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Research Article

On the Solutions of Games in Normal Forms: Particular Models based on Nash Equilibrium Theory

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Abstract

The main objective of this paper is to present in a deductive way, solutions for general games played under normal conditions following competitive paths, applying core principles of Nash equilibrium. Here the normal approach implies strategic choices available for each player, formulated and implemented without any information concerning specific choices to be made by others players. It is convenient to keep in mind that John von Neumann and Oskar Morgenstern outlined a set of conditions for Nash equilibrium for a game in normal form, proposed as the basic framework to analyze the conditions and requirements for a particular Nash equilibrium to be the solution of the game. Theorems that exhibit imbedding relations among the Nash equilibriums of the game are given to examine the role of pre-play communication and the imbedding order in equilibrium selection. A core argument to claim here is that a generic case of Nash equilibriums that are strategically unstable relative to maxi-min strategies is given to emphasize the role of moves of the third kind and pre-play communication in correlated and coordinated solutions and the need to account for cases where Nash equilibriums are not plausible or even desirable as solutions for a game in normal form.

Keywords: *Stable sets, mixed strategy Nash equilibrium instability, maxi-min strategies*

1. Introduction

There are many convincing reasons that suggest that the solution to the non-cooperative has to be a Nash equilibrium. The self-enforcing characteristic of pure strategy Nash equilibriums is one of the major ones. However, many game theorists have observed that such enforcing characteristic is not always present when considering equilibria determined by mixed strategies.

Furthermore, in this paper we will exhibit games with multiple but conflicting Nash equilibria in pure strategies, leaving in some cases no other possibility than to chose a mixed strategy equilibrium as the solution of given game. But then, the possible mixed strategy Nash equilibrium left to be selected is shown to be unstable relative to maximin strategies.

In such cases we are faced with a situation where Nash Equilibrium is neither plausible nor desirable as a solution to the game and it is apparently outperformed by maximin strategies that appear as the only plausible and desirable solution. Thus, for games in normal form where the choice of strategy by each player is to be made without any information concerning the choice to be made by the other players, it appears to be necessary if not convenient to be able to display the interrelations among the different Nash equilibriums of the game.

In that case, it is important to contemplate conditions under which these may be reflected as solutions, as well. We will proceed first by looking at (1) the von Neumann stable set of outcomes which gives the Pareto-optimal set of outcomes of the game and we will introduce the concept of solutions that are "socially stable" in terms of maximum measures of global preference versus those that are strategically or individually stable.

This confrontation is representative of standards of behavior in conflict, namely the standard of individual optimization versus social optimization, where the assumption of one may eliminate the indeterminacy of the other one. Then we will concentrate on (2) the von Neumann stable set of Nash equilibriums (vN-NE).

That is the subset vN-NE of Nash equilibriums of the game such that the corresponding outcomes in utility space are not dominated by any other Nash equilibrium outcome and any Nash equilibrium outcome not in vN-NE is dominated by one in the stable set. The global preference measure is shown to be equivalent to the product of expected utilities (Nash product) not the often taken as social welfare to be the sum of these utilities.

Thus by assuming the global preference as a established standard of behavior in a society, most indeterminacy, undesirability and risk avoidance behaviors would be no longer the consequence of egotistic rational behavior and perhaps a more amenable and also rational society may be achieved.

2. Prime Nash Equilibrium Considerations

As an essential concept, and following Nax exposition (2015), game theory provides a sharp language to formulate mathematical models of underlying interactions that promise clean predictions, now integral parts of the social sciences toolbox.

From a general theoretical perspective, a game is defined by a mapping from various combinations of "strategies" taken by the involved "players" into resulting consequences in terms of "payoffs". A "solution" predicts which outcomes of the game are to be expected. A major issue with traditional/neoclassical game theory, however, has been that its solution concepts, such as the Nash equilibrium (John F. Nash, 1950) or the strong equilibrium (Robert Aumann, 1959), rely on four rather extreme behavioral and informational assumptions. These are:

1) The joint strategy space is common knowledge.
2) The payoff structure is common knowledge.
3) Players have correct beliefs about other players' behaviors and beliefs.
4) Players optimize their behavior so as to maximize their own material payoffs (see Nax, 2915, p. 3).

The number of mixed strategy Nash equilibriums of a game can be shown to be a function of the number of *prime Nash equilibriums.* These are the Nash equilibriums that are not generated by other Nash equilibriums and that can generate all other Nash equilibriums in the game. A pure strategy Nash equilibrium is always a prime Nash equilibrium. However, Nash Equilibriums in mixed strategies may be either prime or not prime.

We show every Nash equilibrium in a square bimatrix (nxn) game to be the composition of at most k prime Nash Equilibriums, with $n \geq k$. The number k of generators of Mixed strategy Nash equilibrium is the order of the equilibrium. Hence prime Nash equilibriums are all of order 1.

2.1 Mixed strategy Nash equilibrium instability

Suppose we have a game with two conflicting Nash equilibriums NE´ and NE´´ competing each to be the solution of a given bi-matrix game, with corresponding payoffs [u1´, u2´] and [u1´´, u2´´] , u1´ > u1´´ and u2´ < u2´´. Then if S = { prime generators of NE´} and T ={prime generators of N´´} then, the order of NE´ is O (NE´) = |S| and the order of NE´´ is O (NE´´) = |T| if there is no other criteria but Nash equilibrium then the solution would be the Nash equilibrium generated by the prime generators in S and in T and the order of this Nash equilibrium "solution" would be |S|+|T| - |S∩T|. Here we point out that the proposed solution may end up not being a reasonable solution. To illustrate this point consider the following:

Example1: Let Γ= (A,B) where $A = \begin{pmatrix} 7 & 1 \\ 6 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 9 & 6 \\ 1 & 7 \end{pmatrix}$. Then N1 = {(1, 1) ; [7, 9]} and N2 = {(2,2) ; [9, 7] } are equilibriums of order 1 and N12 = {((2/3, 1/3), (8/9, 1/9)^t); [6 1/3, 6 1/3] is a mixed Nash equilibrium of order 2.

We might be inclined to think here that in the absence a dominant Nash equilibrium and given the conflicting nature and symmetry of the two Nash equilibriums in pure strategies, the most natural solution would be the Nash equilibrium of order 2 as would be the case in a battle of the sexes case.

Thus, tentatively, let us consider these second order equilibrium as a possible solution: Let the mixed Nash equilibrium strategies be x° for player 1 and y° for player 2; and let x* and y* the corresponding maximin strategies for players 1 and 2 respectively. Then, x° = (2/3, 1/3) and y° = (8/9, 1/9)$^{t.}$ and x* = (1/3, 2/3) and y* = (1/9, 8/9)^t . Immediately, we may observe in the following table the strategic instability of the Nash equilibrium strategies relative to the maximin strategies:

	y* (Maxmin)		y° (NE)	
Maxmin) x*	6,33	6,33	6,33	4
(NE) x°	4	6,33	6,33	6,33

Figure 1. Nash Equilibrium Maxi-Min Strategies

Here, if both players use their maximin strategies independently of each other, they may secure a value of v = 6 1/3. The same value is obtained co-dependently if both use their Nash equilibrium strategies.

However if one of the players uses his maximin strategy and the other one uses his Nash equilibrium, the player using the Nash equilibrium will get an amount inferior to his maximin strategy value. Thus it appears that mixed strategy Nash equilibrium may suffer from strategic instability originated not by a player unilateral deviations, but by risk avoiding deviations of other player´s to other strategies in their strategy sets to which they remain indifferent in terms of value obtained but avoiding any risk of securing the Nash equilibrium obtainable value.

2.2 Conditions for complete stability of the mixed strategy Nash equilibrium

The above analysis suggest that for complete stability of the mixed strategy Nash equilibrium, the stability condition usually stated as : (1) $u_i\,(s^{\circ(i)}, s^{\circ(-i)}) \geq u_i\,(s^{\circ(i)}, s^{\circ(-i)})$ for all players i in N should be complemented with (2) $u_i\,(s^{\circ(i)}, s^{\circ(-i)}) \geq u_i\,(s^{\circ(i)}, s^{*(-i)})$ for all player i in N.

Here, $s^{*(-i)}$ includes, for at least one player, the corresponding maximin strategy. That is mixed Nash equilibrium strategies should be immune to maximin strategy deviations whenever the maximin strategies secure the same value that the mixed strategy Nash equilibrium provides.

It is especially interesting to note that in the 2-player bi-matrix game of example 1, rationalization arguments departing from the assumption that either one of the players is going to use his mixed maximin strategies will lead us to conclude that the only reasonable solution for this game is the Nash equilibrium in pure strategies determined by the maximin pure strategies for both players, namely N1: {(1,1), [7, 9]}.

This example, clearly invite us to rethink where do we stand in considering Nash equilibriums as the only possible solutions for the games in normal form. We note that in the battle of the sexes with pure strategy Nash equilibriums N1: {(1,1); [7, 9]} and N2: {(2,2) ; [9, 7] }, the mixed strategy Nash equilibrium N12 : { ((7/16, 9/16), (9/16, 7/16); [63/16, 63/16]} satisfies the stability condition (2) above and no pure strategies Nash equilibrium can be chosen as consequence of a rationalization process with maximin strategies assumed as point of departure.

2.3 Balanced Strategies and Nash equilibriums

Let A be an m by m non singular square matrix. The matrix A is said to be *row-bala[1]nced* if and only if there exists a positive vector of weights $\gamma^t = (\gamma_1,\ldots\gamma_m)$, $\gamma_i > 0$, i =1, …., m, such that $\gamma^t A = J^t$ where J is the m-dimensional vector of 1´s. Similarly, a matrix A is said to be column-balanced if there exist a positive m-vector $\eta = (\eta_1 \ldots\eta_m)^t$, $\eta_j > 0$,j = 1,…, m , such that $A\eta = J$. See Owen (1995) for similar strategies for 2-person zero sum games. For a concept similar to the one f balanced strategies see Pruzhanski (2011).

Theorem 1 Let Γ = (A,B) be the game in normal form represented by the bi-matrix (A,B) then if both A and B are non singular of dimensions m by m and A is col-balanced and B is row-balanced then there is a full supported mixed strategy Nash equilibrium NE: {(x°,y°); [v_1 , v_2]} where the strategies x° and y° are given respectively by x° = a γ, $\gamma^t = J^tA^{-1}$ with a = $1/\gamma^tJ$, and y° = bη, $\eta = B^{-1}J$ and b = $1/J^t\eta$. And v_1 = a, v_2 = b.

Given (A,B) as above in theorem 1.

Theorem 2 Then if both A and Bare row-balanced and column-balanced each then the maximin and minimax values of A are the same, namely v_A. So are those for B namely v_B.

Theorem 3 If both A and B are row-balanced and column-balanced each, then the full supported mixed strategy Nash equilibrium of the game Γ = (A, B) is given by NE: {(x°, y°) ; [v_A , v_B]} with x° = z* and y° = y* where z* is the minimax strategy for B and y* is the minimax strategy for A.

Theorem 4 The 2x2 sub-matrices subtended by any two pure strategy Nash equilibriums in a bi-matrix game Γ = (A, B) are (1) non-singular (2) row-balanced and column-balanced. They generate a mixed strategy Nash equilibrium. Further, the equilibrium value to the players is the same as that one secured by maximin strategies .and the equilibrium strategies are the cross minimax strategies. That is each player selects the minimax strategy of an imaginary third player in the other player´s payofff matrix.

Theorem 5 The 2x2 sub-matrices subtended by any two Nash equilibriums in a bi-matrix game Γ= (A, B) are (1) non-singular (2) row-balanced and column-balanced. They generate a mixed strategy Nash equilibrium. Further, the equilibrium value to the players is the same as that one secured by maxi-min strategies and the equilibrium strategies are the cross minimax strategies. That is each player selects the minimax strategy of an imaginary third player in the other player´s payoff matrix.

It follows that the sub-matrix subtended by any number of pure strategy Nash equilibriums in a bi-matrix game has the same properties as those of Theorem 4. The same can be said when we combine different Nash equilibriums pure or mixed but with independent support. Hence we may always derived different Nash equilibriums from different prime or composed Nash equilibriums as long as these later ones have independent strategic support.

With the theorems above we are able to construct all Nash equilibriums for small square bi-matrix games. For example, if we have A_{3x3} and B_{3x3} , suppose the game Γ= (A, B) has m =3 pure Nash equilibriums then we may obtain $\binom{3}{2} = 3$ mixed strategy Nash equilibriums of order 2 and $\binom{3}{3} = 1$ equilibrium or order 3 That is, $2^m - 1 - m$ mixed strategy Nash equilibriums for each m

[1] *The concept of balanced collection in Shapley (1967) and Bondareva (1963) for cooperative games is similar to the one distinguished here. Both concepts have decisive implications for the determination strategic equilibrium.*

pure strategy equilibriums.

Here it is convenient seeing Quint and Shubik(1994) for a conjecture on the maxi-min number of Nash equilibriums. At this point, m is clearly the maximum number of pure strategy equilibriums that a game Γ_{mxm} may possibly have.

3. The von Neumann Stable Set of Nash Equilibriums and the Nash Equilibrium Order Partition

Assuming we have obtained all pure and mixed strategy equilibriums for a given bimatrix game we may graph in 2-dimensional utility space all the outcomes of the game. We may assume the game to be in non-negative strategic equivalent form. So, all the outcome pairs will be in the non-negative quadrant.

We may also identify the following sets :

1) Ω = the set of all outcomes of the form $[a_{ij}, b_{ij}]$ where a_{ij} is an element of the matrix and b_{ij} is an element of the matrix B .
2) 2) P = the Pareto optimal set of outcomes given by the von Neumann stable set. of elements in Ω relative to strict domination (dom) and strict domination (s-dom) in terms of utility vectors. We say that $[u_1, u_2]$ dom $[v_1, v_2]$ if and only if $u_1 \geq v_1$ and $v_2 \geq u_2$. See von Neuman and Morgenstern(1967).
3) N = the set of all Nash equilibriums
4) 4) vN-N = The von Neumann stable set of Nash Equilibria. That is the set of Nash equilibriums that are not payoff dominated by any other Nash equilibrium.

Example 2 Let Γ = (A, B) where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The Nash equilibriums in pure strategies are N1={(1,1);[1,3]}, N2={(2,2);[2,2]}, N3={(1,1);[1,3]}. Let Nhk the mixed strategy equilibrium generated by the prime Nash equilibriums Nk and Nh. Then N12 ={((2/5, 3/5, 0), (2/3, 1/3, 0)); [2/3, 6/5]}, N13={((1/4, 0, 3/4), (3/4, 0, 1/4)); [3/4, 3/4]} and N23 ={((0, 1/3, 2/3), (0, 3/5, 2/5)); [6/5, 2/3]} are the mixed Nash equilibriums of order 2; and N123 = {((2/11, 3/11, 6/11), (6/11, 3/11, 2/11)); [6/11, 6/11]}.

In figure 2 we may observe all the Nash equilibriums for the given game. It happens that for the given game all the prime Nash equilibriums are pure strategy equilibriums and they constitute a Pareto optimal set for the game and also a von Neumann stable set of Nash equilibriums.

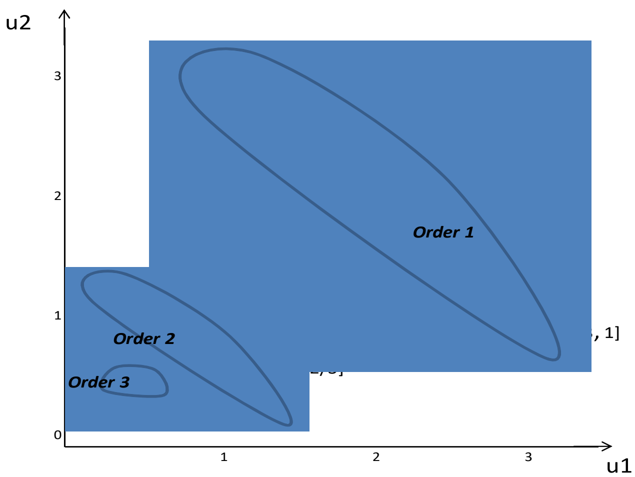

Figure 2. Order partition of Nash equilibriums

For a given game, if the Pareto optimal set contains a unique pure strategy Nash equilibrium, then such dominant equilibrium can be taken as the solution of the game even in adverse risk dominance conditions provided society is subscribed to endorse behavior that prioritize social over individual benefit. This and other dilemmas where pareto optimal dilemmas occur are solved by the Pareto optimal choice if such choice is the one that maximizes global preference.

4. The Global Preference Standard of Behavior

Often in economic, political and day to day activities, decision makers are faced with situations where the individual interest enters in conflict with the interest of society as a whole. Rational behavior understood as one consistent with individual maximization of utility, under conflict of interest, situations where indeterminacy emerges, such may be narrowed when the participants subscribe to higher standards as the one of global preference.

In terms of von Neumann and Morgenstern utilities, Nash (1954) gave a series of axioms to characterize a unique solution to the bargaining problem that can be summarized as the one that maximizes the product of expected utilities. It can be shown that the maximization of such (Nash) product actually gives as a result, the outcome that maximizes a consistent measure of global preference.

It also can be established that most preferent solutions tend to eliminate inequities. If players abide by the global preference standard of behavior, the solution to the game in example 2 above would be the equity solution N2 that gives 2 units of utility to each player.

In the case of a prisoner´s dilemma game, the most preferent solution would be the Pareto optimal outcome. Such outcomes would be the result of players´ rational behavior with independent selection of strategies that act rationally according to certain higher values.

Trying to capture the real world complexity, it is important to introduce –as crucial- the notion of expected utility. Following Nax specific propositions (Nax, 2015) one can assume that an organism must choose from action set of "x" options under assured conditions. There is always uncertainty as to the degree of success of the various alternatives in "x" which means essentially that each x 2 x determines a lottery –for example- that pays i offspring with probability pi.x; for i 0; 1; 2; …n.

In this case, the expected number of offspring from this lottery is .x/ D Pn j D1 jpj .x/. Let L be a lottery on X that delivers xi 2 X with probability qi for i D 1; : : : ; k. The probability of j offspring given L is then Pk iD1 qipj .xi/, so the expected number of offspring given L is Xn j D1 j X k iD1 qipj .xi/ D X k iD1 qi X k iD1 jpj .xi/ D X k iD1 qi .xi/; which is the fundamental aspect of expected value theorem with utility function (see Nax, 2015, and Brekke, K, 2011).

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Appendix A1

1. **Nash Equilibrium instability (NE).** In NE no player can increase his utility by deviating alone (given the other players stay put). However by deviating, alone, (using his maxmin strategy) a player may increase the likelihood (decrease the uncertainty) of obtaining with certainty the same utility level offered by the NE strategy. Other players that stay put with their NE strategy may end up with less utility that the one secured by his maximin strategy That is, a mixed strategy NE may not be risk-preference stable relative to maximin strategies.
2. **Transferring non transferable utility** Consider a utility vector attainable by a coalition v(S) = (u_1, …., u_s).
3. **Global Preference solution** Let Γ= (A,B) where $A = \begin{pmatrix} 9 & 0 \\ 0 & 5 \end{pmatrix} B = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$, .Then N1 = {(1, 1) ; [4, 9]} and N2 = {(2,2) ; [8, 5] } are equilibriums of order 1 and N12 = {((2/3, 1/3), (5/14, 9/14)^t); [2 2/3, 3 3/14] is a mixed Nash equilibrium of order 2.GP(O) = 8 x 5 = 40 GP(F) = 9 x 4 =36 (battle of the sexes (O)pera vs. (F)utball The global-preference solution is N2.
4. **Maximin Stability** Under what conditions are maximin strategies stable? ie. no unilateral deviation would give a higher utility to the deviating player.