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The Unifying Feature of Projection in Model Order Reduction

K. Perev

Key Words: *Balanced truncation; proper orthogonal decomposition; Krylov subspace projection; projection operator; reachable and unobservable subspaces.*

Abstract. *This paper considers the problem of model order reduction of linear systems with the emphasis on the common features of the main approaches. One of these features is the unifying role of operator projection in model reduction. It is shown how projections are implemented for different methods of model reduction and what their properties are. The other common feature is the subspaces where projections are defined. The main approaches for model reduction which are considered in the paper are balanced truncation, proper orthogonal decomposition and the Lanczos procedure from the Krylov subspace methods. It is shown that the range spaces of system gramians for balanced truncation and the range space of the reachability and observability matrices for the Lanczos procedure coincide. The connection between balanced truncation and the proper orthogonal decomposition method is also established. Therefore, the methods for model reduction are similar in terms of general operational principles, and differ mostly in their technical implementation. Several numerical examples are considered showing the validity of the proposed conjectures.*

1. Introduction

The exploration of many physical phenomena requires the creation of complicated and with large number of equations mathematical models. The large scale models dimension and complexity is further increased from the demand of higher accuracy and precision in presenting the system interrelations. The optimization and simulation of such large scale systems is often unbearable task, which determines the necessity of using different approaches for system approximation. One of the most frequently used approaches is model order reduction. Model order reduction is concerned with replacing the original model with a lower order one, while preserving in general its input/output behavior. Model order reduction is successfully applied in VLSC electrical circuits design, air quality exploration, molecular dynamics simulation, deep sea wave propagation, micro electromechanical devices synthesis and many other areas.

The methods for model order reduction can be divided into two main groups [1]: singular value decomposition (SVD) based methods and methods based on projection onto subspaces of Krylov. A main representative of the first group of methods is balanced truncation [15]. A dynamical system is in balanced form when the reachability and observability gramians are identical diagonal matrices. These states which correspond to small diagonal elements are truncated, because they have a small contribution to the input/output behavior of the system. The reduced order system preserves such properties like stability, controllability and observability [15,17]. Another advantage of balanced truncation is that it provides a priory computable upper bound on the error of system approximation [9,7]. A drawback of balanced truncation is that the computational procedure is related to solving large scale matrix Lyapunov equations and if the system is of very high order, it creates computational difficulties. Balanced truncation is developed in time and frequency domain, in deterministic and stochastic settings, for different classes of systems with the restriction for system structure preservation, etc. [10].

The proper orthogonal decomposition (POD) method overcomes the numerical disadvantage of balanced truncation by simplifying to a great extend the computations in the numerical procedures [18]. The method delivers an orthonormal basis for optimal approximation in quadratic sense of certain system characteristics from experimentally obtained data. Especially useful and effective numerically is the snapshots approach [20]. By this approach, the state trajectory is discretized in equally distributed state points called snapshots, which are further employed for low dimensional approximation of system states. The usage of discretized data simplifies the computations which are reduced to the general algebraic operations. POD is data dependent method and does not require previous knowledge of system behavior. The derivation of the reduced model is obtained by projection on a subspace of the original state space. The computational effectiveness allows applying proper orthogonal decomposition to linear as well as nonlinear system problems. Due to the fact that it requires only standard matrix operations, it can be properly applied even to infinite dimensional dynamical systems. Compared to the balanced truncation procedure however, proper orthogonal decomposition does not guarantee stability for the reduced system and does not provides a priory approximation error bound. There are different attempts to connect these two methods and to extend POD to balanced truncation [21,19]. In [11,12] the concept of empirical gramians is used, which are obtained from experimental data and approximate the gramian matrices of the dynamical system. The empirical gramians are easy to compute and give quite accurate approximation when the order of the system is very high. This approach is further extended in [16], where empirical gramians are calculated by using orthogonal polynomial expansions of the state impulse responses of the original system and its dual.

The other group of methods for model order reduction is the group of methods based on projection onto the subspaces of Krylov with their two representatives: the method of Lanczos [13] and the method of Arnoldi [3]. Both methods are iterative in nature, where at each step of the computational procedures the order of the approximated

system increases. A particular feature of these methods is that the reduced system matrices admit a special canonical form, which allows simplifying the computations. The transformation matrices are orthogonal and built by applying the Gram – Schmidt orthogonalization procedure. The Arnoldi algorithm implements Galerkin projection on the subspace of Krylov generated by the columns of the controllability matrix. The two sided Lanczos algorithm realizes Petrov – Galerkin projection onto the Krylov subspaces generated by the columns and rows of the reachability and observability matrices. The Lanczos algorithm can be effectively implemented in the frequency domain, where the procedure of moment matching is successfully applied for model reduction [8]. The methods based on projection onto subspaces of Krylov are computationally efficient and the numerical procedures for obtaining the approximated systems require only matrix – vector multiplications without matrix factorizations or inversions [1]. The Lanczos procedure allows frequency domain implementation, where the selection of frequency points leads to more accurate system approximation in particular frequency ranges. The major drawback of these methods is the loss of orthogonality if the classical Gram – Schmidt procedure becomes unstable, and the algorithms may break down if certain matrices during the computations become singular [1,2].

This paper considers the problem of model order reduction for linear systems. The goal is to derive some common features of the main approaches for system reduction. One of these features is the unifying role of projection in the procedures for model reduction. It is shown how the projection operator is realized into different reduction procedures and regardless the technical implementation, all main approaches include a procedure of subspace projection. The other common feature is the destination subspace, where the states of the original system are projected. It is shown that the reachability and observability operators play decisive role in forming the structure of the projected subspace. The connection between the system gramians and the corresponding reachability and observability matrices is established. The close connection between the main approaches for model reduction is justified by several numerical simulations.

2. The Projection Operator

Projection operators play an important role in approximation theory and for spectral representation of linear operators. A complete characterization of projection operators and their application can be found elsewhere in the literature on operator theory, for example in [6].

Definition 1. Assume that V is a vector space over the field of scalars F. The projection P is a bounded linear operator of the vector space V into itself, which is idempotent, i.e. $P^2 = P$.

Proposition 1. Let V be a vector space over the field of scalars F and P is a projection defined on V. The range and null spaces of the projection operator are supplementary linear subspaces of V, i.e. their intersection contains only the zero element and their sum is the whole space:

$R(P) \cap N(P) = \{0\}$ and $R(P) + N(P) = V$.

Therefore every element of the vector space $v \in V$ can be uniquely expressed as a sum of two elements belonging to the range and null spaces of a projection operator P:

$v = v_1 + v_2$, where $v_1 \in R(P)$ and $v_2 \in N(P)$.

Conversely, if there exist two supplementary subspaces of a given vector space V, then a projection can be defined such that every vector v can be projected onto the range space of the projection operator along its kernel.

Proposition 2. Let V_1 and V_2 are two disjoint linear subspaces of a linear vector space V over the field of scalars F such that $V_1 + V_2 = V$. Then, there exists a projection P defined on V such that $R(P) = V_1$ and $N(P) = V_2$.

If the vector space V is a Hilbert space, i.e. a complete inner product vector space, then projection operators defined on the Hilbert space carry some additional properties due to the orthogonality properties of the elements of this vector space.

Definition 2. A projection P on a Hilbert space H is said to be orthogonal if its range and null spaces are orthogonal, i.e. $R(P) \perp N(P)$.

Proposition 3. Assume that H_1 is a closed subspace of a Hilbert space H. Then for every vector $h \in H$, there exist an orthogonal projection P and unique vectors h_1 and h_2 such that $h = h_1 + h_2$, where $P(h) = h_1$. In this case $h_1 \in R(P)$ and $h_2 \in R(P)^{\perp}$, and therefore $\|h\|^2 = \|h_1\|^2 + \|h_2\|^2$.

Proposition 4. Assume that P is an orthogonal projection on a closed subspace H_1 of a Hilbert space H. Then the following conditions are satisfied:

i) the projection P is a linear bounded self-adjoint operator satisfying $P^2 = P$;

ii) the norm of the projection operator is either zero or one, i.e. either $P = 0$ or $\|P\| = 1$;

iii) the projection operator on the subspace $H_2 = H_1^{\perp}$ is the operator $(I - P)$.

Proposition 4 declares that the orthogonal projection operator is a self-adjoint operator. For finite dimensional real inner product spaces, where the projection operator is a matrix with real elements, this condition means that all projection matrices are symmetric.

Proposition 5. Assume that H_1 is a closed subspace of a Hilbert space H. Then there exists a unique orthogonal projection on H such that $R(P) = H_1$.

Finally, we present one of the most important results concerning orthogonal projections on Hilbert spaces, which form the base for approximation and estimation theory on Hilbert spaces.

Theorem *(The Orthogonal Projection Theorem*). Assume that H_1 is a closed subspace of the Hilbert space H. Let P is the orthogonal projection on the subspace H_1, i.e. $R(P) = H_1$. Then, for any vector $h \in H$ we have $\|h - P(h)\| \le \|h - x\|$ for all vectors $x \in H_1$. In other words: $\|h - P(h)\| = \inf_x \{\|h - x\|: x \in H_1\}$.

Proposition 6. Assume that P_1 and P_2 are projections defined on a Hilbert space H onto closed subspaces H_1 and

H_2, respectively. Then the following statements are equivalent:

i) $P_1 + P_2$ is a projection operator;

ii) $P_1 P_2 = 0$;

iii) the projection subspaces satisfy the orthogonality condition, i.e. $H_1 \perp H_2$.

In the procedures of model order reduction, the state space is a finite dimensional space over the field of real numbers, and therefore the projection operator is presented by a matrix with real elements.

3. The Projection Operator in Model Order Reduction

Consider the stable, linear time-invariant dynamical system:

(1.1) $\dot{x}(t) = Ax(t) + Bu(t),\ t \geq 0,$

(1.2) $y(t) = Cx(t),\ x(0) = x_0,$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$. The reachability and observability gramians at infinity are given by the expression:

(2) $W_r = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ and

$$W_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

The gramians (2) can be computed by solving the following equations of Lyapunov:

(3) $AW_r + W_r A^T + BB^T = 0$ and

$A^T W_0 + W_0 A + C^T C = 0.$

The balanced truncation method is related to transforming the state space coordinates in such a way that the reachability and observability gramians are equal diagonal matrices. The diagonal elements of the gramian matrices called Hankel singular values, give information about the energy needed to reach or observe the corresponding states. The state variables related to large Hankel singular values are easy to reach and to observe and have more influence on the input/output behavior of the system. The state variables associated with small Hankel singular values have less impact on the system behavior and therefore, these states can be truncated from the system description. Applying the square root algorithm, the balancing transformation matrices can be computed as follows [1]: *i)* perform Cholesky decompositions on the gramian matrices $W_r = U U^T$ and $W_0 = L L^T$; *ii)* perform singular value decomposition on the product $U^T L = W \Sigma V^T$; *iii)* compute the similarity transformation matrices to transform the system into a balanced form: $T = \Sigma^{-1/2} V^T L^T$ and $T^{-1} = UW \Sigma^{-1/2}$. The system matrices of the transformed system are obtained as $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, $\widetilde{C} = CT^{-1}$, and $\widetilde{x} = Tx$ is the state of the balanced system. The next step is to truncate these state variables which correspond to small Hankel singular values. Let us assume that the reduced system is of order k and therefore, the other $(n - k)$ state variables are selected for truncation. We partition the similarity transformation matrices as $T = \begin{bmatrix} W^T \\ T_2^T \end{bmatrix}$ and $T^{-1} = [V \quad T_1]$, where $V, W \in R^{n \times k}$ and $T_1, T_2 \in R^{n \times (n-k)}$. Then, system order reduction is achieved by using the first k rows of matrix T and first k columns of matrix T^{-1} to obtain the reduced system matrices as $\widetilde{A}_k = W^T \widetilde{A} V$, $\widetilde{B}_k = W^T \widetilde{B}$ and $\widetilde{C}_k = \widetilde{C} V$. The matrix $P = V W^T$ is projection, since $P^2 = V W^T V W^T = V W^T = P$, where $W^T V = I_k$ follows from the definition of matrices W and V. However, P is oblique (Petrov – Galerkin) projection and not orthogonal (Galerkin), since in general $(V W^T)^T = W V^T \neq V W^T$. The projected state vector is obtained as $\hat{x} = V W^T x = V \widetilde{x}_k$, where $\widetilde{x}_k = W^T$ is the reduced model state vector of the k-th order system. The vector $\hat{x}$ is a projection of x onto the k-dimensional subspace spanned by the columns of matrix V along the kernel of W^T. The vector $\bar{x} = (I_n - V W^T) x$ belongs to the null space of the projection operator and is the unique vector such that $x = \hat{x} + \bar{x}$. In some sense vector $\bar{x}$ is complementary to vector $\hat{x}$ and contains information which has been removed from the original state vector in the procedure of model order reduction.

The proper orthogonal decomposition (POD) method is also part of the group of singular value decomposition based methods and uses as a starting point the measurements performed on system trajectories. The basic idea is to approximate the state trajectory of the system with one corresponding to a system of lower dimension. This goal is achieved by using orthogonal (Galerkin) projection. The proper orthogonal decomposition method uses discretized trajectories data called snapshots, which is obtained from measurement or simulation of the state trajectory. The trajectory data is collected in discrete time for the time moments $t_1, t_2, ..., t_N$, which form the snapshots matrix:

$$(4) \quad X = \begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_N) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_N) \\ \vdots & \vdots & \ddots & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_N) \end{bmatrix}.$$

Then, a set of orthonormal vectors $v_i \in R^n$, $i = 1, 2, ..., n$ are determined, such that the snapshots data is approximated in terms of the orthonormal vectors as $x_i = \sum_{j=1}^{n} \phi_{ji} v_j$, where $x_i = x(t_i)$, $i = 1, 2, ..., N$. These equations are presented as follows:

$$(5) \quad [x_1 \quad x_2 \quad \cdots \quad x_N] = [v_1 \quad v_2 \quad \cdots \quad v_n] \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nN} \end{bmatrix}$$

or in matrix form $X = V * \Phi$ with $V^T V = I_n$. The selection of the orthonormal set of vectors is achieved by using singular value decomposition of the snapshots data matrix

in the form $X = V\Sigma U^T$, where the coefficients matrix is $\Phi = \Sigma U^T$. If the emphasis is on the state trajectories energy, then the matrix XX^T is computed, which in terms of the SVD decomposition of the data matrix can be written as $XX^T = V\Sigma U^T U\Sigma^T V^T = V\Sigma\Sigma^T V^T$. Further, the goal is to select only $k < n$ vectors, such that the approximated state trajectory snapshots $\tilde{x}_i = \sum_{j=1}^{k} \phi_{ji} v_j$ minimize the data matrix difference $\| X - \tilde{X} \|_2$. This is the case when the singular values of matrix X with index larger than k decay rapidly. Then, the state snapshots can be computed by using only k orthonormal vectors as follows:

$$[\tilde{x}_1 \quad \tilde{x}_2 \quad \cdots \quad \tilde{x}_N] = [v_1 \quad v_2 \quad \cdots \quad v_k] \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1} & \phi_{k2} & \cdots & \phi_{kN} \end{bmatrix}. \quad (6)$$

or in matrix form $\tilde{X} = V_k * \Phi_k$, where $\tilde{X} \in R^{n \times N}$, $V_k \in R^{n \times k}$ and $\Phi_k \in R^{k \times N}$, $k < n$. Matrix V_k is built by selecting the first k columns of the orthogonal matrix V. The projection matrix is defined as $P = V_k V_k^T$. The equality $P = P^2$ is satisfied since V is an orthogonal matrix and $V_k^T V_k = I_k$. The projection is orthogonal (Galerkin), since $(V_k V_k^T)^T = V_k V_k^T$. The system matrices of the reduced order model are computed as $\tilde{A}_k = V_k^T \tilde{A} V_k$, $\tilde{B}_k = V_k^T \tilde{B}$ and $\tilde{C}_k = \tilde{C} V_k$. The state vector of the reduced system is computedas $\tilde{x}_k = V_k^T x$ and the projected state vector is obtained as $\hat{x}_k = V_k V_k^T x = V_k \tilde{x}_k$, where $\hat{x}$ is the projection of x onto the span columns of V_k. The complementary state vector $\bar{x} = (I - V_k V_k^T)\, x \in \mathrm{N}(V_k V_k^T)$ and is subadditive to the projected vector to restore the original state vector: $x = \hat{x} + \bar{x}$.

There exists an interesting link between balanced truncation and proper orthogonal decomposition. This link is established by the numerical procedure of computing the system gramians. When the input to the system is a delta impulse, the state trajectory is the state impulse response of the system. If the system is single-input, the state impulse response is $x(t) = e^{At} B$ and the reachability gramian on the time interval $[0, T]$ can be computed as: $W_r(0, T) = \int_0^T x(t)\, x^T(t) dt$. If the system is multi-input, the state impulse response for an input signal $u(t)_j = \delta(t) e_j$, $j = 1, 2, ..., m$, can be obtained as $x_j(t) = e^{At} B_j$, where B_j is the j-h column of matrix B. By using dyadic expansion of matrices, the reachability gramian can be obtained as $W_r(0,T) = \sum_{j=1}^{m} \int_0^T x_j(t) x_j^T(t) dt$. In this case, the snapshots approach of proper orthogonal decomposition can be applied by partitioning the time interval $[0, T]$ on N time moments $0 \le t_1 \le t_2 \le ... \le t_N = T$. Since the state trajectory is Riemann integrable, by applying procedures of numerical integration, the reachability gramian can be approximated by the expression $W_r(0,T) \approx \frac{T}{N} \sum_{j=1}^{m} \sum_{i=1}^{N} x_j(t_i) x_j^T(t_i)$. In terms of the snapshots data matrix, this expression can be presented in the form $W_r(0,T) \approx \frac{T}{N} \sum_{j=1}^{m} X_j X_j^T$, where $X_j = [x_j(t_1)\, x_j(t_2) \ldots x_j(t_N)]$.

Here we have used the simplest quadratures formula for numerical integration namely, the rectangular rule. In general there exist different approaches to approximate the integration in the gramians expression. Some more advanced approach employs the trapezoidal rule and reduces to the calculation of the gramian as [1]:

$$(7) \quad W_r(0,T) \approx \frac{T^3}{4N^3} \sum_{j=1}^{m} \sum_{i=1}^{N} A_\mu^i B_\mu B_\mu^T \left(A_\mu^T\right)^i,$$

where $A_\mu = (A - \mu I_n)^{-1} (A + \mu I_n)$, $B_\mu = (A - \mu I_n)^{-1} B$, and $\mu = \frac{2N}{T}$.

Further, the eigenvalue decomposition procedure can be applied to this gramian approximation, and the Galerkin projection matrix can be selected by employing the leading k eigenvectors of this decomposition:

$$(8) \quad W_r(0,T) = [U_1 \quad U_2] \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} = U_1 \Lambda_1 U_1^T + U_2 \Lambda_2 U_2^T.$$

The projection is defined as $P = U_1 U_1^T$ and since matrix U is orthogonal, its submatrix U_1 satisfies $U_1^T U_1 = I_k$. Then the idempotent condition readily follows $P^2 = U_1 U_1^T U_1 U_1^T = U_1 U_1^T = P$. This projection is orthogonal (Galerkin) since $(U_1 U_1^T)^T = U_1 U_1^T$. The reduced order system is obtained as $\tilde{A} = U_1^T A U_1$, $\tilde{B} = U_1^T B$, and $\tilde{C} = C U_1$.

However, balanced truncation presumes that system reachability as well as observability properties are simultaneously used for the purpose of model order reduction. Similarly to the reachability gramian, the observability gramian can be computed by using the state impulse response of the dual system [4]:

$$(9.1) \quad \dot{p}(t) = -A^* p(t) - C^* \tilde{u}(t), \; t \ge 0$$

$$(9.2) \quad \tilde{y}(t) = B^* p(t), \; p(T) = p_T,$$

where the star superscript denotes the adjoint operator, $p(t) \in R^n$ is the dual system state vector, $\tilde{u}(t) \in R^p$ is the dual system input and $\tilde{y}(t) \in R^m$ is the output. The time of the dual system is running backwards and a change of variables $\tau = -t$ is physically meaningful. Since the system matrices are with real elements, we can use matrix transpose instead of the star superscript. Therefore, the dual system equations can be written in the form:

$$(10.1) \quad \dot{p}(t) = A^T p(\tau) + C^T \tilde{u}(t),$$

$$(10.2) \quad \tilde{y}(\tau) = B^T p(\tau).$$

Using the same arguments as for the reachability gramian, we can present the observabity gramian as $W_o(0,T) \approx \frac{T}{N} \sum_{l=1}^{p} P_l P_l^T$, where $P_1 = [p_1(t_1)\, p_1(t_2) \ldots p_1(t_N)]$ is the dual system state vector snapshots matrix, which is computed for the l-th component of the input function $\tilde{u}(t)_l = \delta(t) e_l$. Then the usual procedures for balancing and balanced state truncation can be applied and the reduced system can be obtained by employing Petrov – Galerkin projection as described above in the balanced truncation method.

Some intermediate results between balanced truncation and proper orthogonal decomposition are presented by transforming the system into output normal form. The output normal form is system description which is characterized with diagonal reachability gramian and identity observability gramian. The numerical procedure for transforming the system into output normal form incorporates the following steps [5]: *i)* compute the eigenvalue decomposition of the system gramians $W_r = R_r\Sigma_r^2 R_r^T$ and $W_o = R_o\Sigma_o^2 R_o^T$, where $R^T R = I$ and $\Sigma^2 = diag\{\lambda_1^2, \lambda_2^2, ..., \lambda_n^2\}$ are the corresponding eigenvalues; *ii)* compute the matrix $H = \Sigma_o^T R_o^T R_r \Sigma_r$; *iii)* compute the singular value decomposition $H = R_H \Sigma_H Q_H$, where $R_H^T R_H = I$ and $Q_H Q_H^T = I$; *iv)* compute the similarity transformation matrices for transforming the system into output normal form as $T = R_H^T \Sigma_o^T R_o^T$ and $T^{-1} = R_o \Sigma_o^{-T} R_H$. The transformed system will have reachability gramian $\widetilde{W}_r = \Sigma_H^2$ and observability gramian $\widetilde{W}_o = I$. The model reduction procedure proceeds by selecting the first k columns of $T^{-1} = [V_k \quad T_1]$ and the first k rows of $T = \begin{bmatrix} W_k^T \\ T_2^T \end{bmatrix}$ and using them as transformation matrices to obtain $A_k = W_k^T A V_k$, $B_k = W_k^T B$ and $C_k = C V_k$. The projection matrix is built as $P = V_k W_k^T$, where $W_k^T V_k = I_k$ follows from the definition of these matrices. Further, $(V_k W_k^T)^T = W_k V_k^T$. Therefore, the projection is Petrov – Galerkin and is almost Galerkin up to scaling by the square value of the diagonal matrix Σ_o elements.

The other group of methods for model order reduction is based on projection on the subspaces of Krylov. A main representative of this group is the method of Lanczos. The method of Lanczos is an iterative procedure for computing an orthonormal basis for the reachability subspace of the system. We consider here the single-input single-output system case. The algorithm proceeds by applying the Gram – Schmidt orthogonalization procedure to a sequence of vectors gradually forming the reachability subspace, where the remaining term serves as a new direction for developing the sequence. Let us assume that the matrix $V_k = [v_1 \quad v_2 \quad ... \quad v_k]$ consists of k column vectors which are orthonormal, i.e. $V_k^T V_k = I_k$. Then matrix A is transformed to a system matrix, where the states are obtained as a projection of the original state vectors onto the subspace spanned by the columns of V_k, i.e. $H_k = V_k^T A V_k$. If $spancol\{v_1 \quad v_2 \quad ... \quad v_k\} = spancol\{v_1 \quad Av_1 \quad ... A^{k-1}v_1\}$ and if $v_1 = B$ then, at each step of the algorithm, the generated subspace is part of the reachability subspace of the system. The iterative procedure is presented in the following form [1]:

(11) $A V_k = V_k H_k + r_k e^T_k$,

where the remaining term r_k is obtained from the Gram – Schmidt orthogonalization procedure as follows:

$$r_k = Av_k - \sum_{j=1}^{k} \langle v_j, Av_k \rangle v_j \quad (12)$$

and therefore, it is orthogonal to the columns of V_k. Furthermore, the next vector from the sequence can be computed by the expression $v_{k+1} = \frac{r_k}{\|r_k\|}$. The matrix $H_k = V_k^T A V_k$, which is obtained from the iterative process (9), is the system matrix of the reduced k-th order system. It has a special tridiagonal structure and is obtained from projection onto the subspace generated by the columns of V_k. The projection is of Galerkin type and is computed as $P = V_k V_k^T$, where $V_k^T V_k = I_k$. The remaining term is $r_k = (I_n - V_k V_k^T) Av_k$ and therefore, $r_k \in N(V_k V_k^T)$ – the null space of the projection operator. If at certain step of the algorithm the remaining term is obtained as $r_k = 0$, then v_{k+1} can not be constructed and the Lanczos procedure comes to an end. This is the case when $spancol\{v_1 \quad v_2 \quad ... \quad v_k\}$ has generated the whole reachable subspace of the system.

The problem with this procedure is that it relies only on the information from the reachable subspace for computing the projection matrix. For obtaining closer approximation of the input/output system behavior it is necessary together with the reachability subspace to consider the set of the observable states as well. This condition is accomplished in the two sided Lanczos algorithm. The two sided Lanczos algorithm consists of two iterative procedures: $AV_k = V_k H_k + r_k e^T_k$ and $A^T W_k = W_k H^T_k + g_k e^T_k$, where the columns of V_k span part of the reachable subspace of the system and the columns of W_k span part of the observable states. Therefore, we can write:

$spancol\ (V_k) = spancol\ \{B \quad AB \quad ... \quad A^{k-1}B\}$ and

$spancol\ (W_k) = spancol\ \{C^T \quad A^T C^T \quad ... \quad (A^T)^{k-1} C^T\}$.

The projection matrix is computed as $P = V_k W_k^T$, where $W_k^T V_k = I_k$ and is oblique (Petrov – Galerkin) since $(V_k W_k^T)^T = W_k V_k^T \neq V_k W_k^T$. The projected state vector is obtained as $\hat{x} = V_k W_k^T x = V_k \tilde{x}_k$, where $\tilde{x}_k = W_k^T x$ is the reduced model state vector of the k-th order system. The vector $\bar{x} = (I_n - V_k W_k^T)x$ belongs to the null space of the projection operator and is the unique vector such that $x = \hat{x} + \bar{x}$.

4. The Reachability and Observability Operators for Linear Systems

Consider the system described by equations (1). In the presentation to follow we consider signals with finite energy and therefore, the input and output signals belong to the Hilbert space $L_2(0, T)$ and its subspaces. The reachability operator $L_r : PC([0,T]) \to R^n$, where $PC([0,T]) \subset L_2(0,T)$ is the set of piecewise continuous functions with finite energy defined on the interval $[0,T]$, maps each admissible input signal $u \in U_{[0,T]} \subset PC([0,T])$ into a state vector by the expression $L_r(u)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$, $t \in [0,T]$ [4]. The range space of the reachability operator $R(L_r)$ is the set containing all reachable states of the system and is called the reachable subspace. The adjoint operator $L^*_r : R^n \to PC([0,T])$, which satisfies the relation

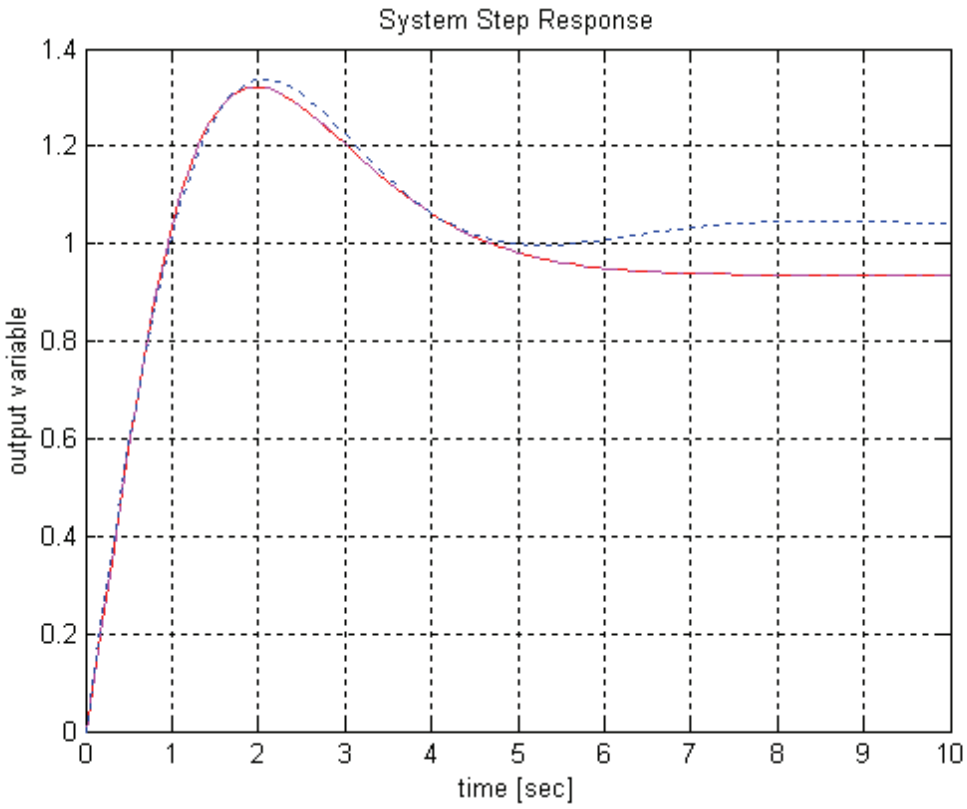

Figure 1. Unit step responses of the full order -----; reduced 4th order -.-.-; reduced 2nd order .....; by BT

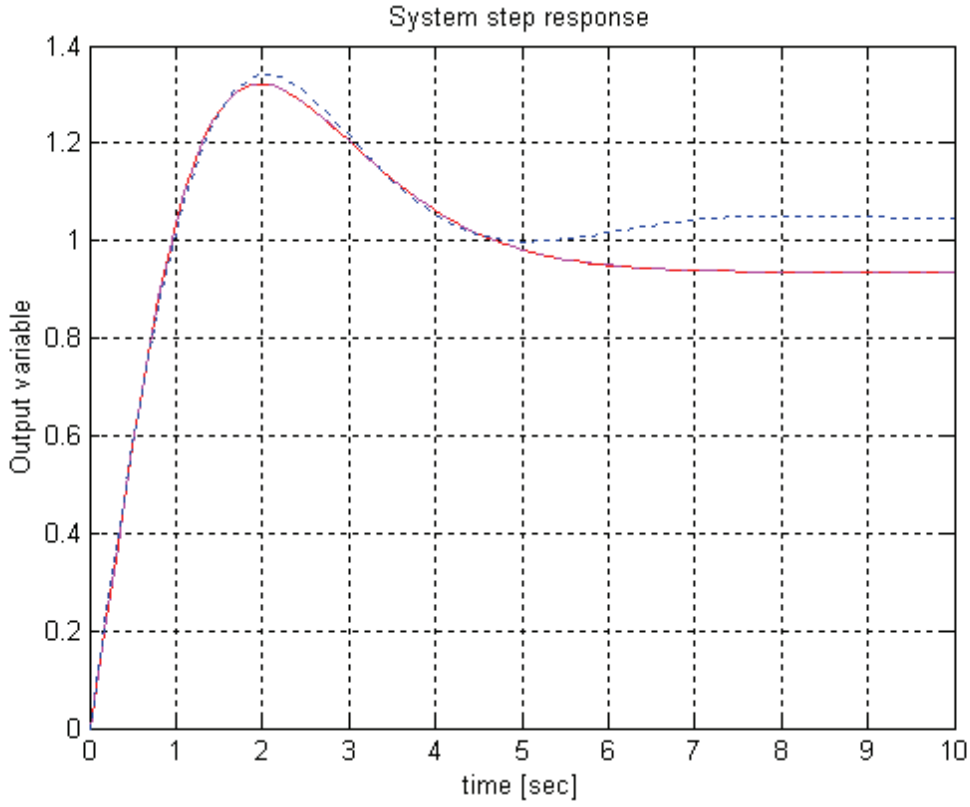

Figure 2. Unit step responses of the full order -----; reduced 4th order -.-.-; reduced 2nd order .....; by POD

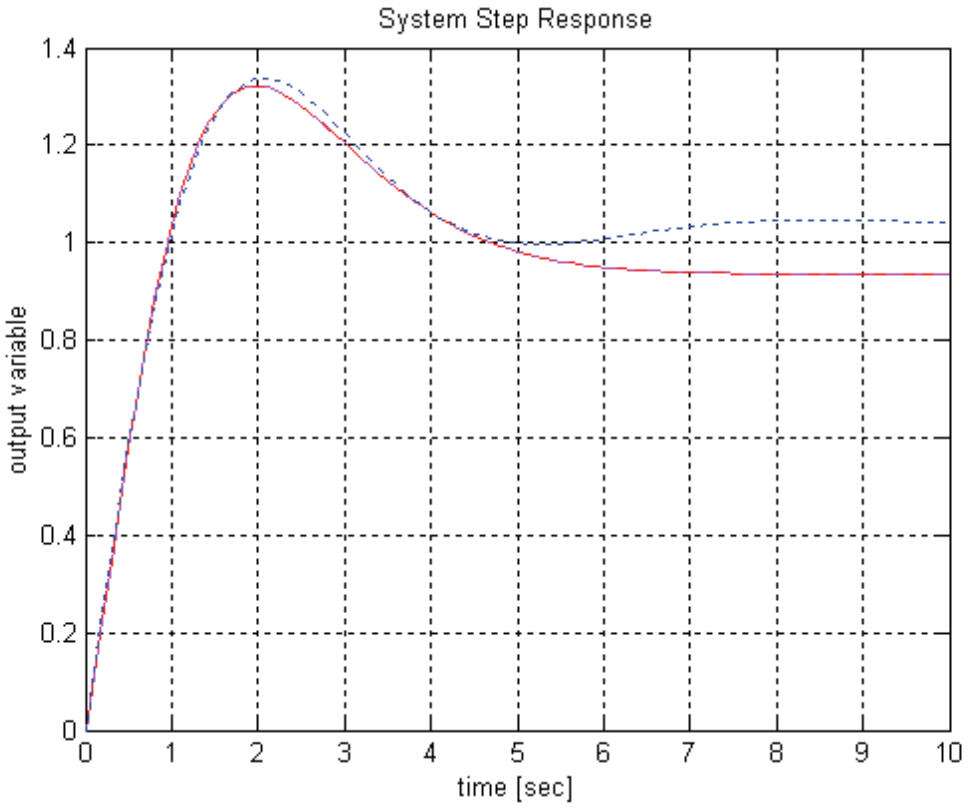

Figure 3. Unit step responses of the full order -----; reduced 4th order -.-.-; reduced 2nd order …..; in ONF

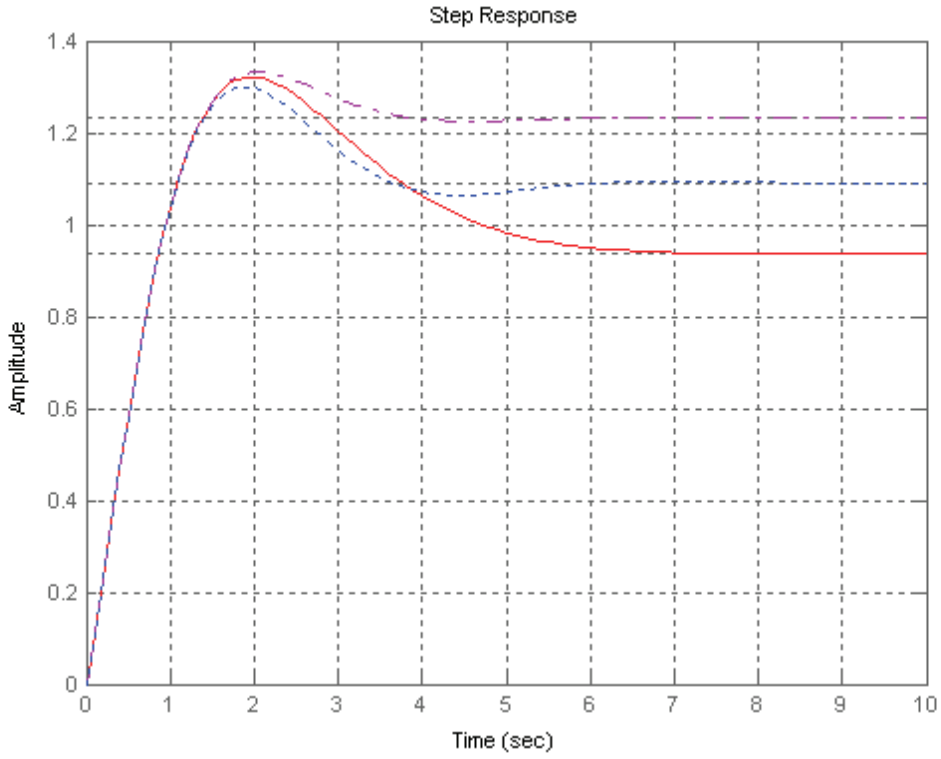

Figure 4. Unit step responses of the full order -----; reduced 5th order -.-.-; reduced 3rd order …..; by L

$\langle z, L_r, u\rangle = \langle L^*_r, z, u\rangle$, can be determined for every vector $z \in R^n$ as $(L^*_r z)(t) = B^T e^{A^T(T-1)}z$, $t \in [0,T]$. The reachability gramian on the interval [0,T] is defined as $W_r(0,T) = \int_0^T e^{At} BB^T e^{A^T t} dt$ and is the matrix representation of the operator $L_r L^*_r : R^n \to R^n$. It is more convenient to work with the gramian than with the reachability operator, since the gramian is a matrix and a map between two state vectors, while the reachability operator maps elements of the infinite dimensional space of admissible inputs into the finite dimensional state space. However, the reachability gramian is not state space basis invariant and its properties depend on the basis of system description. The reachable subspace of the system, which is the range space of the reachability operator coincides with the range space of the gramian $R(L_r) = R(L_r L^*_r)$. Therefore, the reachability condition for all system states $R(L_r) = R^n$ is equivalent to the condition $R(L_r L^*_r) = R^n$ or $\det W_r(0,T) \neq 0$.

Every reachable state can be obtained in the range of the reachability operator for any admissible input signal with finite energy by using Taylor series expansion as $L_r(u)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \sum_{i=0}^{\infty} A^i B \int_0^t \frac{(t-\tau)^i}{i!} u(\tau) d\tau$, and is a linear combination of the vectors B, AB, A^2B, ... for a fixed time moment t. For example, in the single-input case if the input to the system is a delta impulse, then $(L_r \delta)(t) = e^{At}B$ (in the multi-input case we have $(L_r \delta e_j)(t) = e^{At}B_j$, where $u(t) = \delta(t)e_j$, $j = 1, 2, \ldots, m$). The expression $e^{At}B$ can be expanded in Taylor series as $e^{At}B = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} B = \sum_{i=0}^{\infty} A^i B \frac{t^i}{i!}$. By the Cayley – Hamilton theorem, A^i for $i \geq n$ is a linear combination of $\{A^k\}_{k=0}^{n-1}$ and therefore, the infinite sum can be rearranged to contain only a finite number of elements. The reachability matrix of the system (1) is defined as $\Gamma = [B \quad AB \quad \ldots \quad A^{n-1}B]$ and from the above discussion follows that the range space of the reachability operator is spanned by the columns of matrix Γ, or equivalently $R(L_r) = R(L_r L^*_r) = R(\Gamma)$.

This result is even more straightforward in the discrete-time systems case. Consider the linear stable discrete-time system:

(13.1) $x(k+1) = Fx(k) + Gu(k)$,

(13.2) $y(k) = Cx(k) + Du(k)$, $x(0) = x_0$.

The reachability gramian at infinity of system (13) is defined as $W_r = \sum_{k=0}^{\infty} F^k GG^T (F^T)^k$ and the reachability matrix of system (11) is $X = [G \quad FG \quad \ldots \quad F^{n-1}G]$. Let us extend the reachability matrix by adding columns with higher powers of F and so obtain the extended reachability matrix $X_\infty = [G \quad FG \quad \ldots \quad F^{n-1}G \quad \ldots]$. By the Cayley – Hamilton theorem follows that the space generated by the columns of the extended matrix is the same as the space generated by the columns of the original matrix and its dimension is equal to the reachability matrix rank. Since $W_r = X_\infty X^T_\infty$ it is clear that the range spaces of the gramian and the reachability matrix are the same, i.e. $R(W_r) = R(X_\infty) = R(X)$.

The observability operator $L_0 : R^n \to PC([0,T])$, where $PC([0,T])$ is the set of piecewise continuous functions defined on the interval $[0,T]$, maps each state vector $x_0 \in R^n$ into the function $y \in Y_{[0,T]} \subset PC([0,T])$ by the expression $(L_o x_0)(t) = Ce^{At} x_0|_{[0,T]}$, $t \in [0,T]$. The null space of the observability operator $N(L_o)$ is the set containing all state vectors that excite no signal at the system output and is called the unobservable subspace. The adjoint operator $L^*_o : PC([0,T]) \to R^n$ can be determined for every finite energy output signal as $(L^*_o y) = \int_0^T e^{A^T t} C^T y(t) dt$. The observability gramian of system (1) on the time interval $[0,T]$ is determined by the expression $W_o(0,T) = \int_0^T e^{A^T t} C^T C e^{At} dt$ and is the matrix representation of the operator $L^*_o L_o : R^n \to R^n$. Similarly to the case with the reachability gramian, it is more convenient to work with the observability gramian than with the observability operator, since it is a map between vectors on a finite dimensional vector space. Since linear operators always map zeros into zero, the null space of the observability gramian is the same as the null space of the observability operator, i.e. $N(L_o) = N(L^*_o L_o)$. Therefore, the condition that the system (1) is completely observable is $N(L_o) = \{0\}$, which is equivalent to the condition that $N(L^*_o L_o) = \{0\}$. Equivalently the rank of the observability gramian has to be equal to the state space dimension and therefore, $\det W_o(0, T) \neq 0$. If the system (1) is completely observable, the initial state vector can be recovered from the relation $x_0 = (L^*_o L_o)^{-1} L^*_o y$.

Since the output signal is of finite energy and by using the Taylor series expansion formula for a matrix exponent we obtain $L^*_o y = \int_0^T e^{A^T} C^T y(t) dt = \sum_{i=0}^{\infty} (A^T)^i C^T \int_0^T \frac{t^i}{i!} y(t) dt$. Therefore, the range space of the observability operator adjoint can be spanned by the columns of an infinite matrix $O_\infty = [C^T \quad A^T C^T \quad \ldots \quad (A^T)^{n-1} C^T \quad \ldots]$. Then using the relation that $[R(L^*_o)]^\perp = N(L_o)$ [14] and therefore $[R(O_\infty)]^\perp = N(O^T_\infty)$, we obtain that the unobservable states of the system are elements of the kernel of matrix O^T_∞ and therefore, $N(O^T_\infty) = N(L_o)$. By the Cayley – Hamilton theorem, the kernel of matrix O^T_∞ is determined by the first n terms, i.e CA^j, $j = 0,1,2,\ldots, n-1$ and an infinite dimensional matrix can be replaced by a finite dimensional one $O^T = [C^T \quad A^T C^T \quad \ldots \quad (A^T)^{n-1} C^T]^T$. As a conclusion, we obtain that the unobservable subspace of system (1) is determined as $N(L_o) = N(L^*_o L_o) = N(O^T)$.

In the discrete-time system case the observability gramian at infinity of system (11) is defined as $W_o = \sum_{k=0}^{\infty} (F^T)^k C^T C F^k$ and the corresponding infinite observability matrix is $O_\infty = [C^T \quad F^T C^T \quad \ldots \quad (F^T)^{n-1} C^T \quad \ldots]$. It is clear that $W_o = O_\infty O^T_\infty$ and therefore, by aplying the Cayley – Hamilton theorem it follows directly that $N(W_o) = N(O^T_\infty) = N(O^T)$.

A major role for connecting the reachability and observability properties of a dynamical system plays the duality principle. All of the results presented in the above discussion can be summarized as a consequence of the duality principle. The duality principle, applied to the dynamical systems (1) and (10) states that the orhogonal complement of the reachable subspace of (1) coincides with the unobservable subspace of (10) [1]. Furthermore, the system (1) is reachable if and only if system (10) is observable and vice versa. We have shown that the reachable and unobservable subspaces of a linear time-invariant stable system and its dual are interrelated and therefore, the projection on the column and null spaces of the reachability and observability gramians and matrices are closely connected. Regarding the model reduction procedures, different methods give similar results as concerned the projection on common subspaces of the state space. In this sense, the balanced truncation method uses the information from both, the reachability and observability gramians and projects onto the subspace generated by the eigenvectors of their product. The POD method uses information from the state trajectories and projects onto a part of the reachability subspace of the system. The two-sided Lanczos method uses information from both, the reachability and observability matrices and projects onto the subspaces generated by part of their columns and rows. Therefore, the projection operator plays a unifying role in the model reduction procedures and the link between the reachable and unobservable subspaces of the system and its dual one outlines the common features of at first sight completely different methods for system approximation.

5. Numerical Example

Consider the linear stable time-invariant system described by the following equations:

$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0$

$y(t) = Cx(t), \qquad x(0) = x_0$

with the following system matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -390 & -1034 & -1115 & -586 & -155 & -20 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0 \\ 0.001 \\ 0.0 \\ 0.001 \\ 0.0 \\ 0.001 \end{bmatrix}, \quad C = \begin{bmatrix} 390 \\ 1034 \\ 334 \\ 44.1 \\ 0 \\ 0 \end{bmatrix}^T$$

The Hankel singular values for the system are computed as follows:

$\Sigma = (0.8213 \quad 0.3012 \quad 0.0536 \quad 0.0018 \quad 0.0003 \quad 0.0001)$.

Figure 1 presents the unit step responses of the full order model and the obtained by balanced truncation reduced fourth and second order models. It is clearly seen that the step responses of the full order and the reduced fourth order models are almost undistinguishable. Some difference appears in the step response of the reduced second order model and this difference is more apparent in steady state. *Figure 2* presents the unit step responses of the full order model and the obtained by proper orthogonal decomposition based balancing reduced fourth and second order models. The time interval of approximation is [0, T] with T = 8 sec and step size Δ = 0.01 sec. As in the case of balanced truncation, the full and fourth order system are closely related and larger difference appears in the step response of the reduced second order system. Next we explore the model reduction scheme by transforming the system into an output normal form. The reachability gramian of the system into output normal form is diagonal matrix with its elements computed as follows:

$\widetilde{\Sigma} = (0.6745 \quad 0.0907 \quad 0.0029 \quad 3.24 \cdot 10^{-6} \quad 9.26 \cdot 10^{-8} \quad 1.82 \cdot 10^{-8})$.

It is not difficult to notice that the elements of the reachability gramian $\widetilde{\Sigma}$ are the square of the Hankel singular values. The corresponding observability gramian is the identity matrix. *Figure 3* presents the step responses of the full order and corresponding reduced fourth and second order models. It is evident that the obtained responses are similar to those obtained by applying the balanced truncation method.

This observation is not surprising and its explanation is that both systems have the same Hankel singular values.

The step responses of the full order system and the reduced fifth and third order models obtained by applying the Lanczos procedure are shown in *figure 4*. From the figure is clear that the error between the reduced order systems and the original one is larger than in the case of balanced truncation. *Figure 5* presents the unit step responses of the full order model and the reduced fourth order models obtained by balanced truncation, proper orthogonal decomposition based balancing, output normal form based balancing and the Lanczos method. It is obvious that the difference appears mainly in the step response obtained by applying the Lanczos procedure. These results are confirmed by the computed mean square errors between the full order and reduced fourth models. The relative mean quadratic error of approximation between the step responses of the full order model and the reduced fourth order models computed by the balanced truncation, proper orthogonal decomposition based balancing, output normal form based balancing and the Lanczos methods are obtained as follows:

$$\frac{\|y - y_{BT}\|_2}{\|y\|_2} = 1.0374 \cdot 10^{-4}, \qquad \frac{\|y - y_{POD}\|_2}{\|y\|_2} = 2.5626 \cdot 10^{-4},$$

$$\frac{\|y - y_{ONF}\|_2}{\|y\|_2} = 1.0374 \cdot 10^{-4}, \qquad \frac{\|y - y_L\|_2}{\|y\|_2} = 0.1051.$$

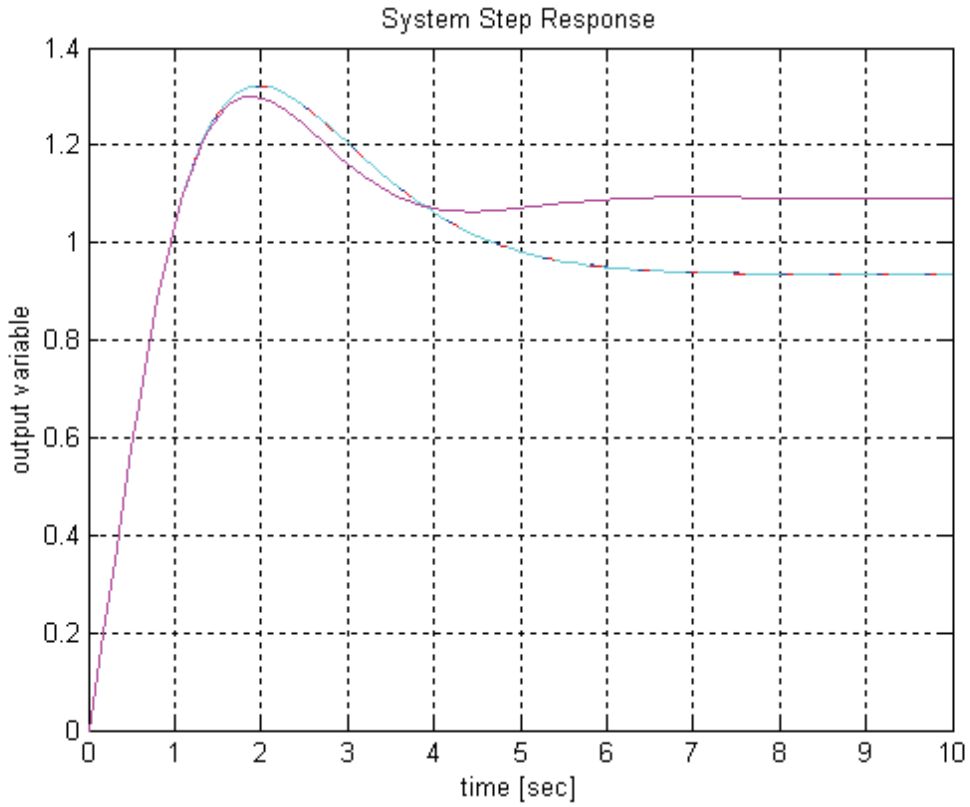

Figure 5. Unit step responses full order -----; reduced 4th order by BT -.-.-; POD .....; ONF -.-.-; L -----

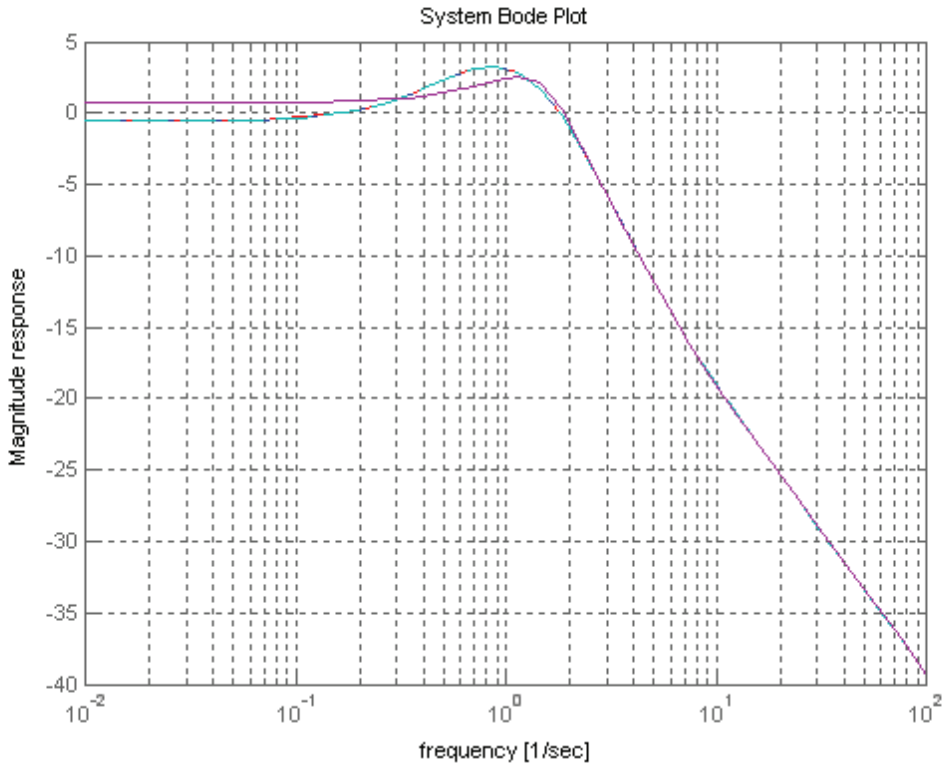

Figure 6. Magnitude responses full order -----; reduced 4th order by BT -.-.-; POD .....; ONF -.-.-; L -----

The obtained relative errors show that the Lanczos procedure gives more inaccurate results than the other and justify the theoretical conjecture that for low order systems the balancing procedures give most accurate results. These results are confirmed in *figure 6*, where the Bode characteristics are presented for the full order and reduced fourth order models obtained by balanced truncation (BT), proper orthogonal decomposition based balancing (POD), output normal form balancing (ONF) and the two sided Lanczos method (L).

6. Conclusion

This paper considers the problem of model order reduction of linear time-invariant stable systems. The emphasis is placed on the common features of several main methods for model reduction. The unifying feature in the procedures of model order reduction is projection. It is shown how projections are computed for different methods of model reduction and what the properties of the projection operators are. Another common feature in model reduction is the determination of the subspaces where the state space vectors are projected. It is shown that the range space of the reachability gramian and the reachability matrix are the same and this observation is related to their connection with the reachability operator. Similarly, the observability gramian and the observability matrix are associated with the observability operator, and therefore the unobservable subspace is determined by the null spaces of these matrices. It is shown that different methods use different schemes for model reduction as concerned the technical details of computation however, all the methods contain similar characteristics determined by the projection on the same subspaces. Several main methods are examined namely: balanced truncation, proper orthogonal decomposition based balancing, output normal form based balancing and the Lanczos procedure. In a numerical example is shown that these methods give similar results in model reduction, although some differences are observed concerning the accuracy of approximation. This observation can be explained by the special features of these methods applied to a specific example.

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*Assoc. Prof. **Kamen Perev** received the engineering degree in Automatics from VMEI – Sofia in 1985. He received the M.S. degree in Control systems and digital signal processing and the Ph.D. degree in Control systems, both from Northeastern University, Boston, MA, USA in 1991 and 1993, respectively. He became an Assistant Professor at the Technical University of Sofia in 1997. In 2006, he was promoted to Associate Professor at the same university. His area of scientific interests includes linear and nonlinear control systems, robust control theory and dynamical system approximation. Associate Professor Perev is a member of the John Atanasoff Society of Automatics and Informatics.*

Contacts:
Systems and Control Department
Technical University of Sofia
8 Kliment Ohridski Blvd.,1756 Sofia
e-mail: kperev@tu-sofia.bg