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BIFURCATION ANALYSIS AND DYNAMIC BEHAVIOUR OF AN INVERTED PENDULUM WITH BOUNDED CONTROL*

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ABSTRACT. This paper presents an investigation on the behaviour of conventional inverted pendulum with an inertia disk in its free extreme. The system is actuated by means of torques applied to the disk by a DC motor, mounted on the pendulum's arm. Thus, the system is underactuated since the pendulum can rotate freely around its pivot point. The dynamical model is given with three ordinary nonlinear differential equations. Using Poincare-Andronov-Hopf's theory, we find a new analytical formula for the first Lyapunov's value at the boundary of stability. It enables one to study in detail the bifurcation behaviour of the above dynamic system. We check the validity of our analytical results on the first Lyapunov's value by numerical simulations. Hence, we find some new results. KEY WORDS: Inverted pendulum, bounded control, bifurcation analysis, dynamic behaviour.

1. Introduction

Dynamical systems are families of motions, determined by evolutionary processes, the evolution of which takes place over time. These systems are

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determined in a variety of ways. Perhaps, the most commonly encountered examples are those which are determined by models of first order ordinary differential equations. The term "dynamical system" first appears in mechanics, where it referred to a mechanical system with a finite number of degree of freedom.

For some critical parameters in the mechanical and mathematical models, significant changes in the properties of the solutions occur when they are modified around critical values called 'bifurcation points'. The dynamical properties of the investigated system can vary, when such modification happen and sustained oscillations can emerge from stable steady states or inversely, oscillations can disappear as damped time limited oscillations, that lead to steady states.

Bifurcation theory describes qualitative changes in phase portraits, that occur as parameters, are varied in the definition of a dynamical system [1]. The computation of Lyapunov quantities (values) is one of the central problems in considering of small limit cycles in the neighbourhood of equilibrium of two-, three-, and four-dimensional dynamical systems [2–7]. These values are also substantial in investigation of the different problems from engineering mechanics, when the corresponding dynamical systems are close to the boundary of stability domain of its parameters. According to [8], the safe (invertible) and dangerous (non-invertible) boundaries take place. Such boundaries correspond to scenario of "soft" and "hard" excitation of oscillations, which was firstly considered by Andronov [9].

Dynamic behaviour and stabilization of the inverted pendulum is one of the most interesting problems in modern nonlinear science [10, 11]. Here, we conduct an investigation of the dynamical behaviour of an inverted pendulum with bounded control (see Fig. 1). The system is of interest both from the point of view of dynamically rich structure that it possesses and also because it is fairly easy to construct a mechanical model of it.

Several investigators have sought to develop nonlinear dynamical models of the inverted pendulum [12, 13]. It is found, that the inverted state stabilizes via alternating 'reverse' subcritical pitch-fork and period-doubling bifurcations, while it destabilizes via 'normal' supercritical period-doubling and pitch-fork bifurcations. On the other hand, in [14] Lozano et al. presented a stabilization of the inverted pendulum around its homoclinic orbit.

In the recent years, the research is focused towards obtaining control algorithms for general underactuated nonlinear mechanical systems. An Underactuated Mechanical System (UMS) is this which possesses fewer control inputs than degrees of freedom [15, 16]. UMSs have been arosen in many different real-life applications, such as: helicopters, aircrafts, spacecrafts, vertical take-off and landing aircrafts, under water vehicles, mobile robots, walking robots and so on. In some UMSs, the lack of action on certain directions can be interpreted as constrains on the acceleration. Some of these systems represent academic benchmarks and are part of a standard control laboratory, like the inverted pendulum, the rotational inverted pendulum, the pendubot [17], the pendulum driven by a spinning wheel. Linear systems can be rendered passive via smooth state feedback and they extended a number of stabilization schemes for global asymptotic stabilization of certain classes of nonlinear systems. The main contribution of the above systems is to exploit their passivity properties to develop appropriate control laws [18].

2. Problem formulation

In order to balance the inverted pendulum at the upper equilibrium position, the control must eventually be switched to a controller, that guarantees (local) asymptotic stability of this equilibrium.

Here, we are interested in the dynamical behaviour of the inverted (inertial wheel) pendulum with bounded control. The system is actuated by means of torques applied to the disk by a DC motor, mounted on the pendulum's arm [19, 20]. The system is underactuated since the pendulum can rotate freely around its pivot point – Fig. 1. In this case (according to [21]), the differential



Fig. 1. The inverted underactuated pendulum

equations of the motion of the pendulum have the form:

(1)
$$\begin{aligned} \varphi_{1} &= \varphi_{2}, \\ \dot{\varphi}_{2} &= q_{1} \sin \varphi_{1} + q_{2} \varphi_{3} - q_{3} u, \\ \dot{\varphi}_{3} &= -q_{1} \sin \varphi_{1} - q_{2} \left(1 + \rho\right) \varphi_{3} + q_{3} \left(1 + \rho\right) u, \end{aligned}$$

where φ_1 is the angle of the arm ($\varphi_1 = 0$ at the upright position), φ_2 is the angular velocity of the arm, φ_3 is the angular velocity of the disk with respect to arm, u is the control input (voltage applied to the motor), and q_1, q_2, q_3 and $\rho > 0$ are constant coefficients, derived from physical parameters [21]. Here, we note that the disk position is not considered as a stable variable, because it is irrelevant for the stabilization of the pendulum in the inverted position.

The control objective is the stabilization of the inverted state of the pendulum, i.e. when the disk velocity is zero. Thus, according to [21], we consider an auxiliary control in the form:

(2)
$$u = \alpha_1 \sin \varphi_1 + \alpha_2 \varphi_2 + \alpha_3 \varphi_3,$$

where α_1, α_2 and α_3 are control gains. It is well-know, that function $\sin \varphi_1$ can be written in Taylor series, i.e.: -1

(3)
$$\sin \varphi_1 = \varphi_1 - \frac{1}{3!}\varphi_1^3 + \frac{1}{5!}\varphi_1^5 - \cdots$$

If we take only the first and the second terms from (3), and after substitution of (2) into (1), Eq. (1) takes the form:

(4)

$$\dot{\varphi}_{1} = \varphi_{2}, \\
\dot{\varphi}_{2} = k_{1}\varphi_{1} - k_{2}\varphi_{2} + k_{3}\varphi_{3} - k_{4}\varphi_{1}^{3}, \\
\dot{\varphi}_{3} = -k_{5}\varphi_{1} + k_{6}\varphi_{2} - k_{7}\varphi_{3} + k_{8}\varphi_{1}^{3},$$

. .

where:

$$k_{1} = q_{1} - \alpha_{1}q_{3}, \quad k_{2} = \alpha_{2}q_{3}, \quad k_{3} = q_{2} - \alpha_{3}q_{3}, \quad k_{4} = \frac{1}{6}(q_{1} - \alpha_{1}q_{3}),$$
(5) $k_{5} = q_{1} - \alpha_{1}q_{3}(1+\rho), \quad k_{6} = \alpha_{2}q_{3}(1+\rho), \quad k_{7} = (1+\rho)(q_{2} - \alpha_{3}q_{3}),$
 $k_{8} = \frac{1}{6}[q_{1} - \alpha_{1}q_{3}(1+\rho)].$

These three dimensional nonlinear ordinary differential equations are autonomous (time does not appear explicitly).

The paper is organized as follows: in Section 3, we obtain the analytical results for first Lyapunov value, when control gains are varied. In Section 4, we present the numerical results. Finally, Section 5 summarizes our results.

3. Qualitative analysis

It is easy to see, that the equilibrium (steady states) values of the system (4) are: (1) (1)

(6)
$$\overline{\varphi}_1^{(1)} = \overline{\varphi}_2^{(1)} = \overline{\varphi}_3^{(1)} = 0,$$

(7)
$$\overline{\varphi}_1^{(2,3)} = \pm \sqrt{6}, \quad \overline{\varphi}_2^{(2,3)} = \overline{\varphi}_3^{(2,3)} = 0$$

We must accomplish some transformations of system (4), before determining analytically first Lyapunov value (L_1) and Routh-Hurwitz conditions for stability of fixed points (6) and (7). First, we express (4) in the form:

(8)
$$\ddot{\varphi}_1 = -c_1\ddot{\varphi}_1 + c_2\dot{\varphi}_1 + c_3\varphi_1 + c_4\varphi_1^3 - c_5\varphi_1^2\dot{\varphi}_1,$$

where:

(9)
$$c_1 = k_2 + k_7, \quad c_2 = k_1 + k_3 k_6 - k_2 k_7, \\ c_3 = k_1 k_7 - k_3 k_5, \quad c_4 = k_3 k_8 - k_4 k_7, \quad c_5 = 3k_4$$

Let us denote:

(10)
$$y = \varphi_1 - \varphi_0.$$

After substitution of (10) into (8), Eq. (8) takes the form:

(11)
$$\ddot{y} = -c_1 \ddot{y} + (c_2 - c_5 \varphi_0^2) \dot{y} + (c_3 + 3c_4 \varphi_0^2) y$$

 $- 2c_5 \varphi_0 y \dot{y} + 3c_4 \varphi_0 y^2 + c_4 y^3 - c_5 y^2 \dot{y}.$

Here, φ_0 is the equilibrium states $\overline{\varphi}_1^{(2)} = \sqrt{6}$ or $\overline{\varphi}_1^{(3)} = -\sqrt{6}$. It is seen, that in linear terms of (11) we have φ_0^2 , therefore, we can conclude that boundary of stability (and qualitative behaviour also) for $\overline{\varphi}_1^{(2,3)}$ will be one and the same.

On the other hand, let:

After substitution of (12) into (11), the third-order ordinary differential equation (11) is reduced to three first-order differential equations:

$$\dot{w}_1 = -c_1 w_1 + c_6 w_2 + c_7 w_3 - c_8 w_2 w_3 + c_9 w_3^2 + c_4 w_3^3 - c_5 w_2 w_3^2, \dot{w}_2 = w_1, \dot{w}_3 = w_2,$$

where:

(14)
$$c_6 = c_2 - c_5 \varphi_0^2, \quad c_7 = c_3 + 3c_4 \varphi_0^2, \\ c_8 = 2c_5 \varphi_0, \quad c_9 = 3c_4 \varphi_0.$$

The flow divergence (13) is:

(15)
$$D_3 = \frac{\partial \dot{w}_1}{\partial w_1} + \frac{\partial \dot{w}_2}{\partial w_2} + \frac{\partial \dot{w}_3}{\partial w_3} = -k_2 - k_7.$$

The system (13) is dissipative and has an attractor, when $D_3 < 0$, i.e. $\alpha_2 \ge \alpha_3 (1 + \rho)$.

Following [8], the Routh-Hurwitz conditions for stability of (7) can be

written in the form:

(16)
$$p = c_1 = k_2 + k_7 > 0,$$

(17)
$$q = -c_6 = k_2 k_7 - k_1 - k_3 k_6 + 3k_4 \varphi_0^2 > 0,$$

(18)
$$r = -c_7 = -c_3 - 3c_4\varphi_0^2 > 0$$

(19) $R = pq - r = -c_1c_6 + c_7 > 0.$

The notations p, q, r and R are taken from [8]. The boundaries of the stability region, according to [2, 8] for three dimensional dynamical system, are two surfaces: $\Psi_1 (R = 0, p > 0, q > 0)$ and $\Psi_2 (r = 0, p > 0, q > 0)$. On the surface R = 0, the characteristic equation has a pair of purely imaginary roots and here, an Andronov-Hopf bifurcation takes place, and at least one zero root on the surface r = 0.

From an analytical point of view, the onset of sustained oscillations generally corresponds to a transition through an Andronov-Hopf bifurcation point. Beyond the bifurcation point, a stable solution arises in the form of a small amplitude limit cycle, surrounding the unstable steady state. This bifurcation can be of two types: (i) supercritical (soft loss of stability) and (ii) subcritical (hard loss of stability).

When the condition (19) is not valid, the steady state (7) becomes unstable. In order to characterize the nature of the bifurcation points in the parameters α_1 , α_2 and α_3 of our model (13), the so-called first Lyapunov value, L_1 , must be computed and analyzed for the system (13) [2, 3, 8, 9, 22]. In the bifurcation points, a positive first Lyapunov value (this is not Lyapunov exponent- see appendix in [23] or for a detailed discussion [2, 9, 22]), represents a hard (subcritical or irreversible) bifurcation and predicts, that the system presents unstable solutions, but may fold back and exhibit unstable periodic oscillations coexisting with stable steady states. In this case, the boundary of stability is considered as "dangerous". In contrast, a negative value for L_1 indicates a soft (supercritical or reversible) bifurcation. Here, the loss of stability takes place when stable self-oscillations emerge from a transition through the bifurcation point. In this case, the boundary of stability is considered "safe".

We summarize the main analytical results for first Lyapunov value here. Further details about the derivations of the formulae can be found in the Appendix. In the case of three first-order autonomous nonlinear differential equa-

tions, this value can be determined analytically by the formula in [8]:

$$L_{1}(\lambda_{0}) = \frac{\pi}{4q} \left[2 \left(A_{33}^{(2)} A_{33}^{(3)} - A_{22}^{(2)} A_{22}^{(3)} \right) + 2A_{23}^{(2)} \left(A_{22}^{(2)} + A_{33}^{(2)} \right) - -2A_{23}^{(3)} \left(A_{22}^{(3)} + A_{33}^{(3)} \right) + 3 \sqrt{q} \left(A_{222}^{(2)} \right) + A_{333}^{(3)} + A_{233}^{(2)} + A_{223}^{(3)} \right] + (20) \qquad + \frac{\pi}{4p\sqrt{q} \left(p^{2} + 4q \right)} \left\{ p^{2} \left[2A_{22}^{(1)} \left(3A_{12}^{(2)} + A_{13}^{(3)} \right) + 2A_{33}^{(1)} \left(A_{12}^{(2)} + 3A_{13}^{(3)} \right) + + 4A_{23}^{(1)} \left(A_{13}^{(2)} + A_{12}^{(3)} \right) \right] + 4p\sqrt{q} \left[\left(A_{22}^{(1)} - A_{33}^{(1)} \right) \left(A_{13}^{(2)} + A_{12}^{(3)} \right) + + 2A_{23}^{(1)} \left(A_{13}^{(3)} - A_{12}^{(2)} \right) \right] + 16q \left(A_{22}^{(1)} + A_{33}^{(1)} \right) \left(A_{12}^{(2)} + A_{13}^{(3)} \right) \right\},$$

where λ_0 is defined as a value of α_1 , α_2 , α_3 , q_1 , q_2 , q_3 and ρ for which the relation R = 0 takes place. The coefficients A_{ij}^n and A_{ijk}^n (i, j, k, n = 1, 2, 3) are defined by corresponding formulas, presented in [8]. According to [3, 8, 23], after accomplishing some transformations and algebraic operations, we obtain the formula describing the first Lyapunov value for the model investigated and calculate its value in the computed bifurcation points (for further details see Appendix).

We focus our analysis on the effect of changes on the bifurcation parameters α_1 , α_2 and α_3 , which represent the control gains of the system. According to experiment work [21], we assume the following intervals of mechanically relevant values for α_1 , α_2 and α_3 : $\alpha_1 \in [15, 400]$, $\alpha_2 \in [13, 50]$, $\alpha_3 \in [0.1, 0.5]$.

In Fig. 2, $L_1(\lambda_0)$ (calculated on the boundary of stability R = 0 in Eq. (20); for further details, see Appendix) is shown for different values of the bifurcation parameters α_1 , α_2 and α_3 . It can be seen that $L_1(\lambda_0)$ passes through regions for which it is negative or positive. Therefore, following the terms introduced in [8, 9], we have a soft and a hard stability loss. Unfortunately, Fig. 2 (left panel) gives only a most general idea about the change of L_1 . A more detailed investigation of $L_1(\lambda_0)$ is shown in the following Fig. 2 (right panel). What could one observe in the figure? Practically, for different values of the parameter α_2 we have different kind boundary of stability. For example, the boundary of stability R = 0 (when $\alpha_2 = 13$ and $\alpha_2 = 50$) passes through different regions of (α_1, α_3) space. In other words, different kind of dynamical behaviour is possible. Hence, we can conclude, that the analysis via Lyapunov-Andronov theory and first Lyapunov value helps us describe and predict the qualitative behaviour of the system for values of α_1 , α_2 and α_3 , higher or smaller than the critical values, established approximated intervals of values for α_1 , α_2 and α_3 in which self-oscillations of the system occur (disappear). However, the precise features of such solutions must be investigated via numerical simulation.

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Here, we note, that the bifurcation theory of control systems focuses on the change of its qualitative properties, such as controllability, stabilizability and the topology of the equilibrium set. In this paper, the second problem is generally investigated. In particular, it is seen in Fig. 2, that only in some bounded (but large) control parameter intervals the limit cycle is stable, i. e. the control of oscillations is bounded effective.

In summary, the results obtained for first Lyapunov value (see Fig. 2) are key because we have defined a computable, finite region of the $\alpha_1 - \alpha_2 - \alpha_3$ space in which sustained oscillations can occur after "soft" or "hard" stability loss and in which we can search for oscillatory solutions. In the following section, we demonstrate numerically different types of behaviour for some particular values of the model parameters.

4. Numerical simulations

In this section, we perform a numerical analysis of the model (4), based on the our previous analytical results obtained in Section 3. The parameter values used in the numerical calculations were selected according to [21], i. e.



Fig. 2. Results for first Lyapunov value L_1 as function of parameters $\alpha_1 \in [15, 400]$, $\alpha_2 \in [13, 50]$ and $\alpha_3 \in [0.1, 0.5]$. It is seen that boundary of stability R = 0 can be 'safe' and 'dangerous' because L_1 is negative or positive. Here we note that the another system parameters are fixed and are: $q_1 = 30$, $q_2 = 0.0245$, $q_3 = 0.0393$, $\rho = 250$

we used the following values for the system parameters:

(21)
$$\begin{array}{l} q_1 = 30 \, \left[{\rm s}^{-2} \right], & q_2 = 0.0245 \, \left[{\rm s}^{-1} \right], & q_3 = 0.0393 \, \left[{\rm s}^{-1} \right], \\ \rho \in \left[{249.4,\ 250.1} \right], & \\ \alpha_1 \in \left[{15,\ 400} \right] \, \left[{\rm s}^{-1} \right], & \alpha_2 \in \left[{13,\ 50} \right], & \alpha_3 \in \left[{0.1,\ 0.5} \right]. \end{array}$$

Here, we note that α_2 , α_3 and ρ are dimensionless parameters and in all our simulations time is in seconds. In Fig. 3, the stable solutions for the angle of the arm, φ_1 [rad], the angular velocity of the arm, φ_2 [s⁻¹], and the angular velocity of the disk with respect to arm, φ_3 [s⁻¹], are shown for $\alpha_1 = 75$, $\alpha_2 = 30$ and $\alpha_3 = 0.5$. It is evident, that after several physically acceptable fluctuations (see Fig. 3 a, b, c), φ_1 , φ_2 and φ_3 approach constant values (equilibrium state). In other words, in this case the conditions (16)–(19) from previous section are satisfied and the system (4) lies in stable zone of its parametric space. Here, we note that the governing equations of the model, represented by (4), were solved numerically using MATLAB [24] with initial



Fig. 3. Stable solutions and phase space of system (4). Time is in seconds

conditions $\varphi_1^0 = \varphi_2^0 = 0$, $\varphi_3^0 = 0.01$. In Fig. 3d, the phase space of the system (4) is shown.

In the next Fig. 4, the periodic solutions and the phase space of the system (4) are shown. In this case, the following values of the model parameters are accepted: $q_1 = 30$, $q_2 = 0.0245$, $q_3 = 0.0393$, $\rho = 250$, $\alpha_1 = 376$, $\alpha_2 = 30$, $\alpha_3 = 0.5$. Our simulation demonstrates that the pendulum inverse state is stabilized near equilibrium (6) and a homoclinic limit cycle occurs. The analysis of the results suggests that in this case the system possesses one unstable fixed point from kind saddle and two stable fixed points from kind focus. Here, we note that these results are in accordance with the analytical results obtained in previous Section 2 and those in [14, 21].

Figure 5 illustrates how the system (4) changes its dynamic behaviour as function of first Lyapunov value. Figure 5 a depicts the solution of the system



Fig. 4. Periodic solution and phase portrait of system (4). Time is in seconds. For details see text

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Fig. 5. (a) Solution of system (4) when the boundary of stability is 'safe'. Here $\alpha_1 = 89.03$, $\alpha_2 = 13.387$ and $\alpha_3 = 0.15$; (b) Solution of system (4) when the boundary of stability is 'dangerous'. Now $\alpha_1 = 159$, $\alpha_2 = 13.387$ and $\alpha_3 = 0.35$. For details see text

(4) in the case when the boundary of stability is 'safe', i. e. $L_1(\lambda_0) < 0$. It is evident, that stable limit cycle with small amplitude occurs. Here, we note that system (4) in this case has reversible behaviour, i. e. a slow drift of the parameters back into the stability region brings a system back into the original response. On the other hand, Fig. 5b shows the case when the boundary of stability is dangerous (i. e. $L_1(\lambda_0) > 0$). This is the case when the system undergoes very fast to large fluctuations and transition through the stability boundary may result in disappearance of the equilibrium state. Therefore, the system is structurally unstable (un-robust).

5. Conclusions

The proposed study treats the problem of dynamical behaviour of a conventional inverted pendulum with an inertia disk in its free extreme is considered. The system is actuated by means of torques applied to the disk by a DC motor, mounted on the pendulum's arm. The system is underactuated since the pendulum can rotate freely around its pivot point. The results found in the present study are in broad agreement with those of [21]. However, some additional dynamical behaviours of the inverted pendulum were shown.

The investigation of this paper is concerned with the question of whether the change of control gains α_1 , α_2 and α_3 alters the dynamic properties (stabilization) of an inverted pendulum with bounded control. General conclusions about design principles and the role of control gains can be suggested, since this system is a classical case study, covering several essential features of inverted pendulum and nonlinear control mechanisms

Our investigation is devoted to the use of the bifurcation analysis [1, 2, 9], as a useful tool to investigate such state transitions when strong perturbation in the system parameters occurs. In a bifurcation point for a given parameter, the dynamical system is always structurally unstable and the stability conditions of limit cycles depend critically on stability conditions of steady states. We are able to define not only the structural stability (robustness) of the steady state, but also the robustness of limit cycles or other types of trajectories, by knowing the sign of the first Lyapunov value at the bifurcation boundary, as we showed in this work. This is a very central point of the bifurcation and stability analysis of steady states, periodic orbits and other types of trajectories. In our case, Lyapunov-Andronov theory were employed for the investigation of the original system (4) and the equivalent system (13). Hence, a new major conclusion of our analysis is the fact that in finite region of the $\alpha_1 - \alpha_2 - \alpha_3$ space the system (4) can have both "soft" and "hard" stability loss, as boundary of stability in first case is "safe" and in second one "dangerous", i. e. reversible and unreversible loss of stability take place.

As a result of our analysis, we found also that in the model (4) may induce the emergence of system configurations with two stable focuses and an unstable saddle point of first order. Under these conditions, the system displays bistability with homoclinic sustained oscillations. In our analysis, we have found at least one possibility to recover bistability; for details see Figs 6–8 in Appendix.

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Appendix

1. Calculation of the first Lyapunov value

In this paper, following [8], we calculate the first Lyapunov value. Generally in accordance with Lyapunov-Andronov theory, we have the following: (i) the sign of Lyapunov's value determines the character (stable or unstable) of equilibrium state at R = 0; (ii) the character of equilibrium state at R = 0qualitatively determines the reconstruction of phase space (including stability or instability of limit cycle) at the transition from R < 0 to R > 0.

After accomplishing some transformations and algebraic operations for the coefficients A_{ij}^n and A_{ijk}^n (i, j, k, n = 1, 2, 3) (into (20)), we obtain the following for the system (13):

$$A_{33}^{(n)} = A_{333}^{(3)} = A_{233}^{(2)} = 0, \quad A_{22}^{(n)} = \frac{1}{\Delta_0} \alpha'_{n1} a_{33}^{(1)}, \quad A_{23}^{(n)} = \frac{1}{\Delta_0} \alpha'_{n1} \alpha_{23} a_{23}^{(1)},$$

(A.1) $A_{12}^{(n)} = \frac{1}{\Delta_0} \alpha'_{n1} \left(a_{33}^{(1)} \alpha_{31} + a_{23}^{(1)} \alpha_{21} \right), \quad A_{13}^{(n)} = \frac{1}{\Delta_0} \alpha'_{n1} \alpha_{23} \alpha_{31} a_{23}^{(1)},$
 $A_{222}^{(2)} = \frac{1}{\Delta_0} \alpha'_{21} a_{333}^{(1)}, \quad A_{223}^{(3)} = \frac{1}{\Delta_0} \alpha'_{31} \alpha_{23} a_{233}^{(1)}.$

Here, we note that:

(A.2)
$$\begin{aligned} \alpha_{11} &= pc_7, \quad \alpha_{21} &= c_7, \quad \alpha_{31} &= p \left(p - c_4 \right) - c_6, \\ \alpha_{12} &= c_6, \quad \alpha_{32} &= 1, \quad \alpha_{23} &= -\sqrt{q}, \quad \alpha_{22} &= \alpha_{13} &= \alpha_{33} &= 0, \\ a_{23}^{(1)} &= -c_8, \quad a_{33}^{(1)} &= c_9, \quad a_{333}^{(1)} &= c_4, \quad a_{233}^{(1)} &= -c_5, \end{aligned}$$

and

(A.3)
$$\Delta_0 = \det \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{11}\alpha_{23}.$$

From the previous formula (A.3), we obtain that:

(A.4)
$$\alpha'_{11} = \alpha_{23}, \quad \alpha'_{21} = -\alpha_{23}\alpha_{31}, \quad \alpha'_{31} = \alpha_{21}.$$

2. Numerical results

Here, we analyze numerically the model constituted by (4). The governing equations of the model were solved numerically using MATLAB [17]. The corresponding numerical values of the model parameters are those in (21). The following Figs 6–8 demonstrate the dependence of the model behaviour on the parameter α_1 , when the another model parameters are fixed, i.e. $\alpha_2 = 30.387$, $\alpha_3 = 0.2$, $q_1 = 30$, $q_2 = 0.0245$, $q_3 = 0.0393$, $\rho = 250$. In other words these



Fig. 6. Stable solution of system (4) near fixed point with coordinate $\overline{\varphi}_1 = -\sqrt{6}$ when $\alpha_1 = 169.3$, $\alpha_2 = 30.387$ and $\alpha_3 = 0.2$. For more details see text



Fig. 7. Stable solution of system (4) near fixed point with coordinate $\overline{\varphi}_1 = -\sqrt{6}$ when $\alpha_1 = 169.5$, $\alpha_2 = 30.387$ and $\alpha_3 = 0.2$. For more details see text



Fig. 8. Stable solution of system (4) near fixed point with coordinate $\overline{\varphi}_1 = -\sqrt{6}$ with very long oscillation-like transient regime. Here, $\alpha_1 = 169.8$, $\alpha_2 = 30.387$ and $\alpha_3 = 0.2$. For more details see text

figures illustrate how the solution of the system (4) changes when parameter α_1 is equal to 169.3, 169.5 and 169.8. It is seen, that for first and second value the system has a stable solution, which is near different fixed point with coordinate $\overline{\varphi}_1 = -\sqrt{6}$ and $\overline{\varphi}_1 = \sqrt{6}$. Under these conditions, the system displays bistability. It can be seen also that in Fig. 8 the oscillation-like transient regime is very long. Therefore, in this case the system lies very close to the boundary region, between two stable zones of attraction of fixed points (7) [18, 19]. Here, we note that similar behaviour can be seen also for another values of α_1 in the interval from 15 to 400.