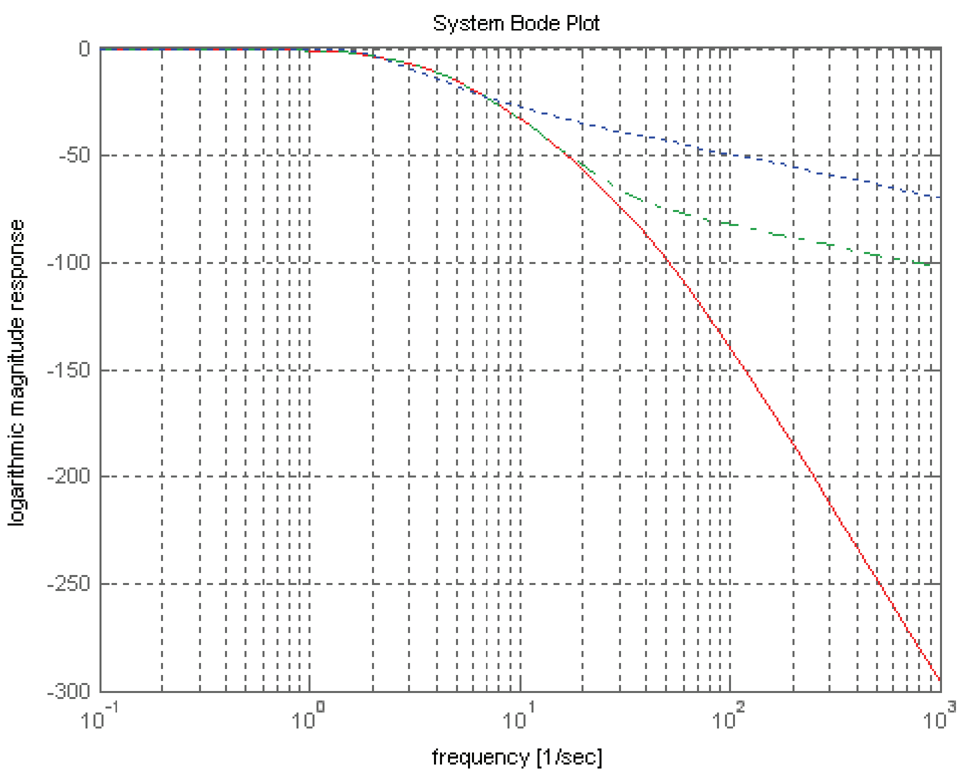

Figure 11. Logarithmic magnitude response full order model ---, Legendre 4th order -.-, and 2nd order .....

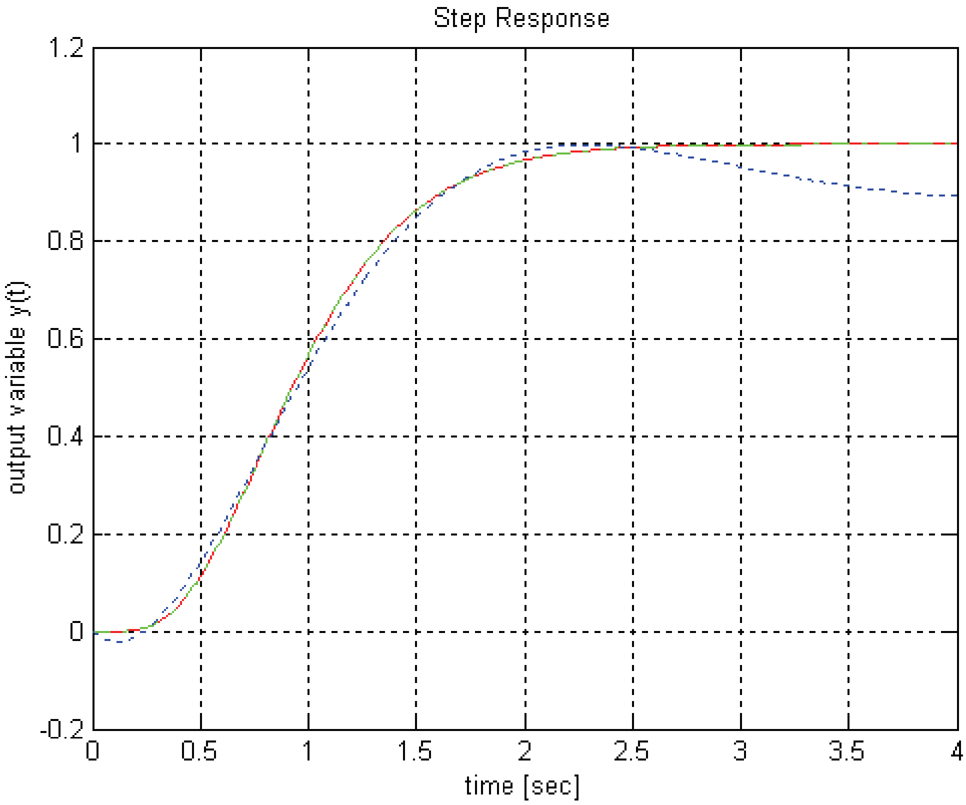

Figure 12. Step responses of full order model ---, Chebyshev I 4th order -.-, and 2nd order .....

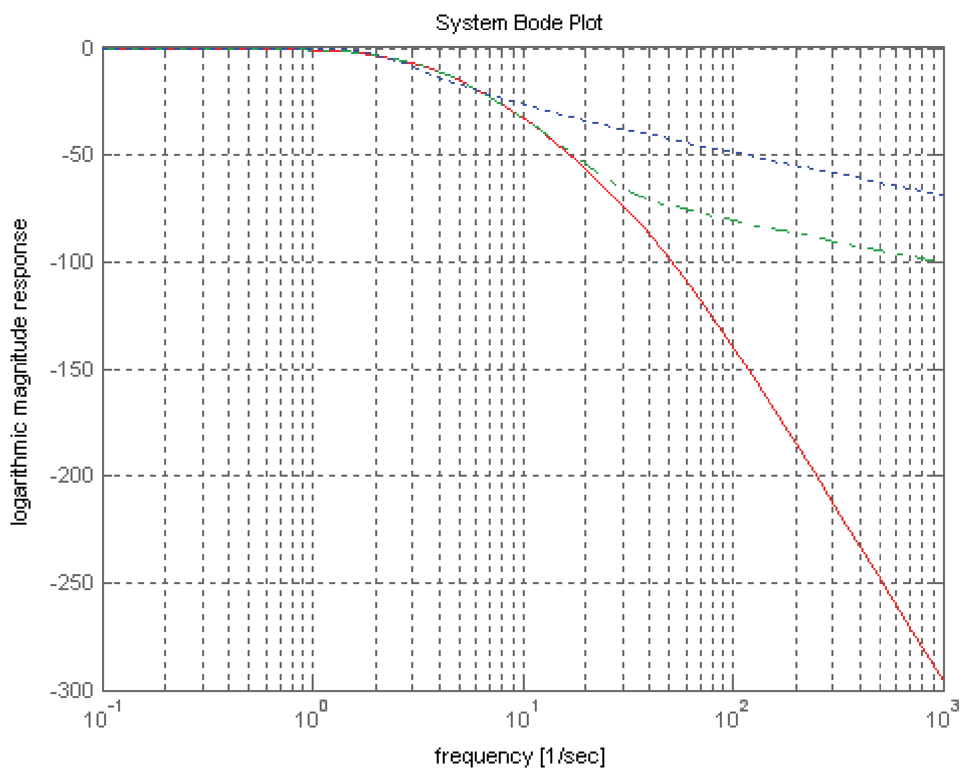

Figure 13. Logarithmic magnitude response full order model ---, Chebyshev I 4th order-.-, and 2nd order .....

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Assoc. Prof. ***Kamen Perev*** *received the Engineering degree in Automatics from VMEI – Sofia in 1985. He received the M.S. degree in Control systems and digital signal processing and the Ph.D. degree in Control systems, both from Northeastern University, Boston, MA, USA in 1991 and 1993, respectively. He became an Assistant Professor at the Technical University of Sofia in 1997. In 2006, he was promoted to Associate Professor at the same university. His area of scientific interests includes linear and nonlinear control systems, robust control theory and dynamical system approximation. Associate Professor Perev is a member of the John Atanasoff Society of Automatics and Informatics.*

Contacts:
Systems and Control Department
Technical University of Sofia
8 Kliment Ohridski Blvd.
1756 Sofia
e-mail: kperev@tu-sofia.bg