GYROSTAT MODEL REGULAR AND CHAOTIC BEHAVIOR[*]

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Abstract. During recent years, the interest in the phenomena of chaos in gyroscopic systems has been increasing. It is well-known, that depending on the speed of rotation, a gyroscopic system may lose or gain stability. Despite the overwhelming number of studies reporting the occurrence of various chaotic structures, little is known yet about the construction details and the generality of the underlying bifurcation scenarios that give rise to such chaotic (complex) behaviour.

In this paper, we report a detailed investigation of the abundance of regular and chaotic behaviour for rigid body (gyrostat) motion. The model contains 6 parameters that may be tuned to produce rich dynamical scenarios. The results confirm that homoclinic and heteroclinic structures with two fixed points from saddle-focus type occur and the emergence of Shilnikov chaos takes place. Finally, we find new results concerning the system's evolution and bifurcation scenarios for its routes to chaos.
Key words: Dynamical systems, chaos, gyrostat, homoclinic and heteroclinic orbits.

1. Introduction

Modelling is a powerful tool in the processes simulation in physics and technologies, dealing with different time and spatial scales, and in the mechan-

ical characterization of system parameters. It is also effective in the interpretation and design of experiments, as well as in the new effects and phenomena prediction. It is clear, that modelling will play an increasing role in improving our understanding of the physical processes in mechanical systems, under normal and abnormal conditions.

In an array of great discoveries in the twentieth century, three of them certainly belong to physics: 1) the theory of relativity of Albert Einstein (without ignoring the great merit of Henri Poincaré); 2) quantum mechanics, associated with a large number of scientists from different countries and 3) chaos theory, associated mostly with the name of the American theorist Edward Lorenz, a meteorologist. Again, Henri Poincaré has a contribution to its development, and later many scholars, the list of whose names cannot fit into several dozen pages. It will be only mentioned that the first to use the word *chaos* in its modern understanding in science are Li and Yorke in 1975 [1], and that the authors of the concept of the strange attractor are Ruelle and Takens in 1971 [2].

There has been a large amount of recent interest in the investigation of gyroscope dynamics. The gyroscope has attributes of great utility to navigational and aeronautical engineering, biology, optics, etc. [3-8]. Different types of gyroscopes (with linear or nonlinear damping, fluid, etc.) are investigated for predicting dynamic responses such as regular and chaotic motions [9–11].

Dissipative systems are a special class of dynamical systems. In general, dissipative mechanical systems in more than two dimensions have bounded and unbounded orbits, depending on the energy. The bounded trajectories of these systems do not converge to an equilibrium point or to a periodic or quasi-periodic orbit. In this case, the flow is essentially aperiodic. A dynamical system is dissipative, if its phase volume contracts continuously, i.e. $D_i < 0$ $(i = 1, \ldots, n)$. Only dissipative dynamical systems have attractors [12, 13].

Attractors are the adequate mathematical (geometrical) representations of time order and chaos that can be: stable equilibrium, stable periodic motions (auto-waves) or strange attractors. Mathematical representations of spatial order and chaos are saddle equilibrium, saddle periodic movements or complex saddle invariant sets.

An important step towards the understanding of the global dynamics of a system of differential equations is the analysis of the existence of homoclinic/heteroclinic orbits (cycles). It is well-known, that a heteroclinic cycle is a sequence of trajectories connecting a set of fixed points in a topological circle. A classical heteroclinic cycle is a loop that consists of saddle equilibrium states connected to one another by their separatrices. The special case of a cycle consisting of one trajectory and one fixed point is usually called a homoclinic

trajectory [14, 15].

A homoclinic trajectory $f(x, t)$, (or a homoclinic (separatrix) loop) is such that the 'inset' to a fixed point of an attractor, x_0, is the same as the 'outset' from the same point. It obeys the rule that $f(x, t) \to x_0$ for $t \to \infty$, $t \to -\infty$ [16]. According to Peixoto's theorem [17, 18], homoclinic bifurcations are structurally unstable and are therefore, destroyed by small perturbations. Consequently, they are more difficult to identify than a local bifurcation, because knowledge of the global properties of the phase space trajectories is required. A systematic characterization of homoclinicity around a saddle-focus equilibrium was provided by Shilnikov [19–21]. In this scenario, reinjection occurs along a well-defined vector associated with a real system eigenvalues, with ejection from the vicinity of the equilibrium, subsequently effected on a spiral path located on a transverse plane. A necessary condition for this mechanism is that the saddle-focus index is $\delta = \left| \mathrm{Re} \left(\chi_2 / \chi_1 \right) \right| < 1$, where χ_1 and χ_2 are the leading eigenvalues. Here, we note that χ_1 is determining the rate of approaching and χ_2 is determining the rate of leaving the stable point. If thus Shilnikov condition is satisfied, an infinite number of nonperiodic trajectories coexist in the vicinity of a homoclinic trajectory bi-asymptotic to the saddle-focus.

A nice example is the Lorenz system [22], which has an important historical relevance in the development of chaos theory. Now, this system is considered as a paradigmatic example of a chaotic system [23]. This system is adaptive, if $\nabla \dot{x}_i$ takes both positive and negative values. Rigid bodies are bodies that cannot deform and change their shape, but they can translate and rotate. In [24, 25], the authors remark that the Lorenz system admits a purely mechanical model of two rigid bodies – lifting and lifted axi-symmetric rotors. The system is firstly suggested by Gluhovsky in [26] and has the following property: when the rotor moves relative to the lifting body, then the distribution mass remains unaltered in space. Thus, the system tensor is constant and such a system is called a gyrostat (see Fig. 1). Further, it is assumed that its center of mass is fixed, and that the ellipsoid of inertia has the form of a rotation ellipsoid.

In case of small angular velocities, the motion of the system can be presented from the following differential equation [9, 27]:

(1) $$\dot{M} = M \times AM + \boldsymbol{B}M,$$

where M is the vector of kinematical moment about the coordinate system of the rigid body, $A = I^{-1} = \mathrm{diag}(a_1, a_2, a_3)$ (where I is the inertial tensor) and $\boldsymbol{B}$ is a constant matrix. For the dissipative case, i.e. div < 0, the condition
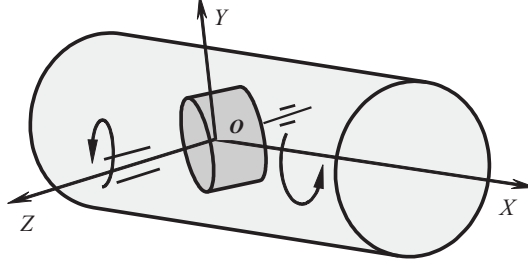
Fig. 1. Two bodies – lifting and lifted axi-symmetric rotor

Tr $B < 0$ is valid. In [28], it is shown for specific values of system parameters, that two attractors have place. It is well known, that system (1) has two particular cases: (i) those of Greenhill and (ii) Klein & Sommerfeld [29]. The trajectories lie in integral surfaces in these two cases. For some complicated cases, system (1) has two strange attractors.

According to Fig. 1, the system (1) can be written in the form:

$$
\begin{aligned}
\dot{M}_1 &= c_1 M_1 + M_2 + c_2 M_2 M_3, \\
\dot{M}_2 &= -M_1 - c_3 M_2 + c_4 M_1 M_3, \\
\dot{M}_3 &= c_5 M_3 - c_6 M_1 M_2,
\end{aligned}
\tag{2}
$$

where M_1, M_2 and M_3 are the projections of the instantaneous angular velocity onto the X-, Y- and Z-axes (see Fig. 1), and $c_1 \div c_6$ are parameters (in form of algebraic complexes), given in [24, 28].

In general, the investigation of such model would start with the computation of steady states. The equilibrium (steady state) points of the system (2) are found by equating the right-hand sides of (2) to zero. Thus, it is easy to see that equilibrium points of the system (2) are:

$$
\mathbf{O}_1 : \overline{M}_1 = \overline{M}_2 = \overline{M}_3 = 0, \qquad \textit{first fixed point}
\tag{3}
$$

$$
(4)\ \mathbf{O}_{2,3,4,5} : \quad \overline{M}_1 = \frac{1}{c_1}\overline{M}_2 \left(1 + \frac{c_2 c_6}{c_1 c_5} \frac{\overline{M}_2^2}{1 - \frac{c_2 c_6}{c_1 c_5 \overline{M}_2^2}} \right), \quad \overline{M}_3 = \frac{c_6}{c_1 c_5} \frac{\overline{M}_2^2}{1 - \frac{c_2 c_6}{c_1 c_5}\overline{M}_2^2},
$$

$$
\overline{M}_2^4 - \frac{1}{c_2^2 c_3 c_6}\left[c_5(c_2 + c_4) + 2 c_1 c_2 c_3 c_5\right] \overline{M}_2^2 + \frac{c_1 c_5^2}{c_2^2 c_3 c_6^2}(1 + c_1 c_3) = 0.
$$

$$
\textit{second, third, fourth and fifth fixed points}
$$

For the reason that $(c_2 + c_4)^2 + 4c_1c_2c_3(c_1c_2c_3 + c_4) > 0$, the system (2) has five real fixed points – see Fig. 2 a, b.
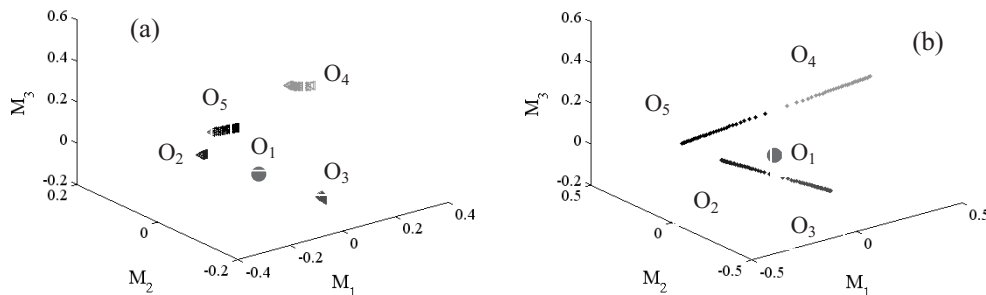

Fig. 2. (a) Fixed points of system (2) in $M_1M_2M_3$ coordinates as a function of c_1;
(b) fixed points of system (2) in $M_1M_2M_3$ coordinates as a function of c_5

The system (2) enjoys the natural symmetry $(M_1, M_2, M_3) \rightarrow (-M_1, -M_2, M_3)$. The M_3-axis is invariant. All trajectories, which start on the M_3-axis remain on it.

The divergence of the flow (2) is:

$$(5) \qquad D_3 = \frac{\partial \dot{M_1}}{\partial M_1} + \frac{\partial \dot{M_2}}{\partial M_2} + \frac{\partial \dot{M_3}}{\partial M_3} = c_5 - (c_1 + c_3).$$

The system (2) is dissipative, when $D_3 < 0$, i.e. $c_5 < c_1 + c_3$. It means, that a volume element $D_3(t_0)$ is contracted by the flow into a volume $D_3(t) = D_3(t_0) \times e^{-(c_1+c_3-c_5)t}$. That is, as $t \rightarrow +\infty$, each volume containing the system trajectories shrinks to zero at an exponential rate (independently of the system states) $D_3 = -(c_1 + c_3 - c_5)$. Hence, all system trajectories will be confined to a specific subset of zero volume. For example, in 1981 Leipnik and Newton [28] found that for $c_5 < 0.8$, the system is dissipative and all volumes in the phase space must contract uniformly, even though cross-sectional and all trajectories except those trapped at the rest points, diverge to infinity. Simulations suggest that for $c_5 = 0.175$ the system (2) has two strange attractors.

The paper is further organized as follows: in Section 2, 3 and 4 we present analytical and numerical results, concerning the system (2) behaviour for different values of bifurcation (control) parameters c_1 and c_5. In Section 5 we discuss and summarize our results.

2. Theoretical analysis

In this section, we investigate the system (2), which presents an autonomous nonlinear 3D dynamical model.

Generally, in order to determine the character of fixed points (Eqs. (3) and (4)), we make the following substitutions into (2):

(6) $$M_1 = \overline{M}_1 + w_1, \quad M_2 = \overline{M}_2 + w_2, \quad M_3 = \overline{M}_3 + w_3.$$

Hence, after some transformations, the system (2) has the form:

(7) $$\begin{aligned}
\dot{w}_1 &= -c_1 w_1 + b_1 w_2 + b_2 w_3 + c_2 w_2 w_3, \\
\dot{w}_2 &= -b_3 w_1 - c_3 w_2 + b_4 w_3 + c_4 w_1 w_3, \\
\dot{w}_3 &= -b_5 w_1 - b_6 w_2 + c_5 w_3 - c_6 w_1 w_2,
\end{aligned}$$

where

(8) $$\begin{aligned}
b_1 &= 1 + c_2 \overline{M}_3, \quad b_2 = c_2 \overline{M}_2, \quad b_3 = 1 - c_4 \overline{M}_3, \\
b_4 &= c_4 \overline{M}_1, \quad b_5 = c_6 \overline{M}_2, \quad b_6 = c_6 \overline{M}_1.
\end{aligned}$$

According to [30], the Routh-Hurwitz conditions for stability of fixed points (3) and (4), can be written in the form:

(9) $$\begin{aligned}
p &= c_1 + c_3 - c_5 > 0, \\
q &= c_1 (c_3 - c_5) - c_3 c_5 + b_1 b_3 + b_2 b_5 + b_4 b_6 > 0, \\
r &= - [c_5 (c_1 c_3 + b_1 b_3) - b_5 (b_1 b_4 + b_2 c_3) + b_6 (b_2 b_3 - b_4 c_1)] > 0, \\
R &= pq - r = (c_3 - c_5) [c_1 (c_1 + c_3 - c_5) - c_3 c_5 + b_4 b_6] + \\
&\quad b_1 b_3 (c_1 + c_3) + b_2 b_5 (c_1 - c_5) - b_1 b_4 b_5 + b_2 b_3 b_6 > 0.
\end{aligned}$$

Here, the notations p, q, r and R are taken from [30]. The characteristic equation of the system (7) (which is equivalent to system (2)), can be written as:

(10) $$\chi^3 + p\chi^2 + q\chi + r = 0.$$

Here, we note that, the five fixed points at bifurcation parameters c_1 and/or c_5 are always from *saddle-focus* type (and not from saddle type as is noted in [39] and many another papers) – negative real eigenvalue and complex eigenvalues with positive real part (unstable focus) (see Fig. 3 – left panel), and positive real eigenvalue and complex eigenvalues with negative real part (stable focus) (see Fig. 3 – right panel). These fixed points can be included in homoclinic and heteroclinic structures with two, three, four and five equilibriums of
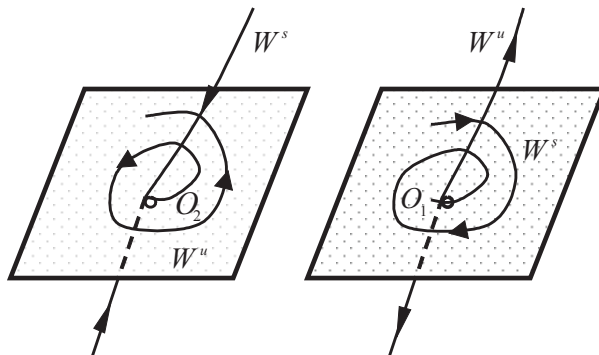
Fig. 3. Fixed points from saddle-focus type of system (2) with: unstable focus (complex eigenvalues with positive real part) – left panel and stable focus (complex eigenvalues with negative real part) – right panel

Shilnikov type, where their invariant manifolds W^s and W^u, are meeting each other in a most intricate manner.

For $c_1 = c_3 = 0.4$, $c_2 = 10$, $c_4 = c_6 = 5$ and $c_5 = 0.3$ the equilibriums and their eigenvalues are given by:

$O_1 = (0, 0, 0)$,   then $(\chi_1, \chi_2, \chi_3) = (0.3, -0.4 \pm i)$;
$O_2 = (-0.0413, 0.1602, -0.1103)$,   then $(\chi_1, \chi_2, \chi_3) = (-0.8, 0.15 \pm 1.1419i)$;
$O_3 = (0.0413, -0.1602, -0.1103)$,   then $(\chi_1, \chi_2, \chi_3) = (-0.8, 0.15 \pm 1.1419i)$;
$O_4 = (0.3129, 0.0403, 0.2103)$,   then $(\chi_1, \chi_2, \chi_3) = (-0.8, 0.15 \pm 1.5831i)$;
$O_5 = (-0.3129, -0.0403, 0.2103)$,   then $(\chi_1, \chi_2, \chi_3) = (-0.8, \ 0.15 \pm 1.5831i)$;

A heteroclinic cycle is one of the common scenarios of the formation or death of a limit cycle, when the limit cycle emanates from, or approaches the heteroclinic cycle as a singular limit, respectively [15, 31]. There are known cases in which a unique limit cycle is born and certain criteria can be used to determine if this cycle must be stable or unstable. In our case here, the known analytical results are not applicable [32, 33], and we are forced to use numerical simulations and specific features of our system. Hence, we find new results for its bifurcation behaviour and routes to chaos.

3. Shilnikov criteria for chaos

In Section 2, we obtained that system (2) has fixed points from saddle-focus type. Therefore, in this case the Shilnikov theory for analyzing of bifurcations of homoclinic/heterclinic orbits and existence of chaos in flow in R^3,

can be used [19, 34-36]. Below, we explain shortly the general results obtained by Shilnikov.

We focus our considerations on bifurcations from homoclinic and heteroclinic orbits between saddle-foci equilibrium in a three dimensional autonomous system:

$$(11) \qquad \dot{x} = \frac{dx}{dt} = f(x, \mu), \quad x \in R^3, \quad \mu \in R^1,$$

where the nonlinearity f is sufficiently smooth for the results to hold. The Shilnikov criteria for saddle-focus equilibrium are summarized in the following theorem [36-38]:

Theorem. *Suppose that system* (11) *has at* $\mu = 0$ *a saddle-focus equilibrium point* $x_0 = 0$ *with eigenvalues* $\chi_1(0) > 0 > \text{Re} \, \chi_{2,3}(0)$ *and a homoclinic orbit* Γ_0. *Assume the following genericity conditions:*

$(H_1) \qquad \sigma_0 = \chi_1(0) + \text{Re} \, \chi_{2,3}(0) < 0;$

$(H_2) \qquad \chi_2(0) \neq \chi_3(0);$

$(H_3) \qquad \beta'(0) \neq 0,$ *where* $\beta(\mu)$ *is the split function;*

$(H_4) \qquad \sigma_0 = \chi_1(0) + Re\chi_{2,3}(0) > 0.$

Hence, (*i*) *if the conditions* $(H_1 - H_3)$ *are valid, then system* (11) *has a unique and stable limit cycle* L_β *in a neighborhood* U_0 *of* $\Gamma_0 \cup x_0$ *for all sufficiently small* $\beta > 0$; (*ii*) *if the conditions* H_2 *and* H_4 *are valid, then system* (11) *has an infinite number of saddle limit cycle in a neighborhood* U_0 *of* $\Gamma_0 \cup x_0$ *for all sufficiently small* $|\beta|$.

Remarks (about part (i) of Theorem):

1. For all sufficiently small $\beta \leq 0$, the system (11) has no periodic orbits in U_0 and the unstable manifold $W^u(x_0)$ tends to the cycle L_β;

2. For $\beta = 0$, the cycle period tends to infinity.

Remarks (about part (ii) of Theorem):

1. For β taking positive or negative values, an infinite number of bifurcations occur, as some of these bifurcations are related to a "basic" limit cycle;

2. The "basic" cycle disappears and appears via tangent/fold bifurcation infinity many times. Moreover, the cycle also exhibits an infinite number of period-doubling bifurcations;

3. The "basic" cycle (as well as the secondary cycles generated by period-doublings) can be stable or repelling, depending on the sign of the divergence of (11) at the saddle-focus, i. e.:

$$(12) \qquad \sigma_1 = (\text{div} f)(x_0, \ 0) = \chi_1 + 2Re\chi_{2,3}.$$

If $\sigma_1 < 0$, the "basic" cycle near the bifurcation is stable - there are only short intervals of β, within which it is stable. If $\sigma_1 > 0$, there are intervals where the "basic" cycle is absolutely unstable (repelling);

4. For $\beta_i \to 0$ $(\beta_i > 0)$, the system (11) has double homoclinic orbits with different (increasing) number of rotations near the saddle-focus.

Initially it was assumed that $n_- = \dim W^s = 2$ and $n_+ = \dim W^u = 1$. We have to reverse the direction of time, in order to apply the above results in the opposite case - $n_- = 1, \quad n_+ = 2$, Hence, the following substitutions are valid: $\chi_j \to -\chi_j, \quad \sigma_i \to -\sigma_i$ and "stable" $\to$ "repelling".

4. Numerical analysis

In the previous two sections, we obtained and showed some analytical results, that we shall use in our numerical analysis in the system (2). According to [28, 39], the corresponding values of the dimensionless parameters $c_1 \div c_6$ are:

$$(13) \qquad \begin{array}{llll} c_1 \in [0.4, 0.9], & c_2 = 10, & c_3 \in [0.4, 0.9], \\ c_4 = 5, & c_5 \in [0, 0.9], & c_6 = 5. \end{array}$$

In order to compare the predictions with the numerical results, the governing equations of system (2) were solved numerically using MATLAB [40]. The initial conditions for all simulations are $(0.349, 0, -0.16)$ or $(0.349, 0, -0.18)$.

Figure 4 shows the bifurcation diagram for system (2): values of M_2 coordinate, $(M_2)_n$ are plotted against c_5 regarded as a continuously varying bifurcation (control) parameter. We see that at $c_5 \in [0.181, 0.2]$ the system (2) has chaotic solution – two strange attractors in the regions of two heteroclinic cycles. It is interesting to note that after $c_5 = 0.2$ (till the end of the interval), the inverse period-doubling bifurcations occur and the system passes from chaotic regime to a regular one. For $c_5 \in [0.28, 0.7]$, the system (2) has periodic solution with period three.

We also conclude, that an apparent sudden collapse in the chaotic attractor size occurs at a value of the control parameter $c_5 \approx 0.149$. Such a sudden qualitative change in a chaotic attractor is known as interior crisis [41,
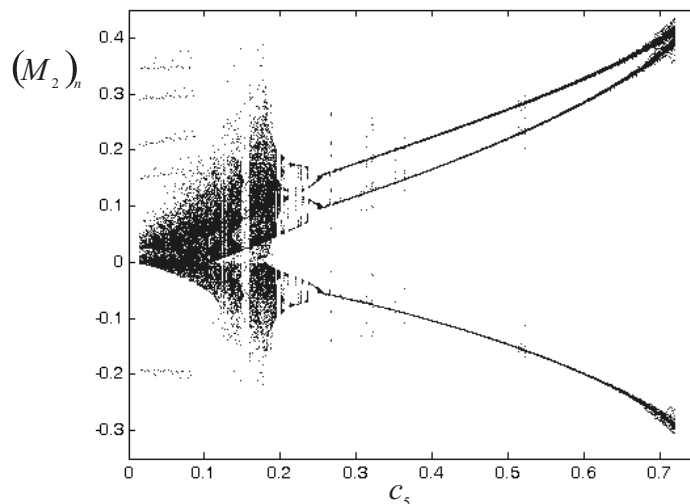
Fig. 4. Bifurcation diagram $(M_2)_n$ versus c_5, generated by computer solutions of the system (2), computed with the parameters:$c_1 = c_3 = 0.4$, $c_2 = 10$, $c_4 = 5$, $c_5 \in [0.01, 0.78]$, $c_6 = 5$

42]. This conclusion is in accordance with the Theorem explained in previous Section 3 and the results in the Appendix– see Figs 7 and 8. For $c_5 \in [0.15, 0.18]$ the system has one homoclinic cycle (for fixed point O_1) and two heteroclinic cycles – $\Gamma_0^{2,3} \subset W^{u(s)}(O_2) \cap W^{s(u)}(O_3)$; $\Gamma_0^{4,5} \subset W^{u(s)}(O_4) \cap W^{s(u)}(O_5)$. For details, see Appendix Fig. 11. From dynamical point of view, as c_5 increases, the fixed points $O_2 - O_5$ change the position (see Fig. 2 a). The saddle-foci $O_2 - O_5$ remove from O_1 and its unstable manifolds do not influence on the saddle-focus O_1, i.e. on the stability of the "basic" cycle. This excludes the possibility of a heteroclinic cycle for any finite c_5 larger than 0.4.

In Fig. 5, the bifurcation diagram of system (2) (as $c_1 \in [0.4, 0.88]$) is shown. It can be seen, that at $c_1 > 0.42$ chaotic solution with two strange attractors occurs. The white zones, seen in Fig. 6, correspond to inverse bifurcations, which are very fast. At about $c_5 \in [0.4, 0.419]$, a single strange attractor takes place, which at the end of the interval bifurcates into two strange attractors. Comparing results presented in Figs 4, 5 and 6, we conclude that the case, when c_1 is a bifurcation parameter, is more interesting from dynamical point of view than those with bifurcation parameter c_5. Discussing the results shown in Fig. 6, it is seen that for $c_5 \in [0.422, 0.428]$ and $c_5 \in [0.758, 0.763]$ the system has periodic solutions and homoclinic cycle (for fixed point O_1) and
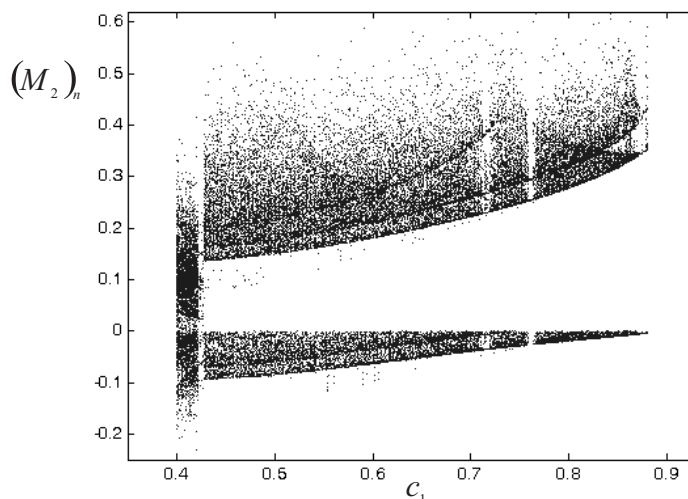
Fig. 5. Bifurcation diagram $(M_2)_n$ versus c_1, generated by computer solutions of the system (2), computed with the parameters: $c_1 \in [0.4, 0.88]$, $c_3 = 0.4$, $c_2 = 10$, $c_4 = 5$, $c_5 = 0.175$, $c_6 = 5$

two heteroclinic cycles - $\Gamma_0^{2,3} \subset W^{u(s)}(O_2) \cap W^{s(u)}(O_3)$; $\Gamma_0^{4,5} \subset W^{u(s)}(O_4) \cap W^{s(u)}(O_5)$, respectively – see Figs 8 and 9 in Appendix.

It is interesting, that in the second interval (see Fig. 6 right panel) the one strange attractor disappears/appears suddenly, as one heteroclinic cycle collapse.

5. Summary and conclusions

An important feature of robust heteroclinic cycles is that they may attract nearby dynamics. What happens, when a cycle loses stability? Such bifurcation may lead to the appearance of long period periodic orbits, other heteroclinic cycles, and more complicated dynamics.

The present paper studies numerically how the dynamics and global behaviour of system (2) vary, when we keep $c_2 = 10$, $c_3 \in [0.4, 0.9]$, $c_4 = 5$, $c_6 = 5$ and change c_1 and c_5. We focused our estimations on the bifurcation behaviour, route to chaos and occurrence of homoclinic and heteroclinic structures (cycles). It is curious, that system (2), which is a purely mechanical interpretation of the Lorenz system, displays some more interesting, from dynamical point of view, behaviours. Our new results suggest that the system
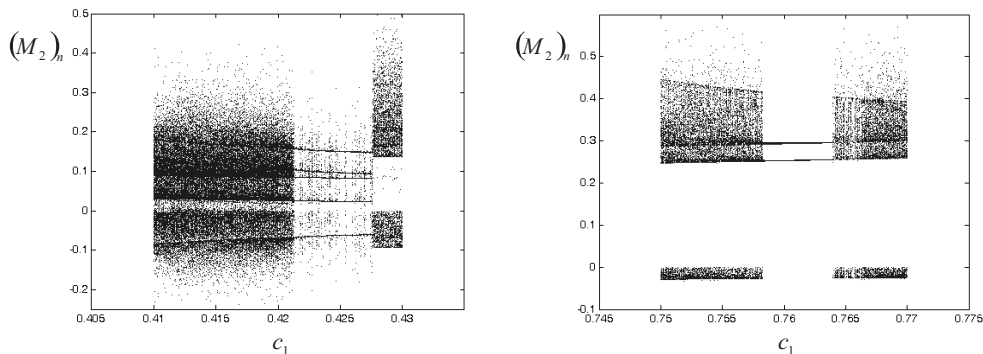
Fig. 6. Bifurcation diagrams of system (2), computed with the parameters:
$c_1 \in [0.41, 0.43]$ – left panel and $c_1 \in [0.75, 0.77]$ – right panel

(2) possesses: (i) five unstable fixed points from saddle-focus type; (ii) a homoclinic structure for fixed point O_1; (iii) two heteroclinic structures $\Gamma_0^{2,3}$ and $\Gamma_0^{4,5}$, including two fixed points; (iv) homoclinic and heteroclinic orbits can exist at the same time. In the case, when the system possesses two strange attractors, two heteroclinic structures, including two fixed saddle-focuses take place, as for $c_5 \in [0.758, 0.763]$ the one strange attractor disappears/appears suddenly. In this case, the corresponding heteroclinic cycle collapses.

According to heteroclinic Shilnikov criteria, if system (11) has heteroclinic orbits of Shilnikov type, which connect two distinct saddle-foci of the system, then it has both Smale horseshoes and horseshoe type chaos.

In conclusion, concerning the system evolution and bifurcation scenarios, we find new results for its routes to chaos.

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Appendix

Additional figures
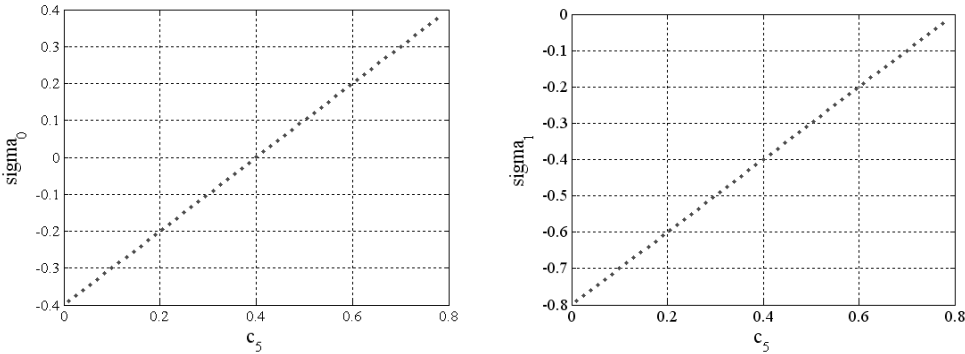

Fig. 7. Results for σ_0 and σ_1 of equilibrium O_1, as functions of control parameter c_5. According to Theorem, there are intervals where the basic cycle is stable
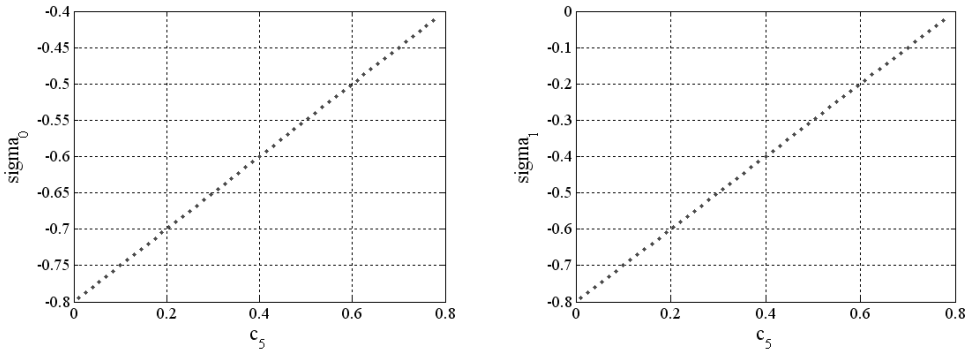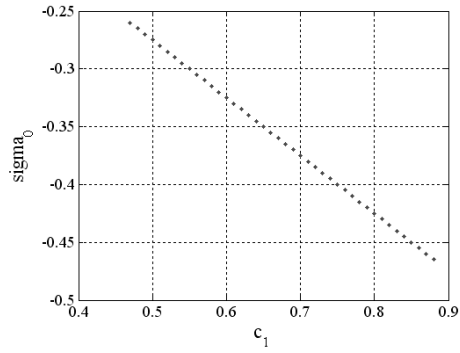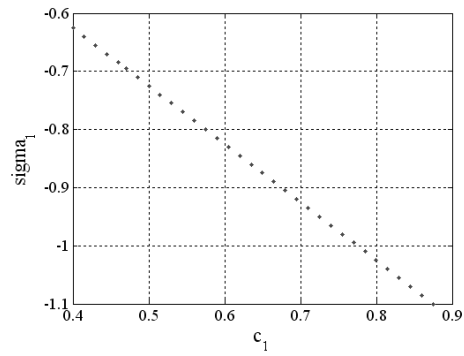

Fig. 8. Results for σ_0 and σ_1 of equilibrium O_2, as functions of control parameter c_5. According to Theorem, the system has infinite number of saddle limit cycles, and there are intervals, where the basic cycle is absolutely unstable (repelling). Here, we note that results for O_3, O_4 and O_5 are similar

Fig. 9. Results for σ_0 of equilibrium O_1, as functions of control parameter c_1. According to Theorem, there are intervals where the one basic cycle is stable
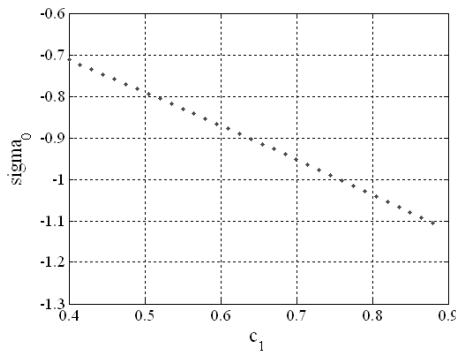

Fig. 10. Results for σ_0 and σ_1 of equilibrium O_2, as functions of control parameter c_1. According to Theorem, the system has infinite number of saddle limit cycles, and there are intervals where the basic cycle is absolutely unstable (repelling). Here, we note that results for O_3, O_4 and O_5 are similar
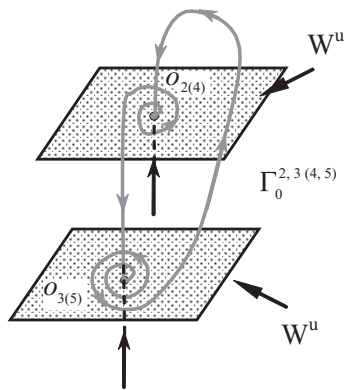

Fig. 11. Heteroclinic structure (cycle) from two saddle-foci